SUPERCYCLIC AND HYPERCYCLIC NON-CONVOLUTION OPERATORS ON HILBERT-SCHMIDT ENTIRE FUNCTIONS

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Abstract

A continuous linear operator $T : X \to X$ is hypercyclic/supercyclic if there is a vector f such that the orbit $\mathrm{Orb}(T, f) = \{T^n f\}$, respectively the set of scalar multiples of the orbit elements, forms a dense set in X. A famous result states that every non-scalar convolution operator on the space $\mathcal{H}(\mathbb{C}^n)$ of n-variable entire functions is hypercyclic (and thus supercyclic). This result has been extended to infinite-dimensional holomorphy and Hilbert-Schmidt entire functions. On the other hand, up to now there are few 'explicit' examples of cyclic type non-convolution operators, both in finite and infinite dimensions. In this note we establish classes of hypercyclic and supercyclic non-convolution operators on the space of Hilbert-Schmidt entire functions. Moreover, we establish the existence of closed infinite-dimensional hypercyclic/supercyclic vector manifolds for the operators in the corresponding classes.

1. Introduction

Let $\mathbb{T} = (T_n)_{n\geq 0}$ be a sequence of operators on a TVS X (here, and in everything that follows, 'operator' refers to a linear and continuous map.) We let $\mathrm{Orb}(\mathbb{T}, f) \equiv \{T_n f\}_{n\geq 0}$ denote the orbit of $f \in X$ under $\mathbb{T}$ and let $\mathrm{Orb}_l\ (\mathbb{T}, f)$ and $\mathrm{Orb}_s\ (\mathbb{T}, f)$ denote the linear hull, respectively the set of scalar multiples of the elements in $\mathrm{Orb}(\mathbb{T}, f)$. Recall that $\mathbb{T}$ is said to be cyclic/supercyclic/hypercyclic if, respectively, $\mathrm{Orb}_l(\mathbb{T}, f)/\,\mathrm{Orb}_s(\mathbb{T}, f)/\,\mathrm{Orb}(\mathbb{T}, f)$ is dense in X for some $f \in X$. (Thus hypercyclic implies supercyclic, which, in turn, implies cyclic.) The vector f is said to be of the corresponding cyclic type (for $\mathbb{T}$). An operator T is cyclic (with cyclic vector f) when $(T_n \equiv T^n)$ is cyclic (with cyclic vector f), and analogously for super- and hypercyclicity. We write, simply, $\mathrm{Orb}(T, f)$ instead of $\mathrm{Orb}((T^n), f)$ etc. The significance of all these notions from the invariant subspace theory is exposed in [9; 11] — the latter reference is a nice survey article. However, let us just remark that the closure of any of the sets $\mathrm{Orb}_\nu(T, f)$, $\nu = l, s$ and $\mathrm{Orb}(T, f)$, is a closed invariant subset for $T : X \to X$ and hence, T lacks closed invariant subsets $M \neq \{0\}, X$ if and only if every vector $f \neq 0$ is hypercyclic. In the same way, if such a non-trivial closed invariant subset M exists, it must be formed by non-hypercyclic vectors, which inspires the study of structures of hypercyclic vectors, which are discussed below.

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A famous theorem due to Godefroy and Shapiro states: Every non-scalar convolution operator on the space $\mathcal{H}(\mathbb{C}^n)$ of n-variable entire functions (compact-open topology) is hypercyclic. (A *convolution operator* is defined as an operator that commutes with all translations, and a *scalar operator* is a scalar multiple of the identity.) This result originates from Birkhoff's classical result [5], saying that any translation operator $\tau_a : f \mapsto f(a+z)$ $(a \neq 0)$ is hypercyclic, and their proof [9] rests on the well-known Hypercyclicity Criterion (HC):

Proposition 1. [4; 9; 11] *Let X be a separable Fréchet space and $\mathbb{T} = (T_n)$ a sequence of operators on X. $\mathbb{T}$ is said to satisfy the* HC *for the sequence (n_k) provided there exist dense subsets $Z, Y \subseteq X$ (not necessarily linear) and a sequence of maps $\mathbb{S} = (S_{n_k} : Y \to X)$ (not necessarily continuous) such that:*

(1) $T_{n_k} z \to 0$ *for all* $z \in Z$,
(2) $S_{n_k} y \to 0$ *for all* $y \in Y$,
(3) $T_{n_k} S_{n_k} y = y$ *for all* $y \in Y$.

If $\mathbb{T}$ satisfies the HC *for (n_k), then $\mathbb{T}$ is* hereditarily hypercyclic *for the sequence (n_k), i.e., $(T_{m_k})_k$ is hypercyclic for every subsequence (m_k) of (n_k).*

In [18] we extended Godefroy-Shapiro's Theorem to infinite-dimensional holomorphy by proving: Every non-scalar convolution operator on the Fréchet space $\mathcal{H}_H(E)$ of Hilbert-Schmidt entire functions on a separable Hilbert space E, is hypercyclic (see also [1]). This and the result of Godefroy and Shapiro motivate us to study cyclic type non-convolution operators — both in finite and infinite dimensions. Such an investigation is also proposed by Aron and Markose in [2], and they and others [8] have obtained the following preliminary result: $Tf \equiv f'(\lambda z + a)$ forms a hypercyclic non-convolution operator on $\mathcal{H}(\mathbb{C})$ for any $a, \lambda \in \mathbb{C}$ provided $|\lambda| > 1$ (a complete proof can be found in [8]). In a recent work [22] we established some classes of hypercyclic and supercyclic non-convolution operators on $\mathcal{H}(\mathbb{C}^n)$. For our purpose we applied results from our study of so called PDE-preserving operators (more on this below). Moreover, we applied Proposition 1 (see below) and our key to show that the required factorization in 3 was the following classical n-variable result of E. Fischer (see [24]): For any homogeneous polynomial $P \neq 0$, $P(D)\bar{P}$ maps the set of m-homogeneous polynomials bijectively for all m. (Here $\bar{P}(z) \equiv \overline{P(\bar{z})}$ and $\bar{P} : f \mapsto \bar{P} \cdot f$, and we say that $(P(D), \bar{P})$ forms a Fischer pair.) This result is purely algebraic in the sense that any such polynomial space is finite-dimensional. However, in [19, theorem 3] we obtained an infinite-dimensional analogue of Fischer's Theorem for Hilbert-Schmidt polynomials $\mathcal{P}_H(^m E)$ (see Proposition 3, below), which will play a major role in this note.

The objective in this article is to establish supercyclic and hypercyclic non-convolution operators on $\mathcal{H}_H(E)$. We shall apply, and develop, the technique of Fischer pairs. We establish, in Section 3.1, a class $\mathcal{O}_{\mathbb{H}}$ of supercyclic operators and a multiplicative closed subclass $\mathcal{O}_{\mathbb{H}}^{\infty}$ formed by hypercyclic operators, where $\mathcal{O}_{\mathbb{H}}^{\infty}$, and thus $\mathcal{O}_{\mathbb{H}}$, is rich with non-convolution operators. Another purpose is the following. A series of papers (see e.g. [3; 10; 12; 13; 15; 16; 17]) have studied when operators on Banach and Hilbert spaces admit supercyclic and hypercyclic subspaces. (Recall that a *hypercyclic subspace* for an operator $T : X \to X$ is a closed infinite-dimensional subspace $H \subseteq X$ whose non-zero vectors are hypercyclic for T. Supercyclic and cyclic subspaces are defined in the same way.) Relatively new results show that it is possible to connect this property with spectral properties. Indeed, in [10] it is proved that the following are equivalent for a hereditarily hypercyclic operator T on a Banach space X:

(i) T has a hypercyclic subspace.
(ii) There is an infinite-dimensional closed subspace $Y \subseteq X$ and a sequence (n_k), for which T is hereditarily hypercyclic, such that $T^{n_k} \to 0$ pointwise on Y.
(iii) There is an infinite-dimensional closed subspace $Y \subseteq X$ and a sequence (n_k) such that $T^{n_k} \to 0$ pointwise on Y.
(iv) The essential spectrum of T meets the closed unit disc.

The result was first obtained in the setting of Hilbert spaces X [13], and that (ii) implies (i) originates from [15]. (That (i) through (iv) are equivalent for both Hilbert and Banach space operators is interesting in view of the fact that the invariant subset problem is solved for Banach but not for Hilbert spaces (see [9]). Indeed, from the discussion above, $T : X \to X$ lacks non-trivial closed invariant subsets if and only if X is a hypercyclic subspace. Similarly, the restriction of T to any of its hypercyclic subspaces, if such exist, has no non-trivial closed invariant subsets.) A spectral sufficient condition for the existence of a supercyclic subspace for Banach space operators is obtained in [16]. It is known that every infinite-dimensional separable Banach space admits an operator with a hypercyclic subspace [12, corollary 2.2]. However, there are indeed supercyclic/hypercyclic operators without supercyclic/hypercyclic subspaces. For instance, the unilateral backward shift $B : \ell_2 \to \ell_2$, defined by $e_{n+1} \mapsto e_n$ and $e_0 \mapsto 0$ where $(e_n)_{n\geq 0}$ is a given orthonormal basis, is supercyclic but lacks supercyclic subspaces [17]. Further, if $|\lambda| > 1$, then λB is hypercyclic but does not admit any hypercyclic subspaces [15]. Now, in this note we extend this topic to the setting of Fréchet spaces. In fact, we complement a parallel work [23] where we prove that (ii) is sufficient for (i) also for non-normable Fréchet spaces that admit a continuous norm. This result was independently obtained by Bonet and co-workers in [6]. We prove in Section 3.2 that every operator in $\mathcal{O}_{\mathbb{H}}$ respective in $\mathcal{O}_{\mathbb{H}}^{\infty}$ has a supercyclic respective a hypercyclic subspace. However, we do not apply the sufficient condition (ii), instead we give a direct proof based on a technique quite different from those used in [6] and in [23].

Finally we remark that the elements of $\mathcal{O}_{\mathbb{H}}$, and $\mathcal{O}_{\mathbb{H}}^{\infty}$ are PDE-preserving in the sense that they map the kernel-set of any homogeneous convolution operator invariantly. Moreover, by virtue of a characterization result for such PDE-preserving operators (see Proposition 4, below), the elements of $\mathcal{O}_{\mathbb{H}}$ and $\mathcal{O}_{\mathbb{H}}^{\infty}$ have explicit representations — in fact, they can be identified with certain sequences of holomorphic functions.

We shall start with some ground work before we prove our main results in Section 3.

2. Fundamentals

We introduce the space of Hilbert-Schmidt entire functions. We shall be quite brief and refer to [18] and [19] for a more comprehensive exposition (see also [7] where a similar type of holomorphy is studied).

E denotes a separable complex Hilbert space and $(\cdot,\cdot)$ the corresponding inner product. By $\mathcal{P}_F({}^nE)$ we denote the space of n-homogenous polynomials on E of finite type, i.e., the subspace of the n-homogenous polynomials on E that is spanned by the elements $(\cdot,y)^n$, $y \in E$. We endow $\mathcal{P}_F({}^nE)$ with the inner product defined by $((\cdot,y)^n,(\cdot,z)^n)_n \equiv n!(z,y)^n$ (well-defined!). The space of n-homogenous Hilbert-Schmidt polynomials, denoted by $\mathcal{P}_H({}^nE)$, is the completion of $\mathcal{P}_F({}^nE)$ w.r.t. the inner product $(\cdot,\cdot)_n$, and we use the symbol $||\cdot||_n$ for the corresponding norm. It follows that $\mathcal{P}_H({}^nE)$ is continuously imbedded into the Banach space of n-homogenous continuous polynomials on E.

Let (e_i) be an orthonormal basis for E. For a given (infinite) multi-index $\alpha \in \mathbb{N}^{\mathbb{N}} \equiv \oplus_{\mathbb{N}}\mathbb{N}$, let $e_\alpha \equiv \prod_{supp\ \alpha}(\cdot,e_i)^{\alpha_i} \in \mathcal{P}_H({}^{|\alpha|}E)$. Here $\mathbb{N} \equiv \{0,1,...\}$, $supp\ \alpha \equiv \{i : \alpha_i \neq 0\}$ and $|\alpha| \equiv \sum \alpha_i$. The elements e_α, $|\alpha| = n$, form an orthogonal basis for $\mathcal{P}_H({}^nE)$ and $||e_\alpha||_n^2 = \alpha! \equiv \alpha_1!...$ (see [7, lemma 1]). Thus $\mathcal{P}_H({}^nE)$ is the Hilbert space formed by all $P = \sum_{|\alpha|=n} P_\alpha e_\alpha$ such that $\sum_{|\alpha|=n}|P_\alpha|^2\alpha! < \infty$ and

$$||P||_n^2 = \sum_{|\alpha|=n} |P_\alpha|^2\alpha!.$$

We define $\bar{P} \equiv \sum \bar{P}_\alpha e_\alpha$ and note that $||\bar{P}||_n = ||P||_n$. Hilbert-Schmidt polynomials are stable under multiplication in the sense that if $P \in \mathcal{P}_H({}^nE)$ and $Q \in \mathcal{P}_H({}^mE)$, then $PQ \in \mathcal{P}_H({}^{n+m}E)$, which is important, as is the following:

$$||P||_n||Q||_m \leq ||PQ||_{n+m} \leq 2^{n+m}||P||_n||Q||_m, \tag{2.1}$$

see [19, lemma 1]. Thus, the set $\mathbb{H} \equiv \cup_{n=0}^{\infty} \mathcal{P}_H({}^nE)$ $(\mathcal{P}_H({}^0E)) \equiv \mathbb{C})$ of homogeneous polynomials is multiplicative closed.

We denote by $\mathcal{F}_H(E)$ the space of all formal power-series $f = \sum f_n$, $f_n \in \mathcal{P}_H({}^nE)$, i.e. $\mathcal{F}_H(E) \equiv \prod_{n=0}^{\infty} \mathcal{P}_H({}^nE)$, and define $\bar{f} \equiv \sum \bar{f}_n$. $\mathcal{F}_H(E)$ is a ring as $\mathbb{H}$ is multiplicative closed. The space of Hilbert-Schmidt polynomials, denoted by $\mathcal{P}_H(E)$, is the subring $\oplus_{n=0}^{\infty} \mathcal{P}_H({}^nE)$ of $\mathcal{F}_H(E)$. The polynomials of degree $\leq m$ is the space $\mathcal{P}_H^m(E) \equiv \prod_{n=0}^{m} \mathcal{P}_H({}^nE)$ — we consider $\mathcal{P}_H^m(E)$ and $\mathcal{P}_H({}^mE)$ as subspaces of $\mathcal{F}_H(E)$ in the obvious way.

Now, the space of entire functions of Hilbert-Schmidt type on E, $\mathcal{F}_H(E)$ is formed by all $f = \sum f_n \in \mathcal{F}_H(E)$ such that

$$||f||_{H:r} \equiv \sum_{n\geq 0} r^n||f_n||_n/\sqrt{n!} < \infty, \quad r > 0, \tag{2.2}$$

equipped with the semi-norms thus defined. $\mathcal{H}_H(E)$ is a Fréchet space, a subring of $\mathcal{F}_H(E)$ and, in particular, $\mathcal{H}_H(\mathbb{C}^n)$= $\mathcal{H}(\mathbb{C}^n)$. The series $\sum f_n$ converges absolutely in $\mathcal{H}_H(E)$ for every $f = \sum f_n \in \mathcal{H}_H(E)$. Thus, $\mathcal{H}_H(E)$ is separable and $\mathcal{P}_H(E)$ forms a dense subspace. (If D_x denotes the directional derivative along x, $\mathcal{H}_H(E)$ can also be described as the space of all Gâteaux holomorphic functions $f \in \mathcal{H}_G(E)$ such that $f_n \equiv D^n_{(\cdot)} f(0)/n! \in \mathcal{P}_H(^nE)$ $n = 0, ...$ and (2.2) hold.)

Given $r > 0$, $\mathrm{EXP}_r(E)$ denotes the Banach space of all $\varphi = \sum \varphi_n \in \mathcal{F}_H(E)$ such that, for some $M > 0$, $||\varphi_n||_n \leq Mr^n/\sqrt{n!}$, $n = 0, ...$, equipped with the norm $|||\varphi|||_{H:r} \equiv \sup_n \sqrt{n!} r^{-n} ||\varphi_n||_n$. The symbol $\mathrm{EXP}_H(E)$ denotes the union $\cup_{r>0}$ $\mathrm{EXP}_r(E)$ provided with the corresponding inductive locally convex topology. From (2.1) we deduce that $\mathrm{EXP}_H(E)$ is a subring of $\mathcal{H}_H(E)$, and every $\varphi \in \mathrm{EXP}_H(E)$ defines an exponential type function, i.e., a holomorphic function with $|\varphi(y)| \leq Me^{r||y||}$ for some $M, r > 0$.

If $y \in E$ we put $e_y \equiv e^{(\cdot,y)} \in \mathrm{EXP}_H(E) \subseteq \mathcal{H}_H(E)$ and have:

Proposition 2. (See [18]) *$\mathcal{H}_H(E)$ is reflexive and the map $\mathcal{F} : \lambda \mapsto \mathcal{F}\lambda$, $\mathcal{F} : \lambda(y) \equiv \overline{\lambda(e_y)}$, is an anti-linear isomorphism between $\mathcal{H}'_H(E)$, provided with the strong topology, and* $\mathrm{EXP}_H(E)$.

We put $\mathcal{H}_H(E)$ and $\mathrm{EXP}_H(E)$ into duality by the sesqui-linear form $\langle f, \varphi\rangle \equiv \mathcal{F}^{-1}\varphi(f) = \sum_{n\geq 0}(f_n, \varphi_n)_n$. Multiplication $\varphi : \psi \mapsto \varphi\psi$ is continuous on $\mathrm{EXP}_H(E)$ for every $\varphi \in \mathrm{EXP}_H(E)$ and we define $\varphi(D) \equiv {}^t\bar{\varphi}:\mathcal{H}_H(E) \to \mathcal{H}_H(E)$. (In particular, $P(D) = \sum_i D^n_{y_i}$ if $P = \sum_i (\cdot, y_i)^n \in \mathcal{P}_F(^nE)$.) By Proposition 2, every $\varphi(D)$ is continuous for the topology on $\mathcal{H}_H(E)$ and, in particular, $e_a(D)$ is the translation operator τ_a, $\tau_a f(x) \equiv f(x + a)$. Thus $\mathcal{H}_H(E)$ is stable under translations, and it follows that $\varphi \mapsto \varphi(D)$ defines a one-to-one correspondence between $\mathrm{EXP}_H(E)$ and the set $\mathcal{C}$ of convolution operators on $\mathcal{H}_H(E)$ [18, proposition 2.2]. (Thus $\mathcal{H}'_H(E)$ $\simeq$ $\mathrm{EXP}_H(E) \simeq \mathcal{C}$.)

Proposition 3. (See [19]) *Let $0 \neq P \in \mathcal{P}_H(^mE)$. Then $P(D) \circ \bar{P}$ is a bijection on $\mathcal{H}_H(E)$ and maps every $\mathcal{P}_H(^nE)$ bijectively. (See [24] for the analogue in finite dimensions.)*

Lemma 1. *Let $0 \neq P \in \mathcal{P}_H(^mE)$. Then $A \equiv \bar{P}(P(D)\bar{P})^{-1}$ maps $\mathcal{P}_H(^nE)$ into $\mathcal{P}_H(^{n+m}E)$ with a norm not greater than $1/||P||_m$.*

PROOF. By the left inequality in (2.1), $||\bar{P}g||_{n+m} \geq ||P||_m||g||_n$ for any $g \in \mathcal{P}_H(^nE)$, since $||P||_m = ||\bar{P}||_m$.

Let $f \in \mathcal{P}_H(^nE)$ and put $g \equiv (P(D)\bar{P})^{-1}f \in \mathcal{P}_H(^nE)$. Then $\bar{P}g = Af$ and Cauchy-Schwartz' Inequality gives

$$||f||_n||g||_n = ||P(D)\bar{P}g||_n||g||_n \geq (P(D)\bar{P}g, g)_n = ||\bar{P}g||^2_{n+m} \geq ||Af||_{n+m}||P||_m||g||_n,$$

and the assertion follows. ■

The algebra of operators on $\mathcal{H}_H(E)$ is denoted by $\mathcal{L} = \mathcal{L}\ (\mathcal{H}_H(E))$. An operator $T \in \mathcal{L}$ is said to be PDE-preserving for a set $\mathbb{E} \subseteq \mathrm{EXP}_H(E)$ if $T \ker\varphi(D) \subseteq \ker\varphi(D)$ for all $\varphi \in \mathbb{E}$. The set of PDE-preserving operators for $\mathbb{E}$, $\mathcal{O}(\mathbb{E})$, is clearly a subalgebra of $\mathcal{L}$ and, in turn, $\mathcal{C}$ forms a commutative subalgebra of $\mathcal{O}(\mathbb{E})$. This is the case for any set $\mathbb{E}$. In view of our purposes, we are most interested in the algebra $\mathcal{O}(\mathbb{H})$ that we now shall describe.

We denote by $\mathcal{S}$ the set of sequences $\Phi = (\varphi_n) = (\varphi_0, ...)$ in $\mathrm{EXP}_H(E)$ such that $||H_m \varphi_n||_m \leq MR^n r^m/\sqrt{m!}$ for some $M, R, r \geq 0$. H_m denotes the projector in $\mathcal{F}_H(E)$ onto $\mathcal{P}_H(^mE)$ defined by $f = \sum f_n \mapsto f_m$. We have the following one-to-one correspondence between $\mathcal{O}(\mathbb{H})$ and $\mathcal{S}$ (the result is of independent interest):

Proposition 4. *Let $T \in \mathcal{L}$, then the following are equivalent:*

(1) $T \in \mathcal{O}(\mathbb{H})$*;*
(2) *For every $m \geq 0$ there is a $T^{(m)} \in \mathcal{L}$ such that $P(D)T = T^{(m)}P(D)$ for all $P \in \mathcal{P}_H(^mE)\setminus \{0\}$;*
(3) *$T = \Phi(D)$ for some $\Phi = (\varphi_n) \in \mathcal{S}$ where $\Phi(D)f \equiv \sum_{n \geq 0} H_n \varphi_n(D)f$.*

The sequence Φ and the operator $T^{(m)}$ are unique and, in fact, $T^{(m)} = \Phi^{(m)}(D) \in \mathcal{O}_{\mathbb{H}}$ where $\Phi^{(m)} \equiv (\varphi_{n+m}) \in \mathcal{S}$.

Proof. Clearly, 2 implies 1 and the uniqueness of $T^{(m)}$ follows by the surjectivity of $P(D)$ (Proposition 3). It is not difficult to prove that any $\Phi(D)$, $\Phi \in \mathcal{S}$, defines an element of $\mathcal{L}$ (cf. [20]). From the fact that if $P \in \mathcal{P}_H(^mE)$, then $P(D)H_n = H_{n-m}P(D)$ if $n \geq m$ and $P(D)H_n = 0$ otherwise, we obtain

$$P(D)\Phi(D) = \sum_{n \geq 0} P(D)H_n\varphi_n(D) = \sum_{n \geq m} H_{n-m}P(D)\varphi_n(D) = \Phi^{(m)}(D)P(D),$$

since $P(D)$ and $\varphi_n(D)$ commute. Hence 3 implies 2.

It remains to prove that any $T \in \mathcal{O}(\mathbb{H})$ is of the form $\Phi(D)$ for some unique $\Phi \in \mathcal{S}$. A proof of an analogous statement for nuclearly entire functions can be found in [20], and, using almost identical arguments to those in the proof of [20, theorem 2], we conclude that there is a sequence (φ_n) in $\mathrm{EXP}_H(E)$ such that $(*)$ $H_n\varphi_n(D) = H_nT$ for all n. (The crucial part in [20] is the surjectivity of any non-zero homogeneous differential operator, which we know holds true for $\mathcal{H}_H(E)$ in view of Proposition 3.) We must prove that $\Phi = (\varphi_n) \in \mathcal{S}$. Let $x \in E$ be any element on the unit sphere, $||x|| = 1$. If we let both sides in $(*)$ act on $(\cdot, x)^n$ and identify homogeneous parts, we deduce that $(\cdot, x)^n H_m(\varphi_n) = H_{n+m}{}^tT(\cdot, x)^n$. We note that $X \equiv \{(\cdot, x)^n/n! : n \geq 0\}$ forms a bounded set in $\mathrm{EXP}_H(E)$. By the reflexivity of $\mathcal{H}_H(E)$ (Proposition 2), the transpose ${}^tT$ is continuous for the topology on $\mathrm{EXP}_H(E)$, hence ${}^tTX$ is a bounded set in $\mathrm{EXP}_H(E)$. Now, one can prove that a set is bounded in $\mathrm{EXP}_H(E)$ if and only if it is contained and bounded in some $\mathrm{EXP}_r(E)$. Thus, for some $M, r > 0$, $||H_m\psi||_m \leq Mr^m/\sqrt{m!}$ for all $\psi \in {}^tTX$ and $m \geq 0$. Hence

$$||H_{n+m}{}^tT(\cdot, x)^n||_{n+m} \leq n!M\frac{r^{n+m}}{\sqrt{(n+m)!}} \leq n!M\frac{r^{n+m}}{\sqrt{m!n!}} \leq Mr^{n+m}\frac{\sqrt{n!}}{\sqrt{m!}} \qquad (2.3)$$

for all n and m. On the other hand, $||(\cdot,x)^n||_n = \sqrt{n!}$ and (2.1) gives

$$\sqrt{n!}||H_m\varphi_n||_m = ||(\cdot,x)^n||_n||H_m\varphi_n||_m \leq ||(\cdot,x)^n H_m(\varphi_n)||_{n+m}.$$

This and (2.3) show that $(\varphi_n) \in \mathcal{S}$. Thus $\sum H_n\varphi_n(D)$ defines an operator $\Phi(D) \in \mathcal{L}$, and we must prove that $T = \Phi(D)$. But, by virtue of Taylor's formula, it is easy to prove that every element of $\mathcal{O}(\mathbb{H})$ must map any $\mathcal{P}^n_H(E)$, and thus $\mathcal{P}_H(E)$, invariantly. In view of this, $(*)$ implies that $T = \Phi(D)$ on $\mathcal{P}_H(E)$ and hence, since $\mathcal{P}_H(E)$ is dense and by continuity, $T = \Phi(D)$. ■

Example 1.

(a) For any $\varphi(D) \in \mathcal{C}$, $\varphi(D) = \Phi(D)$ where $\Phi = (\varphi, \varphi, ...) \in \mathcal{S}$.

(b) Let $\varphi_n \equiv 1$ if $n \leq m$ and $\varphi_n \equiv 0$ otherwise, and put $\Phi \equiv (\varphi_n) \in \mathcal{S}$. Then $\Phi(D)$ is the Taylor projector of order m, i.e., the operator obtained by mapping $f \in \mathcal{H}_H(E)$ onto its Taylor polynomial of order m at the origin. (See [19] for more examples of PDE-preserving projectors.)

(c) With $\Phi = (\varphi_n \equiv n)$, $\Phi(D)$ is the Euler operator $\langle\cdot, D\rangle \equiv \sum_{i=0}^{\infty} x_iD_i \in \mathcal{O}(\mathbb{H})$ (i.e. $f \mapsto \sum x_iD_if(x)$ where $x = \sum x_ie_i$, $D_i \equiv D_{e_i}$ more generally, for any power $m \geq 1$, $\langle\cdot, D\rangle^m = \Phi(D)$ where $\Phi = (\varphi_n \equiv n^m)$.

Theorem 1. *Assume $T = \Phi(D) \in \mathcal{O}(\mathbb{H})$ is cyclic and let f be a corresponding cyclic vector. Then $T^{(m)} = \Phi^{(m)}(D)$ is also cyclic and $P(D)f$ forms a cyclic vector for any m and $P \in \mathcal{P}_H(^mE)\backslash\{0\}$. The analogue holds true for both super- and hypercyclicity.*

PROOF. Put $S \equiv T^{(m)}$. We note that $P(D)T^n = S^nP(D)$ for all $n \geq 0$. Hence $P(D)$ Orb $(T, f) = \{P(D)T^nf\} =$ Orb $(S, P(D)f)$, and from this we also deduce $P(D)$ $\text{Orb}_\nu(T, f) \subseteq \text{Orb}_\nu(S, P(D)f)$, $\nu = s, l$. Since $P(D)$ is surjective (Proposition 3), $P(D)$ maps dense sets onto dense sets and our claim follows. ■

We equip $\mathcal{S}$ with the algebra structure induced by $\mathcal{O}(\mathbb{H})$ so that $(\Phi\Psi)(D) = \Phi(D)\Psi(D)$.

In fact, one can prove [20, theorem 6] that if $(\xi_n) = \Phi\Psi$ in $\mathcal{S}$, then

$$\xi_n = \sum_{i=0}^{\infty} H_i(\varphi_n)\psi_{n+i}, \qquad \Phi = (\varphi_n),\ \Psi = (\psi_n). \tag{2.4}$$

An element $\varphi \in \text{EXP}_H(E)$ is non-degenerate if $\varphi(0) \neq 0$ and a sequence $\Phi = (\varphi_n)$ in $\text{EXP}_H(E)$ is non-degenerate if all φ_n are. From (2.4) we deduce that the product $\Phi\Psi$ of any non-degenerate sequences Φ and Ψ in $\mathcal{S}$ is again non-degenerate ($\xi_n(0) = \varphi_n(0)\psi_n(0)$). Clearly, for *any* sequence $\Phi \in \text{EXP}_H(E)$, $\sum H_n\varphi_n(D)$ is a well-defined map on $\mathcal{P}_H(E)$, and it is convenient to also use the symbol $\Phi(D)$ for this mapping.

Lemma 2. *For any sequence $\Phi = (\varphi_n)$ in $\mathrm{EXP}_H(E)$, $\Phi(D)$ maps every $\mathcal{P}_H^n(E)$ invariantly and continuously. If Φ is non-degenerate, $\Phi(D)$ maps every $\mathcal{P}_H^n(E)$ isomorphically and the restriction of $\Phi(D)$ to $\mathcal{P}_H(E)$ is a linear isomorphism. We assume here that $\mathcal{P}_H^n(E)$ is provided with the topology induced by $\mathcal{H}_H(E)$.*

Proof. That $\Phi(D)$ maps $\mathcal{P}_H^n(E)$ invariantly is obvious since any convolution operator does. Next, the restriction of $\Phi(D)$ to $\mathcal{P}_H^n(E)$ is given by $\sum_{m\leq n} H_m\varphi_m(D)$, which is a finite sum of operators on $\mathcal{P}_H^n(E)$, and the first part follows.

Next, assume Φ is non-degenerate. By the Open-Mapping Theorem and the first part of the proof, we only have to prove that $\Phi(D)$ is a linear isomorphism on every $\mathcal{P}_H^n(E)$. $\Phi(D)1 = \varphi_0(0) \neq 0$, and hence $\Phi(D)$ is surjective on $\mathcal{P}_H^0(E) = \mathbb{C}$. Next we note that if $|\alpha| = m \geq 1$: $(*)$ $\Phi(D)e_\alpha = \varphi_m(0)e_\alpha +$ (terms of degree $< m$). Assume $\Phi(D)$ is surjective on every $\mathcal{P}_H^m(E)$, $m \leq n-1$ and let $P \in \mathcal{P}_H^n(E)$. In view of $(*)$, we may find a $Q_n \in \mathcal{P}_H(^nE)$ such that $\Phi(D)Q_n - P \in \mathcal{P}_H^{n-1}(E)$ and hence, by the inductive hypothesis, $\Phi(D)Q = \Phi(D)Q_n - P$ for some $Q \in \mathcal{P}_H^{n-1}(E)$. Thus $\Phi(D)\mathcal{P}_H^n(E) = \mathcal{P}_H^n(E)$ for all n. To prove that $\Phi(D)$ is one-to-one on any $\mathcal{P}_H^n(E)$, it is clearly enough to prove that $\mathcal{P}_H^n(E)$ is injective on $\mathcal{P}_H(E)$, which is obvious in view of $(*)$. ■

3. The main results

3.1. Supercyclic and hypercyclic operators

Let $\mathcal{S}_{\mathbb{H}}$ denote the set of sequences $\Phi \in \mathcal{S}$ of the form $\Phi = (\psi_n P_n)$ where $\Psi = (\psi_n)$ is a non-degenerate sequence in $\mathrm{EXP}_H(E)$ and $0 \neq P_n \in \mathcal{P}_H(^mE)$, $n = 0, ...$ for some $m \geq 1$. By $\mathcal{O}_{\mathbb{H}}$ we denote the corresponding class of operators $\Phi(D) \in \mathcal{O}(\mathbb{H})$. Clearly, the homogenity degree m is unique and we let $\mathcal{S}_m$ denote the set of sequences $\Phi = (\psi_n P_n)$ in $\mathcal{S}_{\mathbb{H}}$ where $P_n \in \mathcal{P}_H(^mE)$. Thus $\mathcal{S}_{\mathbb{H}} = \cup_{m\geq 1}\mathcal{S}_m$ and $\{\mathcal{S}_m\}$ is a partition of $\mathcal{S}_{\mathbb{H}}$. It is convenient to clarify the following. Let $\Psi = (\psi_n)$ be a non-degenerate sequence in $\mathrm{EXP}_H(E)$ and let $0 \neq P_n, P \in \mathcal{P}_H(^mE)$ where $m \geq 1$, then:

(1) If $\Psi \in \mathcal{S}$ and $||P_n||_m \leq CR^n$ for all n, then $\Phi \equiv (\psi_n P_n) \in \mathcal{S}_{\mathbb{H}}$;

(2) $\Phi \equiv (\psi_n P) \in \mathcal{S}_{\mathbb{H}}$ if $\Psi \in \mathcal{S}$;

(3) $\Phi \equiv (P_n) \in \mathcal{S}_{\mathbb{H}}$ if $||P_n||_m \leq CR^n$ for all n.

(1 and 3 are elementary and 2 is an easy consequence of (2.1).)

Theorem 2. *Every $\Phi(D) \in \mathcal{O}(\mathbb{H})$ is supercyclic. Thus, in particular, any operator $\Phi(D) = \Psi(D)P(D)$, where $\Psi \in \mathcal{S}$ is non-degenerate and $P \in \mathbb{H}\backslash\mathbb{C}$, is supercyclic.*

Proof. Let $\Phi = (\psi_n P_n) \in \mathcal{S}_m$ and put $\Psi \equiv (\psi_n)$. It suffices to prove that there is a sequence (r_n) in $\mathbb{C}$ such that $(T_n \equiv r_n\Phi(D)^n)$ is hypercyclic, and we intend to apply Proposition 1 with $Z = Y = \mathcal{P}_H(E)$. First of all we conclude that, since $m \geq 1$, $\Phi(D)^n f = 0$ for all n sufficiently large if $f \in \mathcal{P}_H(E)$. Thus, for *any* sequence (r_n), $T_n \to 0$ pointwise on $Z = \mathcal{P}_H(E)$. Define B: $\mathcal{P}_H(E) \to \mathcal{P}_H(E)$ by $B \equiv \sum_{n\geq m} P_{n-m}(D)H_n$. Let $\Phi_0 = (\phi_n)$ be a non-degenerate sequence in $\mathrm{EXP}_H(E)$ with $\Phi_0^{(m)} = \Psi$ and consider $\Phi_0(D) : \mathcal{P}_H(E) \to \mathcal{P}_H(E)$ (since Ψ may be outside $\mathcal{S}$, it is possible that $\Phi_0 \notin \mathcal{S}$). We claim that $\Phi(D) = B\Phi_0(D)$ on $\mathcal{P}_H(E)$. Indeed,

$$B\Phi_0(D) = \sum_{n\geq m} P_{n-m}(D)H_n\phi_n(D) = \sum_{n\geq m} H_{n-m}P_{n-m}(D)\phi_n(D) = \Phi(D),$$

since $\phi_n = \psi_{n-m}$ for $n \geq m$. By Lemma 2, $\Phi_0(D)^{-1} : \mathcal{P}_H(E) \rightarrow \mathcal{P}_H(E)$ exists, and, from Lemma 1, we can define a map $A : \mathcal{P}_H(E) \rightarrow \mathcal{P}_H(E)$ by $A \equiv \sum_{n\geq 0} \bar{P}_n(P_n(D)\bar{P}_n)^{-1}H_n$. We deduce that $BA =$ the identity on $Y = \mathcal{P}_H(E)$, so with $C \equiv \Phi_0^{-1}(D)A$, $(*)$ $\Phi(D)C = Id_Y$. Now, in view of Lemma 1 and Lemma 2, C maps $\mathcal{P}_H^n(E)$ into $\mathcal{P}_H^{n+m}(E)$ continuously. Let $\sigma(n)$ denote the operator norm of this map. Here we assume every $\mathcal{P}_H^n(E)$ is provided with the *norm* $||\cdot||_{H:1}$, which clearly generates the topology that is induced by $\mathcal{H}_H(E)$. For $f \in \mathcal{P}_H^k(E)$ and $n \geq k$ we estimate

$$||C^n f||_{H:1} = ||C(C^{n-1}f)||_{H:1} \leq \sigma(k + m(n-1))||C^{n-1}f||_{H:1} \leq ...$$
$$\leq \prod_{i=1}^{n} \sigma(k + m(n-i))||f||_{H:1} \leq \sigma(n + m(n-1))^n||f||_{H:1} \equiv \hat{\sigma}(n)||f||_{H:1},$$

since σ is increasing. Put $r_n \equiv n!\hat{\sigma}(n)$ and define $S_n \equiv r_n^{-1}C^n$. Then $T_nS_n = Id_Y$ in view of $(*)$ and for any continuous semi-norm $||\cdot||_{H:r}$ on $\mathcal{H}_H(E)$,

$$||S_nf||_{H:r} = \sum_{i=0}^{k+nm} \frac{r^i}{\sqrt{i!}}||H_i(S_nf)|_i \leq r^{k+nm}||S_nf||_{H:1} \leq \frac{r^{k+nm}}{n!}||f||_{H:1} \rightarrow 0,$$

as $n \rightarrow \infty$. Thus $S_n \rightarrow 0$ pointwise on $Y = \mathcal{P}_H(E)$ and the proof is complete by virtue of Proposition 1. ■

$\mathcal{O}_{\mathbb{H}}$ is not multiplicative closed. However, let $\mathcal{O}_{\mathbb{H}}^*$ denote the subset of $\mathcal{O}_{\mathbb{H}}$ formed by the special type of operators $\Psi(D)P(D)$ in Theorem 2. (Thus $\mathcal{O}_{\mathbb{H}}^*$ corresponds in $\mathcal{S}$ to the subset $\mathcal{S}_{\mathbb{H}}^*$ of $\mathcal{S}_{\mathbb{H}}$ described in 2 above.) Then, in view of (2.4), it is easily checked that $\mathcal{O}_{\mathbb{H}}^*$ is multipicative closed, i.e. $\mathcal{O}_{\mathbb{H}}^*\ \mathcal{O}_{\mathbb{H}}^* \subseteq \mathcal{O}_{\mathbb{H}}^*$, and $\mathcal{O}_{\mathbb{H}}\mathcal{O}_{\mathbb{H}}^* \subseteq \mathcal{O}_{\mathbb{H}}$. Let us also note the elementary fact that, for any given $m \geq 0$ and $\Phi \in \mathcal{S}_{\mathbb{H}}$; $\Phi^{(m)} \in \mathcal{S}_{\mathbb{H}}$ and, conversely, there is a $\Psi \in \mathcal{S}_{\mathbb{H}}$ such that $\Psi^{(m)} = \Phi$.

Next, a *supercyclic vector manifold* for an operator $T : X \rightarrow X$, is a subspace $S \subseteq X$ whose non-zero vectors are supercyclic for T. Accordingly, a supercyclic subspace is a closed infinite-dimensional supercyclic vector manifold. Hypercyclic vector manifolds are defined in the same way.

Theorem 3. *Assume* $\Phi(D) \in \mathcal{O}_{\mathbb{H}}$ $(\Phi = (\varphi_n) \in \mathcal{S}_H)$. *Then, for any set* $A \subseteq \mathcal{S} \times \mathbb{H}$ *such that* $P \neq 0$, $\Psi^{(m)} = \Phi$ *if* $P \in \mathcal{P}_H(^mE)$, *for all* $(\Psi, P) \in A$*:*

$$\mathcal{I}_A \equiv \bigcup_{(\Psi,P)\in A} \{P(D)f : f \text{ supercyclic for } \Psi(D)\} \tag{3.1}$$

forms an invariant set under $(\Phi(D))$ *of supercyclic vectors for* $\Phi(D)$*. Moreover, for every* $m \geq 0$ *there is a vector* $f \in \mathcal{H}_H(E)$ *such that*

$$\mathcal{M}_m = \{P(D)f : P \in \mathcal{P}_H(^mE)\}$$

forms a supercyclic vector manifold for $\Phi(D)$, and $P \mapsto P(D)f$ is a linear isomorphism between $\mathcal{P}_H(^mE)$ and $\mathcal{M}_m$.

Proof. That (3.1) is formed by supercyclic vectors follows by Theorem 1. We must prove that $\mathcal{I}_A$ is invariant. So let $P(D)f \in \mathcal{I}_A$, $(\Psi, P) \in A$. Then $\Phi(D)P(D)f = P(D)\Psi(D)f$. Since f is supercyclic for $\Psi(D)$, it is elementary that $\Psi(D)f$ also forms a supercyclic vector for $\Psi(D)$, and hence $\Phi(D)P(D)f \in \mathcal{I}_A$.

Next, given m, there is a $\Psi \in \mathcal{S}_{\mathbb{H}}$ with $\Psi^{(m)} = \Phi$. By Theorem 2 we can find a supercyclic vector f for $\Psi(D)$. Theorem 1 gives that $\mathcal{M}_m \equiv \{P(D)f : 0 \neq P \in \mathcal{P}_H(^mE)\}$ is formed by supercyclic vectors for $\Phi(D)$, and we deduce that $\mathcal{P}_H(^mE) \ni P \mapsto P(D)f \in \mathcal{M}_m$ is a linear isomorphism ℓ. Indeed, $P(D)f \neq 0$ for all $P \neq 0$, for otherwise 0 would be a supercyclic vector, so ℓ is one-to-one and hence a bijection. ■

Example 2. *Let $\Phi \in \mathcal{S}_{\mathbb{H}}$ and fix m and $\Psi \in \mathcal{S}_{\mathbb{H}}$ such that $\Psi^{(m)} = \Phi$. Then, with $A \equiv \{(\Psi, P) : 0 \neq P \in \mathcal{P}_H(^mE)\}$, we obtain the invariant set $\mathcal{I}_A = \cup_{P \in \mathcal{P}_H(^mE)\setminus\{0\}} P(D)\mathrm{SC}(\Psi)$ of supercyclic vectors for $\Phi(D)$. Here* SC(Ψ) *denotes the set of supercyclic vectors for $\Psi(D)$.*

Next we shall prove that some of the operators in $\mathcal{O}_{\mathbb{H}}$ are in fact hypercyclic. Indeed, for any $c > 0$, let $\mathcal{S}_m^c$ denote the set of sequences in $\mathcal{S}_m$ of the form (P_n) where $P_n \in \mathcal{P}_H(^mE)$ and $||P_n||_m \geq c$ for all n. Thus, in view of 3 in the beginning of this section, a sequence (P_n) in $\mathcal{P}_H(^mE)$ belongs to $\mathcal{S}_m^c$ if $c \leq ||P_n||_m \leq CR^n$ for some $R, C > 0$. We put

$$\mathcal{S}_{\mathbb{H}}^{\infty} \equiv \bigcup_{\substack{m\geq 1\\ c>0}} \mathcal{S}_m^c \quad (\subseteq \mathcal{S}_{\mathbb{H}}),$$

and let $\mathcal{O}_{\mathbb{H}}^{\infty}$ $(\subseteq \mathcal{O}_{\mathbb{H}})$ denote the corresponding class of operators in $\mathcal{O}(\mathbb{H})$. It is clear that $\mathcal{O}_{\mathbb{H}}^{\infty}$ is multiplicative closed, and stable in the sense that for any given m and $\Phi \in \mathcal{S}_{\mathbb{H}}^{\infty}$; $\Phi^{(m)} \in \mathcal{S}_{\mathbb{H}}^{\infty}$ and $\Psi^{(m)} = \Phi$ for some $\Psi \in \mathcal{S}_{\mathbb{H}}^{\infty}$. (In fact, if $(P_n) \in \mathcal{S}_m^c$ and $(Q_n) \in \mathcal{S}_k^d$, then $(P_n)(Q_n) = (P_nQ_{n+m}) \in \mathcal{S}_{m+k}^{cd}$.) Note also that $\mathcal{O}_{\mathbb{H}}^{\infty} \cap \mathcal{C} = \{P(D) : P \in \mathbb{H}\setminus\mathbb{C}\}$, which in $\mathcal{S}_{\mathbb{H}}^{\infty}$ corresponds to the set of constant sequences $(P, P, ...)$, $P \in \mathbb{H}\setminus\mathbb{C}$.

Theorem 4. *The operators in $\mathcal{O}_{\mathbb{H}}^{\infty}$ are hypercyclic.*

Proof. Let $\Phi = (P_n) \in \mathcal{S}_m^c$. We must then prove that $T \equiv \Phi(D)$ is hypercyclic. Define $Af \equiv \sum_{n\geq 0} A_n f_n$, where $A_n \equiv \bar{P}_n(P_n(D)\bar{P}_n)^{-1}$ and $f_n \equiv H_n f \in \mathcal{P}_H(^nE)$ (as in the proof of Theorem 2). Then,

$$A^n f = \sum_{i\geq 0} A_{i+m(n-1)} ... A_{i+m} A_i f_i, \quad f = \sum f_n \in \mathcal{P}_{\mathbb{H}}(E),$$

and Lemma 1 gives

$$||A^n f||_{H:r} = \sum_{i\geq 0} r^{i+nm} \frac{||A_{i+m(n-1)} \cdots A_i f_i||_{i+nm}}{\sqrt{(i+nm)!}} \leq$$

$$\sum_{i\geq 0} \frac{r^{i+nm}}{||P_i||_m \cdots ||P_{i+m(n-1)}||_m} \frac{||f||_i}{\sqrt{i!}\sqrt{(nm)!}} \leq \frac{r^{nm}c^{-n}}{\sqrt{(nm)!}} ||f||_{H:r} \to 0, \quad n \to \infty.$$

So $S_n \equiv A^n \to 0$ pointwise on $Y \equiv \mathcal{P}_H(E)$, $T^n S_n = Id_Y$, and, since $m > 0$, $T^n \to 0$ pointwise on $Z \equiv \mathcal{P}_H(E)$. Thus T is hypercyclic by Proposition 1. ■

Remark 1. *We recall from the introduction and the result of Aron and co-workers that, when $E = \mathbb{C}$, $T_{\lambda:a} : f \mapsto f'(\lambda z + a)$ is hypercyclic provided $|\lambda| > 1$. We note that $T_{\lambda:a} = \Phi(D) \in \mathcal{O}_{\mathbb{H}}$ if $\lambda \neq 0$ where $\Phi = (\lambda^n z e^{az})$. Thus $T_{\lambda:0} \in \mathcal{O}_{\mathbb{H}}^{\infty}$ if $|\lambda| \geq 1$ and $T_{\lambda:a} \in \mathcal{O}_{\mathbb{H}} \setminus \mathcal{O}_{\mathbb{H}}^{\infty}$ when $a, \lambda \neq 0$ or $0 < |\lambda| < 1$. In particular, $T_{\lambda:a}$ is a hypercyclic operator in $\mathcal{O}_{\mathbb{H}} \setminus \mathcal{O}_{\mathbb{H}}^{\infty}$ when $a \neq 0$ and $|\lambda| > 1$, and it is a natural to ask: what (other) operators in $\mathcal{O}_{\mathbb{H}} \setminus \mathcal{O}_{\mathbb{H}}^{\infty}$ are in fact hypercyclic?*

Corollary 1. *For any non-degenerate sequence of scalars $\Psi = (\psi_n) \in \mathcal{S}$ and $P \in \mathcal{P}_H(^m E) \setminus \{0\}$ where $m \geq 1$, $\Psi(D)P(D)$ is hypercyclic.*

Corollary 2. *$P(D)\langle \cdot, D\rangle^n$ is hypercyclic for any homogeneous $P \in \mathcal{P}_H(^m E)$ and $n \geq 0$ provided $P \neq 0$ and $m \geq 1$.*

PROOF. Recall, from Example 1, that $\langle \cdot, D\rangle^n = \Phi(D)$ where $\Phi = (\varphi_i = i^n)$. Hence, in view of Proposition 4, $P(D)\langle \cdot, D\rangle^n = \Psi(D)P(D)$ where $\Psi = \Phi^{(m)} = (\psi_i = (i+m)^n)$. Thus Ψ is non-degenerate and the statement follows by Corollary 1. ■

The analogue of Theorem 3 holds true for the class $\mathcal{O}_{\mathbb{H}}^{\infty}$, the proof goes parallel:

Theorem 5. *Let $\Phi(D) \in \mathcal{O}_{\mathbb{H}}^{\infty}$. Then, for any set $A \subseteq \mathcal{S} \times \mathbb{H}$ such that $P \neq 0$, $\Psi^{(m)} = \Phi$ if $P \in \mathcal{P}_H(^m E)$, for all $(\Psi, P) \in A$:*

$$\mathcal{I}_A \equiv \cup_{(\Psi,P)\in A}\{P(D)f : f \text{ hypercyclic for } \Psi(D)\}$$

forms an invariant set of hypercyclic vectors for $\Phi(D)$.

For any $m \geq 0$ there is an $f \in \mathcal{H}_H(E)$ such that $\mathcal{M}_m = \{P(D)f : P \in \mathcal{P}_H(^m E)$ forms a hypercyclic vector manifold for $\Phi(D)$ and $\mathcal{P}_H(^m E) \ni P \mapsto P(D)f \in \mathcal{M}_m$ is a linear isomorphism.

3.2. Supercyclic and hypercyclic subspaces

Theorem 5 and $\mathcal{P}_H(^m E) \simeq \mathcal{M}_m$ made us believe that the operators in $\mathcal{O}_{\mathbb{H}}^{\infty}$ admit hypercyclic subspaces. However, we only know that the isomorphism is a linear one, and thus, even though $\mathcal{P}_H(^m E)$ is closed, we do not know if $\mathcal{M}_m$ is. However, instead of proving that the isomorphism is topological, which we do not know to be true, we shall prove that the elements of $\mathcal{O}_{\mathbb{H}}^{\infty}$ indeed have hypercyclic subspaces by applying another technique, based on the theory of basic sequences, see [14].

Theorem 6. *Assume E is infinite-dimensional. Then every operator $T \in \mathcal{O}_{\mathbb{H}}^{\infty}$ has a hypercyclic subspace and, similarly, any $T \in \mathcal{O}_{\mathbb{H}}$ has a supercyclic subspace.*

Proof. We prove that $T \in \mathcal{O}_{\mathbb{H}}^{\infty}$ has a hypercyclic subspace. Let $m \geq 1$ be arbitrary and put $B \equiv \mathcal{P}_H(^mE)$. Let $\mathcal{B}$ denote the Banach space of all $f = \sum f_n \in \mathcal{F}_H(E)$ such that $||f|| \equiv ||f||_{H:1} = \sum_0^\infty ||f_n||_n/\sqrt{n!} < \infty$, equipped with the norm $||\cdot||$. Thus $\mathcal{H}_H(E) \subseteq \mathcal{B}$. Let us note that the topology on $B = (B, ||\cdot||_m)$ coincides with that induced by $\mathcal{B}$ as well as that of $\mathcal{H}_H(E)$, in particular, B is a closed subspace of $\mathcal{B}$. Moreover, $T^n \to 0$ pointwise on B. Since $\mathcal{P}_H(E)$ is dense in $\mathcal{H}_H(E)$, there is a denumerable dense set $\{p_n\}$ in $\mathcal{H}_H(E)$ formed by vectors $p_n \in \mathcal{P}_H(E)$. Choose an orthonormal basis (e_n) for B. Then (e_n) is a basic sequence in B with basic constant $K = 1$ (see [14]). Let (ϵ_n) be a decreasing sequence of real and positive numbers such that $\sum \epsilon_m < \frac{1}{2K} = \frac{1}{2}$, and, for $m, n \geq 1$, put $i(m,n) \equiv \frac{(m+n-1)(m+n)}{2} - n + 1$. The mapping $i = i(m,n)$ is then strictly increasing in both m and n (see [10, p. 173] for further remarks). We claim that there is a map $r = r(i)$, defined for $i = 0$ and for all $i = i(m,n)$, where $r(0) = 0$ and $r(i(m,n))$ is strictly increasing in n for fixed m, and a sequence $(f_m)_{m\geq 1}$ in $\mathcal{H}_H(E)$ such that:

(a) $||T^{r(i(m,n))} f_m - p_n||_{H:n} < \epsilon_m/2^n$ for all $m, n \geq 1$,

(b) $||T^{r(i(k,n))}(f_m - e_m)||_{H:n} < \epsilon_m/2^n$ for all $k, m, n \geq 1$ where $k \neq m$,

(c) $||f_m - e_m|| < \epsilon_m$ for all $m \geq 1$.

Indeed, from the arguments of the proof of [10, theorem 2.1], we deduce that there is a double sequence $(f_{m,n})_{m,n\geq 1}$ in $\mathcal{H}_H(E)$ such that with $f_{m,0} \equiv e_m$:

$$|| \, T^{r(j)}(f_{m,n} - f_{m,n-1})||_{H:n+m} < \epsilon_m/2^{i(m,n)}, 0 \leq j < i(m,n), \tag{3.2}$$

$$|| \, T^{r(i(m,n))} f_{m,n} - p_n||_{H:n} < \epsilon_m/2^{i(m,n)+1}, m, n \geq 1, \tag{3.3}$$

$$|| \, T^{r(i(m,n))} f_{m',n'}||_{H:n} < \epsilon_m/2^{i(m,n)+1}, 1 \leq i(m',n') < i(m,n). \tag{3.4}$$

Hence, in particular, $(f_{m,n})_n$ is a Cauchy sequence in $\mathcal{H}_H(E)$ (take $j = 0$ in (3.2)), and we note that

$$f_m \equiv \lim_i f_{m,i} = f_{m,n} + \sum_{k\geq n}(f_{m,k+1} - f_{m,k})$$

for any $m, n \geq 1$. From this and (3.2) we obtain

$$||f_m - e_m|| = ||f_m - e_m||_{H:1} \leq ||f_{m,1} - e_m||_{H:1} + \sum_{k\geq 1} ||(f_{m,k+1} - f_{m,k})||_{H:1} \leq$$

$$||f_{m,1} - e_m||_{H:1} + \sum_{k\geq 1} ||(f_{m,k+1} - f_{m,k})||_{H:k} \leq \epsilon_m/2^{i(m,1)} + \sum_{k\geq 1} \epsilon_m/2^{i(m,k+1)}$$

$$< \epsilon_m.$$

Hence c), and to see that a) and b) hold, we only have to apply (3.3) and (3.4) and follow the arguments in [10, pp 176–177] (see also [15, pp 424–426]).

Now, we deduce from c) that (f_m) is a basic sequence in $\mathcal{B}$ and equivalent to (e_m). Indeed, $\sum ||f_m - e_m|| \leq \sum \epsilon_m < \frac{1}{2K}$. Hence our claim by virtue of [14, proposition 1.a.9]. Thus the closed linear span F of $\{f_m\}$ is isomorphic to B, and is formed by all convergent series $\sum \alpha_m f_m$ where the expansion is unique. Let L denote the basic constant for (f_m), and let F_0 denote the set of elements $f = \sum \alpha_m f_m$ in F such that the series converges in $\mathcal{H}_H(E)$ (thus $F_0 \subseteq \mathcal{H}_H(E) \cap F$). We shall prove that $H \equiv \overline{F_0}$ (closure in $\mathcal{H}_H(E)$) is a required hypercyclic subspace for T.

The subspace H is of course closed, and, since it contains the elements f_m, infinite-dimensional. By the continuity of the embedding $\mathcal{H}_H(E) \to \mathcal{B}$, $H \subseteq F$, and hence, every $f \in H$ has a representation $f = \sum \alpha_m f_m$ (convergence in $\mathcal{B}$). We must prove that if $f \neq 0$, i.e. $\alpha_k \neq 0$ for some k, then f is hypercyclic for T.

Assume first that $f = \sum \alpha_m f_m \in F_0$ so that the series converges in $\mathcal{H}_H(E)$. Choose k such that $\alpha_k \neq 0$, and we may assume that $\alpha_k = 1$, since a non-zero scalar multiple of a hypercyclic vector is again hypercyclic. Since (f_m) and (e_m) are equivalent, $g \equiv \sum_{m\neq k} \alpha_m e_m$ exists in B—and hence in $\mathcal{H}_H(E)$. Now, if $n \geq \nu$, then a) and b) give

$$\begin{aligned}
&||T^{r(i(k,n))} f - p_n||_{H:\nu} = \\
&= ||T^{r(i(k,n))} f_k - p_n + T^{r(i(k,n))} g + \sum_{m\neq k} \alpha_m T^{r(i(k,n))}(f_m - e_m)||_{H:\nu} \\
&\leq ||T^{i(k,n)} f_k - p_n||_{H:n} + ||T^{r(i(k,n))} g||_{H:\nu} + \sum_{m\neq k} |\alpha_m| ||T^{r(i(k,n))}(f_m - e_m)||_{H:n} \\
&\leq \varepsilon_k/2^n + ||T^{r(i(k,n))} g||_{H:\nu} + 2L||f|| \sum_{m\neq k} \varepsilon_m/2^n \\
&\leq ||T^{r(i(k,n))} g||_{H:\nu} + (L||f|| + 1)/2^n \to 0, \qquad (3.5)
\end{aligned}$$

as $n \to \infty$.

Hence f is hypercyclic. Next, let $f = \sum \alpha_m f_m \in H$ be arbitrary and choose a sequence $(f^s = \sum \alpha_m^s f_m)_s$ in F_0 that converges to f. Again, choose k such that $\alpha_k \neq 0$. We assume $\alpha_k = 1$ and since, clearly, $\alpha_k^s \to \alpha_k$, we may assume $\alpha_k^s = 1$ for all s. For given n and ν, choose constants $M_n(\nu)$ and $m_n(\nu)$ such that $||T^{r(i(k,n))} h||_{H:\nu} \leq M_n ||h||_{H:m_n}$ for all $h \in \mathcal{H}_H(E)$. By the continuity of $\mathcal{H}_H(E) \to \mathcal{B}$, $f^s \to f$ in $\mathcal{B}$ and hence $||f^s|| \leq C$. In fact, $f^s \to f$ in F so, with $g^s \equiv$

$\sum_{m\neq k}\alpha^s_m e_m$, g^s converges in B to some g. Since the norm-topology on B coincides with the subspace topology induced by $\mathcal{H}_H(E)$, $g^s\to g$ in $\mathcal{H}_H(E)$. For given $n\geq\nu$, (3.5) gives that for any s:

$$\begin{aligned}||T^{r(i(k,n))}f-p_n||_{H:\nu} &\leq ||T^{r(i(k,n))}(f-f^s)||_{H:\nu}+||T^{r(i(k,n))}f^s-p_n||_{H:\nu}\\ &\leq M_n||f-f^s||_{H:m_n}+||T^{r(i(k,n))}g^s||_{H:\nu}+\tfrac{(L||f^s||+1)}{2^n}\\ &\leq M_n||f-f^s||_{H:m_n}+||T^{r(i(k,n))}g||_{H:\nu}+M_n||g-g^s||_{H:m_n}+\tfrac{(LC+1)}{2^n}.\end{aligned}$$

We may find an $s=s(n)$ such that $||f-f^s||_{H:m_n}, ||g-g^s||_{H:m_n}\leq 2^{-n}/M_n$, and deduce from this that $||T^{r(i(k,n))}f-p_n||_{H:\nu}\to 0$ as $n\to\infty$ for all ν. Hence f is hypercyclic.

Next, let $T\in\mathcal{O}_{\mathbb{H}}$. From the proof of Theorem 2 we know that there is a sequence (r_n) of scalars such that $(T_n\equiv r_nT^n)$ is hereditarily hypercyclic. Now, the operators T_n commute, and by noting that proposition 2.2 and lemma 2.3 in [10] extend to sequences of commuting operators on a Fréchet space, we deduce that there exist a double sequence $(f_{m,n})$ and a map $r=r(i)$ such that (3.2), (3.3) and (3.4) hold for (T_n). Hence the analogue of (a–c) hold and the arguments above show that (T_n) has a hypercyclic subspace, which clearly forms a supercyclic subspace for T.

4. Concluding remarks

Note that if E is finite-dimensional, then every $\mathcal{P}_H(^mE)$ is finite-dimensional, so the proof of Theorem 6 does not apply. However, if we use the result from [6; 23], saying that (ii) of the Introduction is sufficient for (i) for operators on Fréchet spaces with a continuous norm, we can in fact extend Theorem 6: Any $T\in\mathcal{O}^{\infty}_{\mathbb{H}}$ has a hypercyclic subspace if $\dim E>1$. Indeed, we have seen that any such T is hereditarily hypercyclic, and it is easily checked that $Y\equiv\ker T$ is infinite-dimensional (and closed), and obviously $T^n\to 0$ on Y. Based on the same principle as that in the last part of the proof of Theorem 6, we also conclude that the elements of $\mathcal{O}_{\mathbb{H}}$ has supercyclic subspaces whenever $\dim E>1$.

Finally we remark that, by virtue of the sufficient condition (ii), and with arguments as above, we can conclude that any non-scalar $\varphi(D)\in\mathcal{C}$ has a hypercyclic subspace if $\dim E>1$. This observation complements the main result in [18] — the analogue of Godefroy-Shapiro's Theorem. In particular we obtain that any translation operator τ_a, $a\in E\backslash\{0\}$, has a hypercyclic subspace, which extends the corresponding one-variable result in [3]: $\tau_a:\mathcal{H}(\mathbb{C})\to\mathcal{H}(\mathbb{C})$ has a hypercyclic subspace for any non-zero $a\in\mathbb{C}$. ■

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