BLOCH'S THEOREM FOR ALGEBROID MULTIFUNCTIONS II

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ABSTRACT

Bloch's theorem is extended to multifunctions of the form

$$F(z) = \{w \in \mathbb{C} : w^n + a_1(z)w^{n-1} + \ldots + a_{n-1}(z)w + a_n(z) = 0\} \qquad (|z| < 1),$$

where $a_1(z), \ldots, a_n(z)$ are holomorphic functions on the unit disc and $a_1'(0) = 1$.

1. Introduction and statement of results

Let D be a domain in the complex plane $\mathbb{C}$, and let $n \geq 1$. An *algebroid multifunction* of degree n on D is a set-valued function F of the form

$$F(z) = \Big\{ w \in \mathbb{C} : w^n + \sum_{j=1}^{n} a_j(z) w^{n-j} = 0 \Big\} \qquad (z \in D), \tag{1.1}$$

where $a_1, \ldots, a_n : D \to \mathbb{C}$ are holomorphic functions. It is understood that the roots are counted according to multiplicity. The *image* of F is defined as

$$F(D) := \bigcup_{z \in D} F(z).$$

Given a family $\mathcal{F}$ of algebroid multifunctions of degree n on D, we define

$$L(\mathcal{F}) := \inf_{F \in \mathcal{F}} \Big(\sup\{r > 0 : F(D) \text{ contains a disc of radius } r\} \Big).$$

The main result of [2] gives fairly general conditions on $\mathcal{F}$ ensuring that $L(\mathcal{F}) > 0$. However the result is somewhat unsatisfactory, because it is necessary to assume that $\sup_{F \in \mathcal{F}} \operatorname{diam} F(z_0) < \infty$ for some $z_0 \in D$.

The authors of [2] were thus led to pose the following problem. Let $\mathbb{D}$ be the open unit disc, and let $n \geq 1$. Define $\mathcal{F}_n$ to be the family of algebroid multifunctions F of degree n on $\mathbb{D}$ such that $a_1'(0) = 1$ (in the notation of (1.1) above). Is $L(\mathcal{F}_n) > 0$? As remarked in [2], such a result, if true, would be the most direct generalisation of Bloch's theorem to algebroid multifunctions.

In this paper, we show that this result is indeed true.

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Theorem 1. *With the notation above,* $L(\mathcal{F}_n) > 0$.

For ease of notation, let us set $L_n := L(\mathcal{F}_n)$. Thus, in particular, $L_1 = L$, the usual Landau constant. The next result gives some information on how L_n depends on n.

Theorem 2. *With the notation above,*

$$\frac{1}{L_{n+m}} \geq \frac{1}{L_n} + \frac{1}{L_m} \qquad (n, m \geq 1). \tag{1.2}$$

We immediately deduce the following estimate for L_n.

Corollary 3. $L_n \leq L/n$, *where* L *is Landau's constant.*

Proof. Apply Theorem 2 several times over with $m = 1$. ■

As a further application, we have

Corollary 4. $nL_n \to l$ *as* $n \to \infty$, *where* $l := \inf_{n \geq 1} nL_n$.

Proof. It is well known that if $(\alpha_n)_{n \geq 1}$ is a positive sequence satisfying $\alpha_{n+m} \geq \alpha_n + \alpha_m$ for all $n, m \geq 1$, then $\lim_{n \to \infty}(\alpha_n/n)$ exists and equals $\sup_{n \geq 1}(\alpha_n/n)$ (possibly $+\infty$). The result follows upon applying this with $\alpha_n = 1/L_n$. ■

Evidently $0 \leq l \leq L$. It would be very interesting to narrow these bounds, and in particular, to determine whether $l > 0$.

It is also natural to ask if there is an analogue of Theorem 1 where, instead of using $a_1'(0)$ to normalise F, we use the functions $a_2, \ldots, a_n$. Our next result shows that no such analogue is possible.

Theorem 5. *Let* D *be a domain, let* $n \geq 2$, *and let* $b_2, \ldots, b_n : D \to \mathbb{C}$ *be bounded functions. Given* $\epsilon > 0$, *there exists* $c \in \mathbb{C}$ *such that, if*

$$F_c(z) := \Big\{ w \in \mathbb{C} : w^n + cw^{n-1} + \sum_{j=2}^{n} b_j(z) w^{n-j} = 0 \Big\} \qquad (z \in D),$$

then $F_c(D)$ *contains no disc of radius* ϵ.

Finally, here is a simple application of these ideas to spectral theory. We write $M_n(\mathbb{C})$ for the set of complex $n \times n$ matrices and, given $A \in M_n(\mathbb{C})$, we denote by $\sigma(A), \mathrm{tr}(A)$ the spectrum of A and the trace of A, respectively.

Theorem 6. *Let* D *be a domain containing the closed unit disc and let* $f : D \to M_n(\mathbb{C})$ *be a holomorphic function. Then* $\cup_{z \in D} \sigma(f(z))$ *contains a disc of radius* $L_n |\mathrm{tr} f'(0)|$.

The rest of the paper is devoted to the proofs of these theorems. Theorem 1 is proved in §2, and the proofs of the other theorems, which are much shorter, are gathered together in §3.

2. Proof of Theorem 1

We begin with some elementary remarks about algebroid multifunctions. If F is an algebroid multifunction of degree n on a domain D and if $c \in \mathbb{C} \setminus \{0\}$, then $c + F$ and cF are also algebroid multifunctions of degree n, where

$$(c+F)(z) := \{c + w : w \in F(z)\} \quad \text{and} \quad (cF)(z) := \{cw : w \in F(z)\} \qquad (z \in D).$$

If G, H are algebroid multifunctions on D of degree l, m, respectively, then $(G \cup H)(z) := G(z) \cup H(z)$ is algebroid of degree $l+m$. The following lemma, sometimes known as the localisation principle, gives conditions under which this process can be reversed.

Lemma 7. *Let F be an algebroid multifunction on a domain D. Let U, V be disjoint open subsets of $\mathbb{C}$ such that $F(D) \subset U \cup V$ and $F(D) \cap U \neq \emptyset$ and $F(D) \cap V \neq \emptyset$. Define $G(z) := F(z) \cap U$ and $H(z) := F(z) \cap V$ $(z \in D)$. Then G, H are algebroid multifunctions on D.*

PROOF. Combine [1, theorems 7.1.5 and 7.1.7]. ■

The proof of Theorem 1 depends heavily on the ideas developed in [2]. For the convenience of the reader, we recall here the basic notions and state the necessary lemmas, referring to [2] for the proofs.

Let $\mathbb{C}_\infty$ denote the Riemann sphere. A *meromorphic algebroid multifunction* of degree n on a domain D is a set-valued function F of the form

$$F(z) := \Big\{ w \in \mathbb{C}_\infty : \sum_{j=0}^{n} a_j(z) w^{n-j} = 0 \Big\} \qquad (z \in D),$$

where $a_0, \dots, a_n : D \to \mathbb{C}$ are holomorphic functions on D such that $\sum_{j=0}^{n} |a_j(z)| > 0$ for all $z \in D$. Here we adopt the convention that $\infty \in F(z)$ if and only if $a_0(z) = 0$, and the multiplicity of ∞ in $F(z)$ is the smallest integer k such that $a_k(z) \neq 0$. In this manner, each $F(z)$ becomes an n-tuple in $\mathbb{C}_\infty$.

Let F be a meromorphic algebroid multifunction on a domain D. We define the *reduced image* of F by

$$\check{F}(D) := \Big(\bigcup_{z \in D} F(z) \Big) \setminus \Big(\bigcap_{z \in D} F(z) \Big).$$

The following lemma is a version of the open mapping theorem.

Lemma 8. *Let F be a meromorphic algebroid multifunction on a domain D. Then $\check{F}(D)$ is an open subset of $\mathbb{C}_\infty$.*

Proof. See [2, lemma 2]. ■

Let $(F_k)_{k\geq 1}$ and F be meromorphic algebroid multifunctions of degree n on a domain D. We say that F_k *converges to* F *spherically locally uniformly on* D if $\rho_s(F_k(z), F(z)) \to 0$ locally uniformly, where s is any metric on $\mathbb{C}_\infty$ giving the usual topology, and ρ_s is the matching distance for n-tuples measured according to s, namely

$$\rho_s\big(\{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_n\}\big) := \min_{\sigma\in S_n} \max_{1\leq j\leq n} s(\alpha_j, \beta_{\sigma(j)}).$$

The precise choice of s is irrelevant, since all such metrics are uniformly equivalent. The next lemma is an analogue of Hurwitz's theorem.

Lemma 9. *Let $(F_k)_{k\geq 1}$ and F be meromorphic algebroid multifunctions of degree n on a domain D. Suppose that F_k converges to F spherically locally uniformly on D. Let K be a compact subset of $\check{F}(D)$. Then $K \subset \check{F}_k(D)$ for all large enough k.*

Proof. See [2, lemma 3]. ■

Let $\mathcal{F}$ be a family of meromorphic algebroid multifunctions of degree n on a domain D. We say that $\mathcal{F}$ is a *normal family* if every sequence in $\mathcal{F}$ contains a subsequence that converges spherically locally uniformly on D. The following lemma is an analogue of Montel's theorem. (Again, s denotes any metric on $\mathbb{C}_\infty$ giving the usual topology.)

Lemma 10. *Let $\mathcal{F}$ be a family of meromorphic algebroid multifunctions of degree n on a domain D. Suppose that there exists $\delta > 0$ such that, for each $F \in \mathcal{F}$, one can find $2n+1$ points $w_1, \dots, w_{2n+1} \in \mathbb{C}_\infty \setminus F(D)$ satisfying $s(w_j, w_k) \geq \delta$ for all j, k with $j \neq k$. Then $\mathcal{F}$ is a normal family.*

Proof. See [2, lemma 4]. ■

Proof of Theorem 1. Suppose, if possible, that there exists $n \geq 1$ such that $L(\mathcal{F}_n) = 0$. Fix the smallest such n. Thus $L(\mathcal{F}_m) > 0$ for all m with $1 \leq m < n$. We shall show that this leads to a contradiction.

By definition of $L(\mathcal{F}_n)$, for each $k \geq 1$, there exists $F_k \in \mathcal{F}_n$ such that $F_k(\mathbb{D})$ contains no disc of (Euclidean) radius $1/k$. Translating F_k by a constant, if necessary, we can further suppose that $0 \in F_k(0)$.

We next claim that the $(F_k)_{k\geq 1}$ form a normal family. To see this, let $\Delta_1, \dots, \Delta_{2n+1}$ be $2n+1$ disjoint closed discs in $\mathbb{C}$, each of (Euclidean) radius 1. Let s be a metric on $\mathbb{C}_\infty$ giving the usual topology. By compactness, there exists $\delta > 0$ such that, if w, w' belong to two different Δ_j, then $s(w, w') \geq \delta$. Now, since $F_k(D)$ contains no disc of radius one, it contains no Δ_j and so, picking $w_j \in \Delta_j \setminus F_k(D)$ $(j = 1, \dots, 2n+1)$, we obtain $2n+1$ points outside $F_k(D)$, each pair an s-distance at least δ apart. By Lemma 10, the (F_k) form a normal family, as claimed.

Thus, replacing (F_k) by a subsequence, we can suppose that F_k converges spherically locally uniformly on $\mathbb{D}$ to some meromorphic algebroid multifunction F. By Lemma 8, $\check{F}(\mathbb{D})$ is open in $\mathbb{C}_\infty$, and by Lemma 9, $\check{F}(\mathbb{D})$ can contain no disc of positive radius. The only possibility is that $\check{F}(\mathbb{D}) = \emptyset$. This implies that F is constant. Also, since $0 \in F_k(0)$ for all k, we must have $0 \in F(0)$. Thus $F(z) \equiv \{\alpha_1, \dots, \alpha_n\}$ for all $z \in \mathbb{D}$, where $\alpha_1, \dots, \alpha_n \in \mathbb{C}_\infty$ and $\alpha_1 = 0$.

There are now two possibilities: either all the α_j are zero, or at least one of them is non-zero. We shall show that each of these cases leads to a contradiction.

Suppose first that $\alpha_1 = \ldots = \alpha_n = 0$. Thus F_k converges spherically uniformly to $\{0\}$. Writing F_k in the form (1.1), say

$$F_k(z) = \Big\{w \in \mathbb{C} : w + \sum_{j=1}^{n} a_{j,k}(z) w^{n-j} = 0\Big\} \qquad (z \in \mathbb{D}),$$

it then follows that $a_{j,k} \to 0$ locally uniformly on $\mathbb{D}$ as $k \to \infty$, for $j = 1, \dots, n$. In particular, $a'_{1,k}(0) \to 0$ as $k \to \infty$. However, since $F_k \in \mathcal{F}_n$ for all k, we also have $a'_{1,k}(0) = 1$ for all k. This yields the desired contradiction.

Now suppose that $\alpha_1 = \ldots = \alpha_m = 0$ and that $\alpha_{m+1}, \dots, \alpha_n$ are all non-zero. Let U, V be disjoint open subsets of $\mathbb{C}_\infty$ such that $0 \in U$ and $\alpha_{m+1}, \dots, \alpha_n \in V$. By definition of spherical local uniform convergence, there exists k_0 such that, if $k \ge k_0$ and $|z| \le 1/2$, then $F_k(z) \cap U$ and $F_k(z) \cap V$ contain m and $n - m$ points, respectively (counted according to multiplicity). Define

$$G_k(z) := F_k(z/2) \cap U, \quad H_k(z) := F_k(z/2) \cap V \qquad (z \in \mathbb{D},\ k \ge k_0).$$

By Lemma 7, G_k, H_k are algebroid multifunctions on $\mathbb{D}$, of degree m and $n - m$, respectively. Write them as

$$G_k(z) = \{w \in \mathbb{C} : w + \sum_{j=1}^{m} g_{j,k}(z) w^{m-j} = 0\},$$

$$H_k(z) = \{w \in \mathbb{C} : w + \sum_{j=1}^{n-m} h_{j,k}(z) w^{n-m-j} = 0\}.$$

A simple scaling argument shows that $G_k(\mathbb{D})$ contains a disc of radius $L_m |g'_{1,k}(0)|/2$, where $L_m := L(\mathcal{F}_m) > 0$. Likewise $H_k(\mathbb{D})$ contains a disc of radius $L_{n-m} |h'_{1,k}(0)|/2$, where $L_{n-m} := L(\mathcal{F}_{n-m}) > 0$. Also $g_{1,k}(z) + h_{1,k}(z) = a_{1,k}(z/2)$, so $g'_{1,k}(0) + h'_{1,k}(0) = 1/2$. Putting these facts together, we deduce that, for all $k \ge k_0$, the set $F_k(\mathbb{D})$ contains a disc of radius $\min(L_m/4, L_{n-m}/4)$. This is inconsistent with the original choice of the sequence (F_k). Once again we have arrived at a contradiction, and the theorem is proved. ∎

3. Proofs of the other theorems

Proof of Theorem 2.

Fix $n, m \geq 1$. By definition of L_n, given $\epsilon > 0$, there exists $G \in \mathcal{F}_n$ such that $G(\mathbb{D})$ contains no disc of radius $L_n + \epsilon/2$. Replacing $G(z)$ by $(1/r)G(rz)$, where $r \in (0,1)$ is sufficiently close to 1, we may assume that $G(\mathbb{D})$ is bounded and contains no disc of radius $L_n + \epsilon$. Translating G by a constant, we may further suppose that $G(\mathbb{D})$ lies in the right half-plane. Likewise, there exists $H \in \mathcal{F}_m$ such that $H(\mathbb{D})$ contains no disc of radius $L_m + \epsilon$ and lies entirely in the left half-plane.

Now let $\lambda \in (0,1)$, and define

$$F_\lambda(z) := (1-\lambda)G(z) \cup \lambda H(z) \qquad (z \in \mathbb{D}).$$

Then $F_\lambda \in \mathcal{F}_{n+m}$. Also, $F_\lambda(\mathbb{D}) = (1-\lambda)G(\mathbb{D}) \cup \lambda H(\mathbb{D})$, a union of two disjoint sets, so $F_\lambda(\mathbb{D})$ contains no disc of radius $\max((1-\lambda)(L_n+\epsilon), \lambda(L_m+\epsilon))$. It follows that

$$L_{n+m} \leq \max\bigl((1-\lambda)(L_n+\epsilon), \lambda(L_m+\epsilon)\bigr).$$

This inequality holds for all $\epsilon > 0$ and all $\lambda \in (0,1)$. Taking $\lambda = L_n/(L_n + L_m)$ and letting $\epsilon \to 0$, we obtain

$$L_{n+m} \leq \frac{L_m L_n}{L_n + L_m},$$

which is equivalent to (1.2). ■

Proof of Theorem 5.

Let $\epsilon > 0$. Given $w \in F_c(D)$, there exists $z \in D$ such that

$$w^n + cw^{n-1} + \sum_{j=1}^{n-2} b_j(z) w^{n-j} = 0.$$

If also $|w| \geq \epsilon/2$, then

$$|w^2 + cw| = \Bigl|-\sum_{j=2}^{n} b_j(z) w^{2-j}\Bigr| \leq \sum_{j=2}^{n} \|b_j\|_\infty (2/\epsilon)^{2-j} =: K_\epsilon,$$

say, and hence, completing the square and using the triangle inequality,

$$\bigl(|c|^2/4 - K_\epsilon\bigr)^{1/2} \leq \bigl|w + c/2\bigr| \leq \bigl(|c|^2/4 + K_\epsilon\bigr)^{1/2}. \tag{3.1}$$

In other words, $F_c(D)$ is contained in the union of the disc $\{|w| < \epsilon/2\}$ and the annulus defined by (3.1). In particular, $F_c(D)$ can contain no disc of radius larger than

$$\epsilon/2 + \bigl(|c|^2/4 + K_\epsilon\bigr)^{1/2} - \bigl(|c|^2/4 - K_\epsilon\bigr)^{1/2},$$

and this quantity is less than ϵ if $|c|$ is taken sufficiently large. ■

PROOF OF THEOREM 6.

Set $\beta = -\mathrm{tr} f'(0)$. We can suppose that $\beta \neq 0$, otherwise there is nothing to prove. Choose $R > 1$ such that D contains the disc $\{|z| < R\}$. Define $g(z) := f(Rz)/R\beta$ $(z \in \mathbb{D})$ and $G(z) = \sigma(g(z))$ $(z \in \mathbb{D})$. Then $G(z) = \{w : \det(wI - g(z)) = 0\}$, so G is an algebroid multifunction on $\mathbb{D}$ of degree n, and $\mathrm{tr} g'(0) = -1$, which ensures that $G \in \mathcal{F}_n$. Hence $G(\mathbb{D})$ contains discs of all radii less than L_n, in particular it contains a disc of radius L_n/R. It follows that $\cup_{z \in D} \sigma(f(z))$ contains a disc of radius $(R|\beta|)(L_n/R) = L_n|\beta|$, as desired. ■

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