GENERALISED WEYL'S THEOREM FOR A CLASS OF OPERATORS SATISFYING A NORM CONDITION II

By

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Abstract

For a Banach space operator $T \in B(X)$, it is proved that if either T is an algebraically, totally hereditarily normaloid operator and the Banach space X is separable, or T satisfies the property that its quasinilpotent part $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for all complex numbers λ and some integer $p \geq 1$, then $f(T)$ satisfies generalized Weyl's theorem for every non-constant function f that is analytic on an open neighborhood of $\sigma(T)$.

1. Introduction

A Banach space operator T, $T \in B(X)$, is said to be *Weyl* if it is Fredholm of 0 index. The *Weyl spectrum* $\sigma_w(T)$ of T is the set $\sigma_w(T) = \{\lambda \in \mathcal{C}_1 : T - \lambda$ is not Fredholm of 0 index$\}$, and we say that *Weyl's theorem holds for* T if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, where $\pi_{00}(T)$ is the set of isolated points of the spectrum $\sigma(T)$ of T that are eigenvalues of finite multiplicity. More generally, *g-Weyl's theorem holds for* T provided $\sigma(T) \setminus \sigma_{bw}(T) = E(T)$, where $E(T)$ denotes the isolated points λ of $\sigma(T)$, $\lambda \in \text{iso}\sigma(T)$, which are eigenvalues (no restriction on multiplicity) and $\sigma_{bw}(T)$ is the set of complex numbers λ for which $T - \lambda$ fails to be 'B-Weyl'. Berkani [5] has called an operator $T \in B(X)$ 'B-Fredholm' if there exists a natural number n for which the induced operator $T_n : T^n(X) \longrightarrow T^n(X)$ is Fredholm in the usual sense, and 'B-Weyl' if in addition T_n has 0 index. As Berkani [5] has shown, if g-Weyl's theorem holds for T then so does Weyl's theorem.

Recall that an operator $T \in B(X)$ is said to be *totally hereditarily normaloid*, $T \in HN$, if every part of T (i.e., its restriction to an invariant subspace), and T_p^{-1} for every invertible part T_p of T, is normaloid (see [11]), and that T satisfies the property (H_p) if $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for some integer $p \geq 1$, where $H_0(T-\lambda)$ denotes the *quasinilpotent part* of T (see [3] and [22]). T is said to be *algebraically* HN if there exists a non-trivial polynomial $p(.)$ such that $p(T) \in HN$. This note

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considers operators $T \in B(X)$ that are either algebraically HN or satisfy property (H_p). It is proved that if T satisfies the property (H_p), or if X is separable and T is algebraically HN, then $f(T)$ satisfies g-Weyl's theorem for all (non-constant) functions f that are analytic in a neighborhood of $\sigma(T)$.

The class (H_p) is large; it contains, among others, the classes consisting of *generalized scalar*, *subscalar* and *totally paranormal* operators on a Banach space, *multipliers of semi-simple Banach algebras*, *hyponormal*, *p-hyponormal* $(0 < p < 1)$, *M-hyponormal* operators and totally $*$-paranormal operators on a Hilbert space (see [22], [3], [10], [16] and [19] for further information). The class HN was introduced in [11] (where it was denoted by THN); it properly contains, in particular, the classes consisting of paranormal operators on a Banach space [17, p.229] (and is properly contained in the class of normaloid operators) [12]. Our results generalize the results of [3], [10], [11], [16] and [22].

2. Notation and terminology

Let $T \in B(X)$. In addition to the notation (and terminology) already introduced, we shall use $\sigma_p(T)$, $\sigma_a(T)$, $\Pi(T)$, $\Pi_0(T)$ and $E^a(T)$ to denote the point spectrum, the approximate point spectrum, the set of poles (no restriction on rank), the set of poles of finite rank and the set of eigenvalues of T in $\text{iso}\sigma_a(T)$, respectively. Recall that T is said to be *isoloid* if the isolated points of $\sigma(T)$ are eigenvalues of T.

Following Grabiner [15], let $(c_n(T))$, $(c'_n(T))$ and $(k_n(T))$ be the sequences:

(i) $c_n(T) = \dim(T^n(X)/T^{n+1}(X))$.
(ii) $c'_n(T) = \dim(T^{-(n+1)}(0)/T^{-n}(0))$.
(iii) $k_n(T) = \dim\left[(T^n(X) \cap T^{-1}(0))/(T^{n+1}(X) \cap T^{-1}(0))\right]$.

Then the *descent* $\text{dsc}(T)$ and the *ascent* $\text{asc}(T)$ of T are defined by:
$\text{dsc}(T) = \inf\{n : c_n(T) = 0\} = \inf\{n : R(T^n) = R(T^{n+1})\}$,
$\text{asc}(T) = \inf\{n : c'_n(T) = 0\} = \inf\{n : T^{-n}(0) = T^{-(n+1)}(0)\}$,
with $\inf \emptyset = \infty$.

Let $d \in \mathcal{N}$ (=the set of natural numbers). We say that T has a *uniform descent for $n \geq d$* if $R(T) + T^{-n}(0) = R(T) + T^{-d}(0)$ for all $n \geq d$ (equivalently, $k_n(T) = 0$ for all $n \geq d$)). If, in addition, $R(T) + T^{-d}(0)$ is closed, then T is said to have a *topological uniform descent* for $n \geq d$. Following Berkani-Koliha [7] we say that a point $\lambda \in \sigma_a(T)$ is a *left pole* (resp., *left pole of finite rank*) of T, denoted $\lambda \in \Pi^a(T)$ (resp., $\lambda \in \Pi_0^a(T)$), if $T - \lambda \in LD(X)$ (resp., $T - \lambda \in LD(X)$ and $\alpha(T - \lambda) = \dim((T - \lambda)^{-1}(0)) < \infty$), where $LD(X)$ is the *regularity*

$$LD(X) = \{T \in B(X) : d = \text{asc}(T) < \infty \text{ and } T^{d+1}(X) \text{ is closed}\}.$$

If $\lambda \in \Pi^a(T)$, then $T - \lambda$ is an operator of topological uniform descent [7, Remark 2.7], and hence $\lambda \in \text{iso}\sigma_a(T)$ [15].

An operator $T \in B(X)$ has *the single-valued extension property* at $\lambda_0 \in \mathbf{C}$, SVEP at $\lambda_0 \in \mathbf{C}$ for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to X$ that satisfies

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\mathbf{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in \text{iso}\sigma(T)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \boldsymbol{C}$.* The *quasinilpotent part* $H_0(T-\lambda)$ and the *analytic core* $K(T-\lambda)$ of $(T-\lambda)$ are defined by

$$H_0(T-\lambda) = \{x \in X : \lim_{n \longrightarrow \infty} ||(T-\lambda)^n x||^{\frac{1}{n}} = 0\}$$

and

$$K(T-\lambda) = \{x \in X : \text{there exists a sequence } \{x_n\} \subset X \text{ and } \delta > 0 \\ \text{for which } x = x_0, (T-\lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ...\}.$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T-\lambda)$ such that $(T-\lambda)^{-p}(0) \subseteq H_0(T-\lambda)$ for all $p = 0, 1, 2, ...$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$ [20]. Recall that if $\lambda \in \text{iso}\sigma(T)$, then $H_0(T-\lambda) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the *glocal spectral subspace* consisting of all $x \in X$ for which there exists an analytic function $f : \mathbf{C} \setminus \{\lambda\} \longrightarrow \mathbf{X}$ that satisfies $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbf{C} \setminus \{\lambda\}$ (see [19, p. 241]). An operator $T \in B(X)$ is said to be *semi-regular* if $T(X)$ is closed and $T^{-1}(0) \subset T^{\infty}(X) = \cap_{n \in \mathbf{N}} T^n(X)$; T admits a *generalized Kato decomposition*, *GKD* for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. We say that T is of *Kato type* at a point λ if $(T-\lambda I)|_M$ is nilpotent in the GKD for $(T-\lambda I)$. Fredholm operators are Kato type [18, Theorem 4], and operators $T \in B(X)$ satisfying property (H_p) are Kato type at isolated points of $\sigma(T)$ (but not every Kato type operator T satisfies property (H_p)).

An operator T is Drazin invertible if both $\text{asc}(T)$ and $\text{dsc}(T)$ are finite [8]. If $\lambda \in \Pi(T)$, then $T-\lambda$ is Drazin invertible, and hence B-Fredholm [6, Theorem 2.3]. Recall from [7] that an operator T is *B-semi-Fredholm*, denoted $T \in \Phi_{BSF}$, if $T^n(X)$ is closed for some $n \in \mathcal{N}$ and the induced operator T_n is either *upper semi-Fredholm* or *lower semi-Fredholm* (in the usual sense). For a $T \in \Phi_{BSF}$, the index of T is defined by $\text{ind}(T) = \text{ind}(T_d)$, where $d \in \mathcal{N}$ is *the degree of stable iteration* of T (see [7, Definition 2.2]). Let $\Phi_{BSF_+^-}$ denote those $T \in \Phi_{BSF}$ that are upper B-semi-Fredholm with $\text{ind}(T) \leq 0$, and let $\sigma_{BSF_+^-}(T) = \{\lambda \in \mathcal{C}_1 : T-\lambda \notin \Phi_{SBF_+^-}\}$. Following Berkani-Koliha [7] we say that T *satisfies g-Browder's theorem* (resp., *generalized a-Browder's theorem*) if $\sigma_{bw}(T) = \sigma(T) \setminus \Pi(T)$ (resp., $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$): generalized a-Browder's theorem $\Longrightarrow$ g-Browder's theorem.

3. Operators with property (H_p)

Recall that an operator T is a quasi-affine transform of an operator S, denoted $T \prec S$, if there exists a quasi-affinity Y such that $SY = YT$. Let $\mathcal{H}(\sigma(T))$ denote the set of functions f that are non-constant and analytic on a neighborhood of $\sigma(T)$.

Lemma 3.1. *If the operator $S \in B(X)$ has SVEP and $T \prec S$, then g- Browder's theorem holds for $f(T)$ (consequently, Browder's theorem hold for $f(T)$) for all $f \in \mathcal{H}(\sigma(T))$. Moreover, if X is a Hilbert space, then generalized a-Browder's theorem holds for $f(T)$.*

PROOF. If $S \in B(X)$ has SVEP, then a straightforward argument shows that T has SVEP (see, for example, [10, lemma 3.1] or [3]). Consequently, $f(T)$ has SVEP (see [9, theorem 1.5 of chapter 1]). We prove that a Banach space operator with SVEP satisfies g-Browder's theorem: this would then imply the result for $f(T)$. Observe that if $\lambda \in \Pi(T)$, then $T - \lambda$ is Drazin invertible, and hence B-Fredholm of zero index. Thus $\Pi(T) \subseteq \sigma(T) \setminus \sigma_{bw}(T)$. For the reverse inclusion, assume that $\lambda \in \sigma(T) \setminus \sigma_{bw}(T)$. Then $T - \lambda$ is B-Fredholm, and hence an operator of uniform topological descent (see [4, proposition 2.6]). We claim that $\lambda \in \operatorname{iso}\sigma(T)$. If $\lambda \notin \operatorname{iso}\sigma(T)$, then there exists a sequence $\{\mu_n\} \subset \sigma(T)$ such that $\mu_n \to \lambda$. But then $\alpha(T - \mu_n) = c_0'(T - \mu_n) = c_0'(T - \lambda) = \alpha(T - \lambda) > 0$ (see [15, theorem 4.7]), so that λ is a point of accumulation of $\sigma_p(T)$. Since this contradicts the fact that T has SVEP (see [14, theorem 10]), our claim is proved. Hence, $\lambda \in \operatorname{iso}\sigma(T)$, which by [6, theorem 2.3] implies that λ is a pole of the resolvent of T. Thus $\lambda \in \Pi(T)$, and T satisfies g-Browder's theorem.

To complete the proof, we now let X be a Hilbert space. Then, since T has SVEP, it follows from [7, remark 2.7 and theorem 2.8] that $\lambda \in \Pi^a(T) \Longrightarrow \lambda \notin \sigma_{SBF_+^-}(T)$. Observe that if $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then the operator $T - \lambda$ has a representation $T - \lambda = T_1 \oplus T_2$ on $X = X_1 \oplus X_2$, where T_1 is upper semi-Fredholm and T_2 is nilpotent (see [7, proposition 2.9]). The operator $T_1 \in B(X_1)$ being the restriction of an operator with SVEP to an invariant subspace has SVEP; hence T_1 is upper semi-Fredholm, and, consequently, Kato type. This implies that $\sigma_a(T_1)$ does not cluster at 0 (see [3, theorem 2.6]). The operator T_2 being a nilpotent, it follows that $\lambda \in \operatorname{iso}\sigma_a(T)$, and hence (see [7, theorem 2.8]) that $\lambda \in \Pi^a(T)$. Consequently, $\sigma_a(T) \setminus \Pi^a(T) = \sigma_{SBF_+^-}(T)$, i.e. T satisfies generalized a-Browder's theorem. ■

We say that the B-Fredholm operator T has *stable index* if $\operatorname{ind}(T - \lambda)\operatorname{ind}(T - \mu) \geq 0$ for every λ, μ in the B-Fredholm region of T.

Lemma 3.2. *Let $T \in B(X)$, and let $f \in \mathcal{H}(\sigma(T))$. Then $\sigma_{bw}(f(T)) \subset f(\sigma_{bw}(T))$, and if the B-Fredholm operator T has stable index, then $\sigma_{bw}(f(T)) = f(\sigma_{bw}(T))$.*

PROOF. Let $T \in B(X)$, let $f \in \mathcal{H}(\sigma(T))$, and let $g(T)$ be an invertible function such that $f(\mu) - \lambda = (\mu - \alpha_1) \cdots (\mu - \alpha_n) g(\mu)$. If $\lambda \notin f(\sigma_{bw}(T))$, then $f(T) - \lambda = (T - \alpha_1) \cdots (T - \alpha_n) g(T)$ and $\alpha_i \notin \sigma_{bw}(T)$, $i = 1, \ldots, n$. Consequently, $T - \alpha_i$ is a B-Fredholm operator of zero index for all $i = 1, \ldots, n$, which, by [5, theorem 3.2], implies that $f(T) - \lambda$ is a B-Fredholm operator of zero index. Hence, $\lambda \notin \sigma_{bw}(f(T))$.

Suppose now that T has stable index, and that $\lambda \notin \sigma_{bw}(f(T))$. Then, $f(T)-\lambda = (T-\alpha_1)\cdots(T-\alpha_n)g(T)$ is a B-Fredholm operator of zero index. Hence, by [4, corollary 3.3], the operators $g(T)$ and $T-\alpha_i$, $i=1,\dots,n$, are B-Fredholm and

$$0 = \text{ind}(f(T)-\lambda) = \text{ind}(T-\alpha_1) + \cdots + \text{ind}(T-\alpha_n) + \text{ind}g(T).$$

Since $g(T)$ is an invertible operator, $\text{ind}(g(T)) = 0$; also $\text{ind}(T-\alpha_i)$ has the same sign for all $i=1,\dots,n$. Thus $\text{ind}(T-\alpha_i)=0$, which implies that $\alpha_i \notin \sigma_{bw}(T)$ for all $i=1,\dots,n$, and hence $\lambda \notin f(\sigma_{bw}(T))$. ■

Lemma 3.3. *If $T \in B(X)$ has SVEP, then* $\text{ind}(T-\lambda) \le 0$ *for every $\lambda \in \mathbf{C}$ such that $T-\lambda$ is B-Fredholm.*

PROOF. If T has SVEP, then $T_{|M}$ has SVEP for every invariant subspace $M \subset X$ of T. Recall from [4, Theorem 2.7] that if $T-\lambda$ is a B-Fredholm operator, then there exist $T-\lambda$ invariant closed subspaces M and N of X such that $X = M \oplus N$, $(T-\lambda)_{|M}$ is a Fredholm operator (with SVEP) and $(T-\lambda)_{|N}$ is a nilpotent operator. Since $\text{ind}(T-\lambda)_{|M} \le 0$ (see [22, proposition 2.2]), it follows that $\text{ind}(T-\lambda) \le 0$. ■

Remark 3.4. *If T is a B-Fredholm operator such that $T^n(X)$ is closed and the induced operator T_n is Fredholm, then one defines* $\text{ind}(T) = \text{ind}(T_n)$ *[5]. If $T-\lambda$ is B-Fredholm, then there is $n \in \mathcal{N}$ such that $(T-\lambda)^n(X)$ is closed, the induced operator $T_{n(\lambda)}$ is Fredholm and the* deficiency indices $\alpha(T_{n(\lambda)}) = \dim(T_{n(\lambda)}(X) \cap (T-\lambda)^{-1}(0))$ *and* $\beta(T_{n(\lambda)}) = \text{codim}((T-\lambda)(X) + (T_{n(\lambda)})^{-1}(0))$ *are finite [5, theorem 3.1]. Since T has SVEP at λ implies $T_{n(\lambda)}$ has SVEP (at 0), and since $T_{n(\lambda)}$ is Fredholm,* $\text{asc}(T_{n(\lambda)}) < \infty$ *[1, theorem 2.6]. Hence* $\text{ind}(T-\lambda) = \text{ind}(T_{n(\lambda)}) \le 0$. *This provides an alternative proof of Lemma 3.3.*

Theorem 3.5. *If $T \in B(X)$ has property (H_p), then $f(T)$ satisfies g-Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$.*

PROOF. It is apparent from the definition that operators T satisfying property (H_p) have finite ascent, and hence SVEP. Thus g-Browder's theorem holds for T (see Lemma 3.1). Recall that if $T \in (H_p)$, then so does any restriction of T to an invariant subspace; recall also that operators $T \in (H_p)$ satisfy Weyl's theorem (see [22] and [3]). Hence the restriction of T to an invariant subspace satisfies Weyl's theorem.

Observe that if $\lambda \in E(T) \cap (H_p)$, then $\lambda \in \text{iso}\sigma(T)$ and $X = (T-\lambda)^{-p} \oplus (T-\lambda)^p(X)$, λ is a pole of order p of the resolvent of T and $\dim \chi_T(\{\lambda\}) = \dim H_0(T-\lambda) < \infty$. This, taken along with the fact that the restriction of T to an invariant subspace satisfies Weyl's theorem, implies that generalized Weyl's theorem holds for T (see [11, lemma 2.4]). Observe also that if f is an analytic function that is nonconstant on every component of $\sigma(T)$, then $\sigma_{bw}(f(T)) = f(\sigma_{bw}(T))$ (by Lemmas 3.3 and 3.2). Thus to prove the theorem it will suffice to prove that $\sigma(f(T)) \setminus E(f(T)) = f(\sigma(T) \setminus E(T))$.

Let $\lambda \in \sigma(f(T)) \cap E(f(T))$. Since $\lambda \in \text{iso}\sigma(f(T)) \equiv \text{iso}f(\sigma(T))$, there exists a $\mu \in \text{iso}\sigma(T)$ such that $\lambda = f(\mu)$. Since T is an isoloid operator (see [11, lemma 2.1]), $\mu \in E(T)$. Hence $\sigma(f(T)) \setminus E(f(T)) \supseteq f(\sigma(T) \setminus E(T))$.

To show the reverse inclusion, let $\lambda \in \sigma(f(T)) \setminus E(f(T))$. We have two possibilities:

Case I: $\lambda \notin \text{iso}\sigma(f(T))$. Then there exists a sequence $\{\lambda_n\} \subset \sigma(f(T)) = f(\sigma(T))$ and a sequence $\{\mu_n\} \subset \sigma(T)$ such that $f(\mu_n) = \lambda_n \to \lambda$. Since $\sigma(T)$ is a compact subset of the complex plane, we can suppose, without loss of generality, that $\mu_n \to \mu \in \sigma(T)$, and then $\lambda = f(\mu) \in f(\sigma(T) \setminus E(T))$.

Case II: $\lambda \in \text{iso}\sigma(f(T))$ and $\lambda \notin E(f(T))$. Then $f(T) - \lambda = (T - \mu_1) \cdots (T - \mu_n)g(T)$, where $g(T)$ is an invertible operator. Since $(f(T) - \lambda)^{-1}(0) = \{0\}$, and $T - \mu_i$, $i = 1, \ldots, n$, are commuting operators, μ_i is not an eigenvalue of T for all $i = 1, \ldots, n$. Hence $\lambda \in f(\sigma(T) \setminus E(T))$. ■

As we noted in the introduction, a number of the commonly considered classes of Hilbert space operators satisfy property (H_p), and hence they satisfy Weyl's theorem. Theorem 3.5 generalizes these extant results to prove that these operators satisfy g-Weyl's theorem. Recall that if an operator $T \in B(H)$, H a Hilbert space, is either hyponormal ($|T^*|^2 \leq |T|^2$) or p-hyponormal ($|T^*|^{2p} \leq |T|^{2p}$ for some $0 < p < 1$) or M-hyponormal (there exists a scalar $M \geq 1$ such that $||(T-\lambda)^*x|| \leq M||(T-\lambda)x||$ for all complex numbers λ and $x \in H$) or totally $*$-paranormal ($||(T-\lambda)^*x||^2 \leq ||(T-\lambda)^2x||$ for all complex numbers λ and unit vectors $x \in H$) or totally paranormal ($||(T-\lambda)x||^2 \leq ||(T-\lambda)^2x||$ for all complex numbers λ and unit vectors $x \in H$), then $H_0(T-\lambda) = (T-\lambda)^{-1}(0)$ (see [3], [22] and [16]): Theorem 3.5 says that $f(T)$ satisfies g-Weyl's theorem, hence it also satisfies Weyl's theorem, for every $f \in \mathcal{H}(\sigma(T))$ for all such operators T.

4. Algebraically HN operators

The following lemma is immediate from the definition of the class HN.

Lemma 4.1. *Let $T \in B(X)$ be an algebraically HN operator, and let $X_1 \subset X$ be a T-invariant subspace. Then $T_{|X_1}$ is an algebraically HN operator.*

Lemma 4.2. *Isolated points of $\sigma(T)$ of an algebraically HN operator are simple poles of the resolvent of T.*

Proof. If $\lambda \in \text{iso}\sigma(T)$, then T has a direct sum decomposition $T = T_1 \oplus T_2$ on $X = X_1 \oplus X_2$ such that $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Let p be a non-constant polynomial such that $p(T)$ is an HN operator. Then X_1 is a $p(T)$-invariant subspace, and, hence, $p(T_1)$ is an HN operator such that $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$. But then $p(\lambda) \in \Pi_0(p(T_1))$ (see [11, lemma 2.1]), and $\lambda \in \Pi_0(T_1)$ (see [13, theorem 1]). Hence, since $\lambda \notin \sigma(T_2)$, $\lambda \in \Pi_0(T)$. ■

Remark 4.3. *It is apparent from the proof of Lemma 4.2 that* $\dim X_1 < \infty$.

Corollary 4.4. *If $T \in B(X)$ is an algebraically HN operator, then*

$$\Pi_0(T) = \pi_{00}(T) = E(T).$$

Proof. Since $\Pi_0(T) \subset \pi_{00}(T) \subset E(T)$ for every operator $T \in B(X)$, we have to show only the opposite inclusion. But this is obvious, since $\lambda \in E(T) \Longrightarrow \lambda \in \mathrm{iso}\sigma(T) \Longrightarrow \lambda \in \Pi_0(T)$ (by Lemma 4.2). ■

The following lemma is a slight generalization of [11, lemma 3.1 (i)].

Lemma 4.5. *If $T \in B(X)$ is an algebraically HN operator, and if X is a separable Banach space, then T has SVEP.*

Proof. Suppose that $\sigma_p(T)$ is not countable. Then the set of non-zero eigenvalues of $p(T)$ is not countable. Since the eigenspaces corresponding to different non-zero eigenvalues of $p(T)$ are orthogonal (see [11, lemma 2.2]), it follows that X is not separable. This contradiction shows that $\sigma_p(T)$ is countable, and hence, since every operator with countable point spectrum has SVEP (see [9]), $p(T)$ has SVEP. Recall from [2, theorem 5] that $p(T)$ has SVEP at $\lambda \in \mathbf{C}$ if and only if T has SVEP at every $\mu \in \sigma(T)$ such that $p(\mu) = \lambda$. Hence T has SVEP. ■

Corollary 4.6. *Let $T \in B(X)$, X a separable Banach space, be an algebraically HN operator. Then $\mathrm{ind}(T-\lambda) \leq 0$ for every $\lambda \in \mathbf{C}$ such that $T-\lambda$ is a B-Fredholm operator.*

Proof. The algebraically HN operator T has SVEP; hence $\mathrm{ind}(T-\lambda) \leq 0$ (by Lemma 3.3). ■

Lemma 4.7. *Let $T \in B(X)$, X a separable Banach space, be an algebraically HN operator. Then for every $f \in \mathcal{H}(\sigma(T))$, $\sigma_w(f(T)) = f(\sigma_w(T))$ and $\sigma_{bw}(f(T)) = f(\sigma_{bw}(T))$.*

Proof. If $f \in \mathcal{H}(\sigma(T))$, then $\sigma_w(f(T)) \subset f(\sigma_w(T))$ for any operator $T \in B(X)$. For the reverse inclusion, let $\lambda \notin \sigma_w(f(T))$. Then $f(T)-\lambda$ is Weyl and $f(T)-\lambda = c(T-\lambda_1)\cdots(T-\lambda_n)g(T)$, where $g(T)$ is invertible. By the mutual commutativity of the operators $(T-\lambda_1),\ldots,(T-\lambda_n)$ and $g(T)$, we have that each of the operators $T-\lambda_i$ is Fredholm. Since T has SVEP (by Lemma 4.5), it follows from [1, corollary 2.7] that $\mathrm{ind}(T-\lambda_i) \leq 0$ for each $i = 1,\ldots,n$. Therefore $\lambda \notin f(\sigma_w(T))$ and, hence $\sigma_w(f(T)) = f(\sigma_w(T))$. To conclude the proof we recall from Corollary 4.6 that T has stable index. Hence $\sigma_{bw}(f(T)) = f(\sigma_{bw}(T))$ (by Lemma 3.2). ■

Corollary 4.8. *Let $T \in B(X)$ be an algebraically HN operator on a separable Banach space X. If $f \in \mathcal{H}(\sigma(T))$, then $f(\sigma(T) \setminus E(T)) = \sigma(f(T)) \setminus E(f(T))$.*

Proof. This follows as in the proof of Theorem 3.5, since T is also an isoloid operator with stable index. ■

Theorem 4.9. *Let $T \in B(X)$ be an algebraically HN operator on a separable Banach space X. Then g-Weyl's theorem holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. We start by proving that Weyl's theorem holds for T. Let p be a polynomial such that $p(T) \in HN$. Then it follows from Lemma 4.5 that T has SVEP. Thus Browder's theorem holds for T (see Lemma 3.1), i.e.

$$\sigma_w(T) = \sigma(T) \setminus \Pi_0(T) \supset \sigma(T) \setminus \pi_{00}(T).$$

Observe that if $\lambda \in \pi_{00}(T)$, then $\lambda \in \mathrm{iso}\,\sigma(T) \Longrightarrow \lambda \in \Pi_0(T)$ (see Lemma 4.2). Hence,

$$\sigma_w(T) = \sigma(T) \setminus \Pi_0(T) = \sigma(T) \setminus \pi_{00}(T),$$

i.e. Weyl's theorem holds for T. We prove next that g-Weyl's theorem holds for T.

Let M be a T-invariant subspace of X. Then $T_{|M}$ is an algebraically HN operator; hence Weyl's theorem holds for $T_{|M}$. Let $\lambda \in E(T)$. Then, by Corollary 4.4, $\lambda \in \Pi_0(T)$, which implies that $\dim \chi_T(\{\lambda\}) = \dim H_0(T-\lambda) < \infty$. But then T satisfies g-Weyl's theorem (see [11, lemma 2.4]).

To complete the proof, we now appeal to Lemma 3.2 and Corollary 4.8, when it follows that $f(T)$ satisfies g-Weyl's theorem. ■

As we pointed out earlier on, an important class of Banach space operators in HN is that of paranormal operators (i.e. operators $T \in B(X)$ such that $||Tx||^2 \leq ||T^2x||$ for every unit vector $x \in X$). Algebraically paranormal operators (on a separable Hilbert space) have been considered by Curto and Han in [10], where it is shown that such operators satisfy Weyl's theorem. Our Theorem 4.9 extends the results of [10] to prove that algebraically paranormal operators on a separable Banach space satisfy (the more general) g-Weyl's theorem.

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