# ON LEFT–RIGHT CONSISTENCY IN RINGS

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[Received 29 October 2003. Read 12 July 2005. Published 31 January 2006.]

## Abstract

A bounded linear operator $T$ is said to be ‘left–right consistent’ if the products $ST$ and $TS$ always have the same spectrum: this notion lies behind a spectral property of positive operators. Extended to Banach algebras, the same notion helps to delineate the closure of the invertible group.

## 1. Introduction

In a $C^*$-algebra the spectrum of the product of a positive and a self adjoint element is always real. This simple observation is the tip of a curious iceberg, built on a sort of ‘left–right consistency’ relative to invertibility.

Suppose $A$ is a semigroup, assumed by default to have an identity 1, with invertible group $A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}$; more generally much of what we have to say extends to an abstract category. Elements $a \in A$ induce left and right multiplications on $A$,

$$L_a : x \mapsto ax \ ; \ R_a : x \mapsto xa \ .$$

It is the relationship between these operators that gives rise to the left–right consistency behind the positive operator phenomenon.

**Definition 1.** *If $K \subseteq A$ is arbitrary, write*

$$\varpi(K) = \{a \in A : L_a^{-1}(K) = R_a^{-1}(K)\} \tag{1}$$

*for the set of $K$-(left–right) consistent elements of $A$.*

Evidently $\varpi(K)$ is always a sub-semigroup:

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*Mathematical Proceedings of the Royal Irish Academy*, **106**A (1), 11–17 (2006)

$$\varpi(K)\cdot\varpi(K)\subseteq\varpi(K)\ . \tag{2}$$

In this note we determine $\varpi(K)$ for the invertible group $K = A^{-1}$ and for the semigroups $A^{-1}_{left}$ and $A^{-1}_{right}$ of left and of right invertibles.

**Theorem 2.** *For an arbitrary semigroup $A$ there is equality*

$$\varpi(A^{-1}) = A^{-1}\cup\big(A\setminus(A^{-1}_{left}\cup A^{-1}_{right})\big) = \varpi(\{1\})\ . \tag{3}$$

*If $K\subseteq A^{-1}$ is a normal subgroup then also*

$$\varpi(K)=\varpi(A^{-1})\ . \tag{4}$$

Proof. We claim, if $K\subseteq A^{-1}$ is a normal subgroup,

$$a\in A^{-1}\Longrightarrow L_a^{-1}(K)=a^{-1}K=Ka^{-1}=R_a^{-1}(K)\ ; \tag{5}$$

$$a\in A\setminus(A^{-1}_{left}\cup A^{-1}_{right})\Longrightarrow L_a^{-1}(A^{-1})=\emptyset=R_a^{-1}(A^{-1})\ ; \tag{6}$$

$$a\in A^{-1}_{left}\setminus A^{-1}_{right}\Longrightarrow L_a^{-1}(1)=A^{-1}_{left}\neq\emptyset=R_a^{-1}(A^{-1})\ ; \tag{7}$$

$$a\in A^{-1}_{right}\setminus A^{-1}_{left}\Longrightarrow L_a^{-1}(A^{-1})=\emptyset\neq A^{-1}_{right}=R_a^{-1}(1)\ . \tag{8}$$

To see all this, observe

$$\begin{gathered}L_a^{-1}(1)\subseteq A^{-1}_{left}\Longrightarrow R_a^{-1}(A^{-1}_{left})\neq\emptyset\Longrightarrow a\in A^{-1}_{left}\\ \Longrightarrow L_a^{-1}(A^{-1}_{left})=A^{-1}_{left}\subseteq R_a^{-1}(A^{-1}_{left})\end{gathered} \tag{9}$$

and

$$\begin{gathered}R_a^{-1}(1)\subseteq A^{-1}_{right}\Longrightarrow L_a^{-1}(A^{-1}_{right})\neq\emptyset\Longrightarrow a\in A^{-1}_{right}\\ \Longrightarrow R_a^{-1}(A^{-1}_{right})=A^{-1}_{right}\subseteq L_a^{-1}(A^{-1}_{right}).\end{gathered} \tag{10}$$

■

For example, it is trivial that whenever the semigroup $A$ is commutative

$$\varpi(A^{-1})=A\ ; \tag{11}$$

this also holds [11] in finite dimensional rings $A$, where $A^{-1}_{left}\cup A^{-1}_{right}=A^{-1}$.

This is very simple ([5, corollary 1.2]) in a $C^*$-algebra:

**Corollary 3.** *If $A$ is a $C^*$-algebra then, that*

$$\textit{either } \{a^*a, aa^*\}\subseteq A^{-1}\ \textit{ or } \{a^*a, aa^*\}\cap A^{-1}=\emptyset \tag{12}$$

*is necessary and sufficient for $a \in \varpi(A^{-1})$. In particular, normal elements are left–right consistent.*

PROOF. $a \in A^{-1}_{left} \Longleftrightarrow a^*a \in A^{-1}$. $\blacksquare$

Djordjevic ([4, theorem 2.1]) has obtained Theorem 2 for the ring $A = BL(X, X)$ of bounded operators on a Banach space $X$, and also ([4, theorem 2.4]) for the Calkin algebra $A = BL(X, X)/KL(X, X)$. Gong and Han ([5, theorem 1.1]) had the same result for Hilbert space; they seem to have been motivated by the observation of Hladnik and Omladic ([10, proposition 1])— which ironically does not use Corollary 3. Of course in a linear algebra $A$, where the Jacobson lemma guarantees that if $0 \neq \lambda \in \mathbf{C}$, then

$$\textit{either } \{\lambda - ba, \lambda - ab\} \subseteq A^{-1} \textit{ or } \{\lambda - ba, \lambda - ab\} \cap A^{-1} = \emptyset \ , \tag{13}$$

the condition $a \in \varpi(A^{-1})$ can be reproduced in terms of the spectrum $\sigma$:

$$x \in A \Longrightarrow \sigma(ax) = \sigma(xa) \ . \tag{14}$$

Corollary 3 therefore says that certain relatives of $a \in A$, derived from its 'polar decomposition', share its spectrum: whenever we can write $a = u|a|$ with $|a| = (a^*a)^{\frac{1}{2}}$ and $u = uu^*u$, then

$$\sigma(u|a|) = \sigma(|a|u) = \sigma(|a|^{1/2}u|a|^{1/2}) \ . \tag{15}$$

The normal subgroup condition (4) applies in particular if the semigroup $A$ has a topology for which multiplication is separately continuous, and inversion is continuous, when $K = A^{-1}_0$ is the connected component of $1 \in A^{-1}$.

There is an interaction between left–right consistency and generalised inverses. We recall [6; 7] the 'regular' elements of a semigroup, or more generally a category:

$$\overset{\sqcap}{A} = \{a \in A : a \in aAa\} \ , \tag{16}$$

elements $a = aba$ with *generalised inverses* $b \in A$, together with the 'decomposably regular' elements

$$\overset{\ulcorner}{A} = \{a \in A : a \in aA^{-1}a\} \ , \tag{17}$$

elements with invertible generalised inverses $b \in A^{-1}$ .

**Theorem 4.** *If $A$ is a semigroup with identity, then there is inclusion*

$$\overset{\ulcorner}{A} \subseteq \varpi(A^{-1}) \ . \tag{18}$$

PROOF. In a general semigroup or category we have

$$\overset{\ulcorner}{A} \ \cap (A^{-1}_{left} \cup A^{-1}_{right}) = A^{-1} \ . \tag{19}$$

By (19) and (3)

$$\overset{\sqcup}{A} \cap \varpi(A^{-1}) = A^{-1} \cup \big(\overset{\sqcup}{A} \setminus (A^{-1}_{left} \cup A^{-1}_{right})\big) = A^{-1} \cup (\overset{\sqcup}{A} \setminus A^{-1}) = \overset{\sqcup}{A} \ . \tag{20}$$

■

If, for example, $A = BL(X,X)$ is the bounded operators on a Banach space, then [9]

$$\{a \in A : a^{-1}(0) \cong X/\mathrm{cl}\ a(X)\} \subseteq \varpi(A^{-1}) \ : \tag{21}$$

operators 'of index zero' are left–right consistent.

In a ring $A$ we can look for a stabilised version of this, in particular relative to a special kind of two sided ideal.

**Definition 5.** *We shall call the two sided ideal $J \subseteq A$ completely regular if there is inclusion*

$$J \subseteq \overset{\sqcap}{A}; \tag{22}$$

*completely decomposably regular if there is inclusion*

$$J \subseteq \overset{\sqcup}{A}; \tag{23}$$

*regular if there is inclusion*

$$1 + J \subseteq \overset{\sqcap}{A}; \tag{24}$$

*and decomposably regular if there is inclusion*

$$1 + J \subseteq \overset{\sqcup}{A} \ . \tag{25}$$

*If $A$ is a Banach algebra we shall say that $J$ is weakly Riesz if there is inclusion*

$$1 + J \subseteq \mathrm{cl}\, A^{-1}. \tag{26}$$

The archetype is the finite rank operators in the ring of all bounded operators on a normed space. With the help of a lemma of Atkinson ([7, theorems 7.3.2, 7.3.3]) it follows that

$$J \ \textit{completely regular} \Longrightarrow \overset{\sqcap}{A} + J \subseteq \overset{\sqcap}{A}; \tag{27}$$

the analogue for completely decomposably regular ideals is not so clear [8]. In the other direction it follows from (26) that also

$$J \subseteq \mathrm{cl}\ A^{-1} \ , \tag{28}$$

since (26) puts $\lambda + J \subseteq \mathrm{cl}\ A^{-1}$ for arbitrary scalars $\lambda$. The point about weakly Riesz ideals is that in a Banach algebra $A$ there is ([6, theorem 1.1; 7 theorem 7.3.4]) equality

$$\overset{\sqcap}{A} \cap \mathrm{cl}\ A^{-1} = \overset{\sqcup}{A} \ . \tag{29}$$

Since ([7, theorems 3.10.5, 3.10.6]) the boundary of the invertibles lies among the topological zero divisors, we also have

$$(A^{-1}_{left} \cup A^{-1}_{right}) \cap \mathrm{cl}\ A^{-1} = A^{-1}\ . \tag{30}$$

From (30) and (3) it is clear that, in a Banach algebra $A$,

$$\mathrm{cl}\ A^{-1} \subseteq \varpi(A^{-1})\ . \tag{31}$$

We can improve on this:

**Theorem 6.** *If $J \subseteq A$ is a completely regular weakly Riesz two-sided ideal in a Banach algebra $A$, then there is inclusion*

$$\mathrm{cl}\ A^{-1} \subseteq \overline{A} \cup \ \left(A \setminus \ \overline{A}\right) \subseteq \bigcap_{d \in J} (\varpi(A^{-1}) + d)\ . \tag{32}$$

PROOF. The first inclusion follows at once from (29). Towards the second observe ([8, theorem 7]) that if $J \subseteq A$ is a two-sided ideal then

$$J \textit{ regular and weakly Riesz} \Longrightarrow J \textit{ decomposably regular}\ ; \tag{33}$$

$$\begin{matrix} J \textit{ completely regular and weakly Riesz} \\ \Longrightarrow J \textit{ completely decomposably regular} \end{matrix}\ ; \tag{34}$$

$$J \textit{ weakly Riesz and completely decomposably regular} \Longrightarrow \overline{A} + J \subseteq \overline{A}\ \ . \tag{35}$$

This mostly also follows from (29): immediately in the case of (33), while for (34) recall (28):

$$1 + J \subseteq \mathrm{cl}\ A^{-1} \Longrightarrow J \subseteq \mathrm{cl}\ A^{-1} + J \subseteq \mathrm{cl}(A^{-1} + J) \subseteq \mathrm{cl}\ A^{-1} = \mathrm{cl}\ A^{-1}\ .$$

Finally, for (35) (cf [3, theorem 4]) note that from (26) it also follows $A^{-1} + J \subseteq \mathrm{cl}\ A^{-1}$, so that

$$\overline{A} + J \subseteq \overline{A} \cap \mathrm{cl}(A^{-1} + J) \subseteq \overline{A} \cap \mathrm{cl}\ \mathrm{cl}\ A^{-1} = \overline{A}\ .$$

Now for (32), if $a \in \overline{A}$ then $a + J \subseteq \overline{A} \subseteq \varpi(A^{-1})$ using (18), while if $a \in A \setminus \overline{A}$ then $a + J \subseteq A \setminus \overline{A} \subseteq A \setminus (A^{-1}_{left} \cup A^{-1}_{right}) \subseteq \varpi(A^{-1})$. ■

In the particular case of the finite rank operators $J \subseteq A$ among the bounded operators on a separable Hilbert space both these inclusions become equality; the first was noticed by Bouldin ([1, theorem 3; 2, (3.5)]), while the second is given by Gong and Han ([5, theorem 3.2]). Another characterisation, due to Wu ([12 theorem 1.1; 5 theorem 3.1]), says that the same set consists of all finite products of normal operators.

We can also consider left–right consistency separately relative to left and to right invertibility. We need to recall the ‘mixed invertible’ elements of $A$,

$$A^{-1}_{mixed} = \{a \in A : 1 \in AaA\}\ . \tag{36}$$

**Theorem 7.** *For arbitrary $A$ there is equality*

$$\varpi(A^{-1}_{left}) = A^{-1} \cup \big(A \setminus A^{-1}_{mixed}\big) = \varpi(A^{-1}_{right})\ . \tag{37}$$

Proof. It is clear that if $a \in A^{-1}$ is invertible then

$$L_a^{-1}(A^{-1}_{left}) = A^{-1}_{left} = R_a^{-1}(A^{-1}_{left}), \tag{38}$$

while if $a \in A \setminus A^{-1}_{mixed}$ then

$$L_a^{-1}(A^{-1}_{left}) = \emptyset = R_a^{-1}(A^{-1}_{left})\ . \tag{39}$$

If $a \in A^{-1}_{left} \setminus A^{-1}$ then

$$a'a = 1 \Longrightarrow a' \in R_a^{-1}(A^{-1}_{left}) \setminus L_a^{-1}(A^{-1}_{left})\ ; \tag{40}$$

finally if $a \in A^{-1}_{mixed} \setminus A^{-1}_{left}$ then $R_a^{-1}(A^{-1}_{left})$ is empty while $L_a^{-1}(A^{-1}_{left})$ is not. This gives the first equality in (37), and therefore also the second. ■

Notice that if $J \subsetneq A$ is a proper two-sided ideal then $A^{-1}_{mixed}$ is disjoint from the '$J$ inessential elements'

$$\mathrm{Hull}(J) = \{a \in A : a + J \in \mathrm{Radical}(A/J)\ \ : \tag{41}$$

certainly if $1 \in AaA$ and also $a \in \mathrm{Radical}(A)$, so that also $1 - AaA \subseteq A^{-1}$, then $1 = 0$ giving $A = \{0\}$: now apply this to the quotient $A/J$. Thus, in particular when $A = B(X)$, then [4] $A^{-1}_{mixed}$ excludes all *strictly singular* and *strictly co-singular* operators.

## Acknowledgement

Supported by Enterprise Ireland Basic Research Grant SC/2002/0266

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