A SPATIAL DECAY ESTIMATE FOR A NONLINEAR ELLIPTIC PROBLEM

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Abstract

A Dirichlet boundary value problem for a nonlinear elliptic p.d.e. is considered in a right cylinder, the boundary value on the lateral surface being independent of the axial coordinate. An inequality estimate, in terms of data, is obtained for the spatial rate of convergence — as one recedes from the plane ends — of the solution of the problem, to the solution of the corresponding two-dimensional solution (induced by the lateral boundary condition). The estimate is for a suitably defined, cross-sectional measure, which is positive-definite in the perturbation (i.e. the difference between the solution and that of the two-dimensional state). The estimate is obtained by establishing a differential inequality for the cross-sectional measure. The cross-sectional measure is analogous to a Liapunov functional that has been used in time-dependent, initial boundary value problems. The paper concludes with a discussion of the estimate obtained.

1. Introduction

In previous papers, a novel, and very effective, Liapunov functional was introduced and used to derive decay and asymptotic stability estimates (with respect to time) in a variety of nonlinear thermal and thermo-mechanical contexts [3; 4; 5]. Moreover, it was shown in [6] that the versatility of this functional extends to certain non-linear elliptic boundary value problems in a right cylinder, the axial coordinate in this context replacing the time variable in the previous one: a non-linear steady-state temperature problem in a right cylinder was considered (i) where the transverse diffusivity is supposed to depend on temperature, while that in the axial direction is supposed homogeneous; (ii) where Dirichlet conditions are specified on the surfaces of the cylinder, the condition on the lateral boundary being independent of the axial coordinate; and an estimate was obtained, in terms of data, for a cross-sectional measure of the error committed in approximating the temperature field by the two-dimensional field induced by the boundary condition on the lateral surface. Stated otherwise, an estimate is obtained for the spatial rate of convergence — as one recedes from the plane ends — of the solution to that of the corresponding two-dimensional problem. The estimate in question is based on a second order differential inequality.

The aforementioned estimate was based on the assumption that the transverse diffusivity (depending on temperature) is bounded below by a given positive constant and, possibly, bounded above by a positive constant. The present work addresses the same issue as [6] but the assumption on the *transverse* diffusivity is

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different: it is supposed to depend on a positive power of the temperature (as in the case of the porous medium equation); moreover, the solutions (in the original temperature variable) are supposed to be classical and positive.

The analysis arising here is reminiscent of a recent proof of the global convergence (in time) of the solution of an initial boundary value problem — with time-independent, non-null boundary conditions — for the porous medium equation, to that of the corresponding steady state. The proof, given in [7], supposes positive solutions, and depends crucially on an algebraic inequality. The aforementioned inequality is also central to the issue discussed in this paper.

To put this work in context, the spatial decay estimate considered here is of a type similar to that associated with Saint-Venant's principle in elasticity. There has been a large volume of work in this area in the past forty years or so, much of which is well summarized in [8; 9; 10]. Much of the work related to elasticity, both linear and nonlinear, but the work expanded to cover boundary value problems for nonlinear (i.e. quasilinear) partial differential equations of other types. The present work is similar in spirit to that of Horgan and Payne [11], and related papers by these and other authors, in the following respect: they are concerned with the spatial rate of convergence of the solution to a boundary value problem for a nonlinear partial differential equation to the solution of a problem of lower dimensionality. The nonlinear p.d.e. considered in the present paper differs from that considered in [11] and the approach is also different.

2. A spatial decay estimate

Let (x, y, z) denote rectangular cartesian coordinates and let ∇_1 denote the gradient operator in the xy plane. We shall consider a right cylinder $D(z) \times 0 < z < L$, where $D(\cdot)$ denotes the cross-section of the cylinder and L its length, the boundary of the cylinder being supposed sufficiently smooth to allow application of the divergence theorem.

We consider the solution — assumed smooth and positive — of

$$\nabla_1 . \{k(u)\nabla_1 u\} + u_{zz} = 0 \text{ in } D(z) \ \times 0 < z < L, \tag{2.1}$$

where $k(u)$ is a prescribed function — which is (unless otherwise, and exceptionally, indicated) assumed to be

$$k(u) = u^{n-1}, \tag{2.2}$$

n being a constant such that $n > 1$, subject to the boundary conditions

$$u = u_0(x, y) \text{ on } \partial D(z) \times 0 < z < L, \tag{2.3}$$

and such that

$$u = \left.\begin{array}{l} \bar{u}_0 \ (x, y) \text{ on the end } z = 0, \\ \bar{u}_L \ (x, y) \text{ on the end } z = L, \end{array}\right\} \tag{2.4}$$

where $u_0, \bar{u}_0, \bar{u}_L$are all assumed positive. In (2.1) and subsequently, subscripts denote partial differentiation with respect to the appropriate variable.

Consider the corresponding two-dimensional problem: $U(x,y)$ (assumed smooth and positive) satisfies

$$\nabla_1.\{U^{n-1}\nabla_1 U\} = 0 \text{ in } D(\cdot) \tag{2.5}$$

subject to

$$U = u_0(x,y) \text{ on } \partial D(\cdot). \tag{2.6}$$

The question arises as to how well U approximates u, and the analysis given below seeks to elucidate the matter.

We define the perturbation θ as follows:

$$\theta = u - U; \tag{2.7}$$

and we define a function $\Phi(\theta, U)$ — central to the ensuing analysis — as follows:

$$\Phi(\theta, U) = \int_0^\theta d\bar{\theta} \int_0^{\bar{\theta}} (\tau + U)^{n-1} d\tau\,. \tag{2.8}$$

Bearing in mind that (2.1),(2.5) may, respectively, be written in the forms

$$\nabla_1^2 \left\{ \int_0^u \bar{u}^{n-1} d\bar{u} \right\} + u_{zz} = 0, \quad \nabla_1^2 \left\{ \int_0^U \bar{u}^{(n-1)} d\bar{u} \right\} = 0,$$

one finds that the perturbation θ satisfies

$$\nabla_1^2 \Phi_\theta + \theta_{zz} = 0 \text{ in } D(z) \times 0 < z < L, \tag{2.9}$$

subject to

$$\theta = 0 \text{ on } \partial D(z) \times 0 < z < L \tag{2.10}$$

and

$$\theta = \bar{u}_0 - U \text{ on the end } z = 0,\ \theta = \bar{u}_L - U \text{ on the end } z = L. \tag{2.11}$$

The function Φ, defined by (2.8), is central to the ensuing analysis, and its salient properties are now discussed. Explicit calculations yield

$$\Phi(\theta; U) = \{n(n+1)\}^{-1} \left[(\theta + U)^{n+1} - U^{n+1} - (n+1)U^n \theta \right] \tag{2.12}$$

$$\Phi_\theta(\theta; U) = n^{-1} \left[(\theta + U)^n - U^n \right], \tag{2.13}$$

and, in particular,

$$\Phi(0; U) = \Phi_\theta(0; U) = 0. \tag{2.14}$$

Using the foregoing, Taylor's Theorem (remainder form) gives

$$\Phi(\theta; U) = [\psi(\theta + U) + (1 - \psi)U]^{n-1}\, \theta^2/2 \tag{2.15}$$

where ψ is such that $0 < \psi < 1$. It follows from this, on assuming $U + \theta \geq 0, U \geq 0$

(although strict inequalities were assumed earlier) that Φ is *positive definite* in θ. We note the following important property proved in [7] and referred to in Appendix (a):

$$\left.\begin{aligned} &\Phi_\theta^2 \geq K_n \Phi^{2n(n+1)^{-1}} \\ \text{where} \quad & \\ &K_n = (n+1)^{2n(n+1)^{-1}} n^{-2}. \end{aligned}\right\} \tag{2.16}$$

Let us now define the *cross-sectional measure* of the perturbation θ, as follows:

$$F(z) = \int_{D(z)} \Phi(\theta; U)dA. \tag{2.17}$$

In view of the positive definiteness of Φ, discussed in the last paragraph, (2.17) represents a reasonable global cross-sectional measure of the 'size' of the perturbation θ. Differentiation of (2.17) gives

$$F'(z) = \int_{D(z)} \Phi_\theta \theta_z dA, \tag{2.18}$$

$$F''(z) = \int_{D(z)} \left[\Phi_{\theta\theta}\theta_z^2 + \Phi_\theta \theta_{zz}\right] dA, \tag{2.19}$$

primes denoting ordinary differentiation with respect to z, the θ subscripts denoting partial differentiation with respect to this variable. Now, using (2.9), we obtain

$$\int_D \Phi_\theta \theta_{zz} dA = -\int_D \Phi_\theta \nabla_1^2 \Phi_\theta dA = \int_D (\nabla_1 \Phi_\theta)^2 dA, \tag{2.20}$$

using the divergence theorem and (2.10).

We now note the well known inequality

$$\int_D (\nabla_1 \chi)^2 dA \geq \lambda_1 \int_D \chi^2 dA$$

for arbitrary smooth functions χ vanishing on the boundary ∂D, where λ_1 is the lowest (positive)'fixed membrane' eigenvalue of

$$\nabla_1^2 \chi + \lambda\chi = 0 \text{ in } D, \chi = 0 \text{ on } \partial D. \tag{2.21}$$

We use this inequality with $\chi = \Phi_\theta$,bearing in mind (2.14), to obtain

$$\int_D (\nabla_1 \Phi_\theta)^2 dA \geq \lambda_1 \int_D \Phi_\theta^2 \, dA. \tag{2.22}$$

Using this together with (2.16) yields

$$\int_D (\nabla_1 \Phi_\theta)^2 dA \geq \lambda_1 K_n \int_D \Phi^{2n(n+1)^{-1}} dA,$$

and an application of Hölder's inequality gives

$$\int_{D(z)} (\nabla_1 \Phi_\theta)^2 dA \geq \lambda_1 K_n A^{-(n-1)(n+1)^{-1}} F^{2n(n+1)^{-1}}, \tag{2.23}$$

where A denotes the area of the cross-section, bearing in mind (2.17).

Thus (2.19), (2.20), (2.23) — bearing in mind the non-negativity of $\Phi_{\theta\theta}$ — gives the (elementary) differential inequality for $F(z)$

$$F''(z) - J_n \left[F(z)\right]^{1+\sigma} \geq 0 \tag{2.24}$$

where, for convenience, we write

$$J_n = \lambda_1 K_n A^{-\sigma}, \sigma = \sigma(n) = (n-1)/(n+1). \tag{2.25}$$

This inequality is adequate for small u.

An improved inequality is obtainable on taking into account the following:

$$(F^\beta)'' = \beta F^{\beta-2} \left[FF'' - (1-\beta)F'^2\right] \tag{2.26}$$

for constant β;

$$\Phi\Phi_{\theta\theta} \geq (u_m/u_M)^{n-1}\Phi_\theta^2/2, \tag{2.27}$$

where u_m, u_M denote the minimum and maximum values of u prescribed on the boundary (see Appendix(c)).

Using Schwarz's inequality together with (2.17), (2.18), (2.19), (2.23), (2.26), (2.27), we obtain the following, strengthened differential inequality

$$\Im'' - M_n \Im^{1+\nu} \geq 0, \tag{2.28}$$

where we define

$$\Im = F^\beta, \tag{2.29}$$

with

$$\nu = \nu(n) = \sigma\beta^{-1}, \beta = 1 - (u_m/u_M)^{n-1}/2 \tag{2.30}$$

and

$$M_n = \beta J_n. \tag{2.31}$$

It should be noted that (2.28) reduces to (2.24) when $\beta = 1$ (rather than the value given in (2.30)). Indeed, (2.24) is a reasonable approximation to (2.28) for small u.

We can obtain upper bounds for $\Im(z)$, in terms of data, using a comparison theorem (e.g. [13]):

$$\Im(z) \leq G(z), \tag{2.32}$$

where G satisfies

$$G'' - M_n G^{1+\nu} \leq 0, \tag{2.33}$$

together with

$$G(0) \geq \Im(0), G(L) \geq \Im(L). \tag{2.34}$$

[This comparison theorem is geometrically plausible in the case $M_n = 0$].

To identify a satisfactory G, we proceed in a constructive, step by step, manner, the better to understand how it is arrived at. First, seek a function G, taking the value $\Im(0)$ at $z = 0$ and satisfying (2.33) with the *equality* sign, of the form:

$$G(z) = \Im(0)(1 + \alpha_0 z)^{-p}, \tag{2.35}$$

$\alpha_0(\geq 0), p\,(\geq 0)$ being constants; (2.34_2) is regarded as a constraint. Substitution of (2.35) in (2.33) with the equality sign, gives (on comparison):

$$p = 2\nu^{-1}, \alpha_0 = \sqrt{M_n \nu^2 (2\nu + 4)^{-1} \{\Im(0)\}^{\nu}}. \tag{2.36}$$

Thus

$$\Im(z) \leq \Im(0)(1 + \alpha_0 z)^{-p}, \tag{2.37}$$

where p,α_0 are defined by (2.36), *provided that*

$$\Im(L) \leq \Im(0)(1 + \alpha_0 L)^{-p}. \tag{2.38}$$

This would be satisfied, say, if, formally, $\Im(L) = 0$.

To deal with the case where (2.38) is not satisfied, observe first that a similar analysis gives

$$\Im(z) \leq \Im(L) \left\{1 + \alpha_L (L - z)\right\}^{-p} \tag{2.39}$$

provided that

$$\Im(0) \leq \Im(L)(1 + \alpha_L L)^{-p}, \tag{2.40}$$

where p is defined as in (2.36)and α_L is defined thus

$$\alpha_L = \sqrt{M_n \nu^2 (2\nu + 4)^{-1} \left\{\Im(L)\right\}^{\nu}} \tag{2.41}$$

(similar to α_0).

The two estimates obtained can be combined to cater for any combination of end conditions. Denote the two G functions obtained, and occurring on the right hand sides of (2.37),(2.39), by $G_0(z), G_L(z)$, respectively. It is easily verified that

$$G(z) = G_0(z) + G_L(z) \tag{2.42}$$

satisfies (2.33) and

$$\left.\begin{array}{l} G(0) = \Im(0) + G_L(0) \geq \Im(0), \\ G(L) = G_0(0) + \Im(L) \geq \Im(L). \end{array}\right\} \tag{2.43}$$

Thus $G(z)$ defined by (2.42) satisfies the conditions of the comparison theorem. We thus have the following theorem:

Theorem 1. *The (modified) cross-sectional measure $\Im(z)$, defined by (2.17), (2.29), (2.30), of the perturbation θ, defined by (2.9), (2.10), (2.11), satisfies*

$$\Im(z) \leq \Im(0)(1+\alpha_0 z)^{-p} + \Im(L)\left\{1+\alpha_L(L-z)\right\}^{-p} \tag{2.44}$$

where p, α_0, α_L are defined by (2.30), (2.36), (2.41) together with (2.21), (2.25).

A result of the Phragmèn–Lindelöf type, may easily be deduced from Theorem 2.1 when $L \to \infty$:

Corollary 1. *The modified cross-sectional measure $\Im(z)$ satisfies*

$$\Im(z) \leq \Im(0)(1+\alpha_0 z)^{-p}$$

for $0 < z < \infty$, provided that

$$\lim_{L\to\infty} \Im(L)(1+\alpha_L L)^{-p} = 0.$$

Remark 1. *The inequality (2.44) is, in a sense, optimal in the limit $n \to 1$: one may verify, noting the result*

$$(1+sz)^{-1/s} \to e^{-z} \text{ as } s \to 0,$$

that, in the limit $n \to 1$, (2.44) reduces to

$$F^{1/2}(z) \leq F^{1/2}(0)\exp(-\sqrt{\lambda_1}z) + F^{1/2}(L)\exp\left[-\sqrt{\lambda_1}(L-z)\right],$$

while the p.d.e. reduces to Laplace's equation and the cross-sectional measure to $\int(1/2)u^2 dA$. It is easily verified (e.g. [2]) that one has optimality, in these circumstances, when $L \to \infty$ and $F(L) \to 0$ $(L \to \infty)$.

3. Futher discussion of the decay result

(a) It may be objected that rather than considering the p.d.e. (2.1), it would be more reasonable to consider an equation with isotropic diffusivity (depending on the dependent variable)

$$\nabla \cdot \{f(u)\nabla u\} = 0.$$

However, a moment's reflection will show that the spatial rate of approach to the steady state in this case is essentially that arising in the linear case. Hence this would not be interesting from a mathematical point of view.

It is of interest to note that Antontsev, Díaz and Shmarev [1] consider anisotropic nonlinear elliptic equations somewhat similar to (2.1), which give rise to interesting phenomena. However, there is no overlap between this work and that presented here.

It should be noted also that Horgan and Payne [12] consider spatial decay issues for an inhomogeneous (linear) analogue of Laplace's equation, and recover estimated decay rates quite different from those (exponential ones) associated with the standard Laplace equation.

(b) Perhaps the most notable feature of the decay implied by (2.44) is that the spatial decay rates, away from the ends $z = 0, z = L$, depend on the relevant end values of the perturbation $\Im(0), \Im(L)$, respectively. In this respect, the spatial rate of approach to the two-dimensional state resembles the time-rate of approach to the steady state for an initial boundary value problem for the porous medium equation, as estimated by a Liapunov functional analogous to the cross-sectional measure (2.17). The latter issue is discussed in [7].

(c) From Theorem 1 it is possible to deduce spatial decay estimates for cross-sectional measures that are arguably more tangible than that given by (2.17). One may prove (see Appendix (b)) the optimal inequality

$$\Phi(\theta, U) \geq (n+1)^{-1}\theta^2 U^{n-1}. \tag{3.1}$$

One may deduce a spatial decay estimate for the L_2 cross-sectional norm

$$\|\theta\|_2 = \left[\int \theta^2 dA\right]^{1/2},$$

as follows: Using (2.44) together with the fact— using the maximum principle—that if $U \geq m(> 0)$ on ∂D, m being a constant, one has $u \geq m$ everywhere, then

$$\|\theta\|_2^2 \leq (n+1)m^{-(n-1)}F(z). \tag{3.2}$$

The required estimate follows since an upper estimate for $F(z)$ follows from (2.44).

Similar estimates may be obtained for other norms.

(d) An alternative estimate may be obtained for $\Im(z)$ using a result proved in [6], and it is instructive to investigate—at least in tractable cases—the circumstances in which each estimate is best.

In [6] the general issue addressed in this paper is examined under a different constitutive assumption: under the assumptions

$$k_M \geq k(u) \geq k_0$$

where k_M, k_0 are positive constants, it is proved *inter alia* that, for $0 < z < \infty$,

$$\Im(z) \leq \Im(0) \exp\left\{-\sqrt{2k_0\lambda_1\beta}\, z\right\} \tag{3.3}$$

provided

$$\Im(L) \to 0 \text{ as } L \to \infty, \tag{3.4}$$

where, in the definition (2.29), one has

$$\beta = 1 - (k_0/k_M)/2. \tag{3.5}$$

Adapting this result to the present case where the assumption (2.2) holds, one deduces, in view of the maximum principle, that, for $0 < z < \infty$,

$$\Im(z) \leq \Im(0) \exp\left\{-\sqrt{2\lambda_1 (u_m)^{n-1}\beta}\, z\right\} \tag{3.6}$$

where one now takes (as in Theorem 2.1)

$$\beta = 1 - \frac{1}{2}(u_m/u_M)^{n-1}, \tag{3.7}$$

and where (3.4) again holds.

The corresponding result following from Theorem 1 is: for $0 < z < \infty$,

$$\Im(z) \leq \Im(0)(1 + \alpha_0 z)^{-p} \tag{3.8}$$

where (3.4) again holds, and where α_0 is again given by (2.36). The issue to be addressed is: in what circumstances is the (upper) estimate (3.8) better (i.e. smaller) than (3.6)? This will be so provided that

$$M_n(\nu + 2)^{-1} \{\Im(0)\}^{\nu} \geq \lambda_1 u_m^{n-1}\beta \tag{3.9}$$

(where β is given by (3.7)) *and* provided that $0 < z < z_c$, where z_c is an easily obtainable (but somewhat cumbersome) number whose size increases as the difference between the two sides of (3.9) increases. In words, (3.9) states that the deviation of the end-value of u, from that of the two-dimensional state, exceeds a critical value. Indeed, the superiority of estimate (3.8) compared to (3.2), becomes more marked the bigger $\Im(0)$ is and the smaller u_m.

4. Closing Comments

It is believed likely that the result (2.44) continues to hold when the boundary values of the original problem are allowed to take the value zero. In this case, it would be necessary to consider suitably defined weak solutions, as the ellipticity of the operator degenerates in these circumstances; alternatively, a suitable limiting process could be used. Moreover, in these circumstances, the exponential estimate (3.6) would not arise as u_m would then be zero.

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APPENDIX

(a) Inequality (2.16) is optimal for all possible θ, U such that $\theta + U \geq 0$. The proof, which is given in [11], essentially amounts to showing that the minimum value of

$$h(p) = (p^n - 1)^2 / \left[p^{n+1} - 1 - (n+1)(p-1)\right]^{2n(n+1)^{-1}}$$

for $p \geq 0$ ($p \neq 1$ w.l.o.g.) is

$$n^{-2n(n+1)^{-1}}.$$

(b) The inequality (3.1) is now proved. One has

$$\Phi(\theta; U) \geq (n+1)^{-1}\theta^2 U^{n-1}$$

for all θ, U such that $\theta + U \geq 0$; the constant therein is optimal.

We assume w.l.o.g. that $n > 1$. In view of (2.12), we need to prove that

$$h(p) \overset{\text{defn}}{=} p^{n+1} - 1 - (n+1)(p-1) - n(p-1)^2 \geq 0.$$

We do this by considering two cases separately:
(i) $p \geq 1$,
(ii) $0 \leq p < 1$.

Case (i) follows on noting that

$$h(1) = h'(1) = 0,$$

while

$$h''(p) = n[(n+1)p^{n-1} - 2] \geq 0$$

(since $n > 1$).

Case (ii) follows on noting that

$$h(0) = h(1) = 0, h'(0) > 0, h'(1) = 0,$$

while

$$h''(p) \geq 0 \ \text{ if } \ p \geq [2/(n+1)]^{(n-1)^{-1}},$$

and

$$h''(p) \leq 0 \ \text{ if } \ p \leq [2/(n+1)]^{(n-1)^{-1}}.$$

(c) It follows from the maximum (minimum) principle, for the p.d.e. (2.1), that

$$u_M \geq u \geq u_m$$

where u_M, u_m are the maximum and minimum values of u prescribed on the boundary. In view of (2.8),

$$\Phi_{\theta\theta} = u^{n-1}$$

and thus

$$u_M^{n-1} \geq \Phi_{\theta\theta} \geq u_m^{n-1}\,. \tag{4.1}$$

On multiplying across by Φ_θ and integrating etc., the upper bound therein yields

$$\Phi_\theta^2 \leq 2u_M^{n-1}\Phi. \tag{4.2}$$

Thus (4.1), (4.2) yield

$$\Phi\Phi_{\theta\theta} \geq (u_m/u_M)^{n-1}\Phi_\theta^2/2.$$