

FACTORISATION AND WEAK AMENABILITY OF $\mathcal{A}(X)$

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ABSTRACT

We investigate weak amenability of the Banach algebra $\mathcal{A}(X)$ of approximable operators on a Banach space X and its relation to factorisation properties of operators in $\mathcal{A}(X)$. We show that if $\mathcal{A}(X)$ is weakly amenable, then either $\mathcal{A}(X)$ is self-induced (a nice factorisation property), or X is very special, combining some of the exotic properties of the spaces of Gowers and Maurey [9] and of Pisier [19]. In the class of self-induced Banach algebras we show that weak amenability is preserved under an equivalence of Morita type. Using this we extend some results of A. Blanco [1; 2] about weak amenability of $\mathcal{A}(X)$.

1. Introduction

Recall that a Banach algebra \mathfrak{A} is called *weakly amenable* if every bounded derivation $D: \mathfrak{A} \rightarrow \mathfrak{A}^*$ is inner, or equivalently if the first bounded Hochschild cohomology group $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$ vanishes. Recently weak amenability has been investigated for algebras of the type $\mathcal{A}(X)$ for an infinite-dimensional Banach space X . In [4] it is shown that $\mathcal{A}(X)$ is weakly amenable, when $X = \ell_p(Y)$ with Y reflexive and having the approximation property, or when $X = E \oplus C_p$, where E is a Banach space with the bounded approximation property and C_p denotes any of the universal spaces introduced by W.B. Johnson in [16]. In [1] Blanco introduced a technical property of X , so-called trace unbounded triples, that allows for taking averages of matrix-like representations of a given finite rank operator. Using this Blanco established weak amenability of $\mathcal{A}(X)$ for a wide range of Banach spaces X . In [2] Blanco studied hereditary properties of as well as necessary conditions for weak amenability for algebras $\mathcal{A}(X)$. In this work factorisation properties of $\mathcal{A}(X)$ play a crucial role.

In the present paper, we shall take an approach almost exclusively related to factorisation properties. We show that, if $\mathcal{A}(X)$ is weakly amenable, then either X is pathological (probably non-existing), or $\mathcal{A}(X)$ is ‘self-induced’, which we shall define below. Self-induced Banach algebras constitute the class of Banach algebras for which a Morita theory can be developed naturally. Hence our approach will be to transfer Hochschild cohomology from a few key examples by means of Morita equivalence, i.e. by means of factorisation properties. In this way we give a unified approach to some of Blanco’s results, with shorter and less technical proofs and in some cases improvements of the statements. We emphasise though that our aim is to view the question of weak amenability from a more general stand i.e. that of Morita

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equivalence. Blanco's results serve here as a source of test cases. In particular, the result [1] on weak amenability of $\mathcal{A}(T)$, with T the Tsirelson space, remains a challenge.

2. Preliminaries

For Banach spaces X and Y we consider the following spaces of operators:

$$\begin{aligned}\mathcal{F}(X, Y) &= \{\text{finite rank operators } X \rightarrow Y\}; \\ \mathcal{A}(X, Y) &= \{\text{approximable operators } X \rightarrow Y\}; \\ \mathcal{N}(X, Y) &= \{\text{nuclear operators } X \rightarrow Y\}; \\ \mathcal{I}(X, Y) &= \{\text{integral operators } X \rightarrow Y\}; \\ \mathcal{B}(X, Y) &= \{\text{bounded operators } X \rightarrow Y\}.\end{aligned}$$

We write $|\cdot|_{\mathcal{N}}$, $|\cdot|_{\mathcal{I}}$, and $\|\cdot\|$ for the nuclear, integral, and uniform norm on $\mathcal{N}(X, Y)$, $\mathcal{I}(X, Y)$, and $\mathcal{B}(X, Y)$, respectively. Recall that $\mathcal{A}(X, Y)$ is the $\|\cdot\|$ -closure of $\mathcal{F}(X, Y)$. As it is customary, we shall write $\text{Operators}(X)$ for $\text{Operators}(X, X)$. The identity operator on X is denoted by $\mathbf{1}_X$, or, if the context is clear, simply by $\mathbf{1}$.

For Banach spaces E and F their projective tensor product is denoted $E \widehat{\otimes} F$. Let X be a Banach space. The *tensor algebra* of X is $X \widehat{\otimes} X^*$, with multiplication given by

$$(x \otimes x^*)(\xi \otimes \xi^*) = x^*(\xi)x \otimes \xi^*, \quad x, \xi \in X, x^*, \xi^* \in X^*.$$

The trace $\text{tr}: X \widehat{\otimes} X^* \rightarrow \mathbb{C}$ and operator trace $\text{Tr}: X \widehat{\otimes} X^* \rightarrow \mathcal{N}(X)$ are given by

$$\text{tr}(x \otimes x^*) = x^*(x), \quad \text{Tr}(x \otimes x^*)(\xi) = x^*(\xi)x, \quad x, \xi \in X, x^*, \xi^* \in X^*.$$

Note that Tr maps onto $\mathcal{N}(X)$.

For Banach spaces X and Y we denote the statement ' X is isomorphic to Y ' by $X \cong Y$. For spaces in duality we shall use $\langle \cdot, \cdot \rangle$ to denote the corresponding bilinear form, in particular we shall write $\langle x, x^* \rangle = x^*(x)$ for $x \in X$ and $x^* \in X^*$. We note in particular the *trace duality*

$$\langle F, T \rangle = \text{tr}(F^*T), \quad F \in \mathcal{F}(X), T \in \mathcal{B}(X^*),$$

which isometrically identifies $(X \widehat{\otimes} X^*)^*$ and $\mathcal{A}(X)^*$ with $(\mathcal{B}(X^*), \|\cdot\|)$ and $(\mathcal{I}(X^*), |\cdot|_{\mathcal{I}})$ respectively.

For any normed space E the unit ball is denoted by E_1 .

Let $(X_n : n \in \mathbb{N})$ be a sequence of Banach spaces. We denote the ℓ_p -sums of this sequence by $(\bigoplus_1^\infty X_n)_p$ for $p = 0, 1 \leq p \leq \infty$. If $X_n = X$ for all $n \in \mathbb{N}$ we write simply $c_0(X)$, or $\ell_p(X)$, $1 \leq p \leq \infty$.

We shall frequently without further reference use the fact (see [8]) that, for a

Banach space X , we have

$$\begin{aligned} \mathcal{A}(X) \text{ has a bounded left approximate identity} &\iff \\ X \text{ has the bounded approximation property.} \end{aligned}$$

Hence, if X has the bounded approximation property, we may use Cohen's factorisation theorem [3, theorem 11.10] on the Banach algebra $\mathcal{A}(X)$.

The definitions of Banach (co)homological concepts are standard and can be found for example in [14] and [15]. We shall here point only to the following:

Definition 2.1. *Let \mathfrak{A} be a Banach algebra, let X be a right Banach \mathfrak{A} -module, and let Y be a left Banach \mathfrak{A} -module. We define*

$$X \widehat{\otimes}_{\mathfrak{A}} Y = X \widehat{\otimes} Y / N$$

where $\widehat{\otimes}$ is the projective tensor product and $N = \text{clspan}\{x.a \otimes y - x \otimes a.y \mid x \in X, y \in Y, a \in \mathfrak{A}\}$. Thus, $X \widehat{\otimes}_{\mathfrak{A}} Y$ is the universal object for linearising bounded, \mathfrak{A} -balanced, bilinear maps $X \times Y \rightarrow Z$: If Z is a Banach space and $\phi: X \times Y \rightarrow Z$ is a bounded bilinear map which is \mathfrak{A} -balanced, that is,

$$\phi(xa, y) = \phi(x, ay), \quad x \in X, y \in Y, a \in \mathfrak{A},$$

then there is a unique bounded linear map $\tilde{\phi}: X \widehat{\otimes}_{\mathfrak{A}} Y \rightarrow Z$ with

$$\tilde{\phi}(x \otimes_{\mathfrak{A}} y) = \phi(x, y), \quad x \in X, y \in Y.$$

Here $x \otimes_{\mathfrak{A}} y$ is the notation for the elementary tensor $x \otimes y + N$.

We start by recalling some facts about bounded derivations $D: \mathcal{A}(X) \rightarrow \mathcal{A}(X)^*$. As said above, we identify $\mathcal{A}(X)^*$ with $\mathcal{I}(X^*)$, and $(X \widehat{\otimes} X^*)^*$ with $\mathcal{B}(X^*)$ via trace duality. Consider the diagram

$$\begin{array}{ccc} X \widehat{\otimes} X^* & \xrightarrow{\text{Tr}} & \mathcal{A}(X) \\ \delta \downarrow & & \downarrow D \\ \mathcal{B}(X^*) & \xleftarrow{\text{Tr}^*} & \mathcal{I}(X^*) \end{array},$$

where $\delta = \text{Tr}^* \circ D \circ \text{Tr}$. Then δ is a continuous derivation from $X \widehat{\otimes} X^*$ into its dual space. Since $X \widehat{\otimes} X^*$ is biprojective [21] and, in particular, weakly amenable, the derivation δ is inner. Hence we have the following result, already noted by Blanco.

Proposition 2.2. *Let $D: \mathcal{A}(X) \rightarrow \mathcal{A}(X)^*$ be a bounded derivation. Corresponding*

to D there exists $T \in \mathcal{B}(X^*)$ such that

$$\langle F, D(G) \rangle = \text{tr}((FG - GF)^*T), \quad F, G \in \mathcal{F}(X).$$

Consequently D is inner if and only if $T \in \mathcal{I}(X^*) + \mathbb{C}\mathbf{1}_{X^*}$, that is, if and only if there are $\lambda \in \mathbb{C}$ and $K > 0$ such that

$$|\text{tr}(F^*T - \lambda F^*)| \leq K\|F\|, \quad F \in \mathcal{F}(X).$$

3. Factorisation properties and weak amenability

An important aspect of Morita theory is to provide tools to compare homological properties of Banach algebras using ‘good factorisation properties’. In this section we shall extract such factorisation properties in order to compare $\mathcal{H}^n(\mathfrak{A}, \mathfrak{A}^*)$ and $\mathcal{H}^n(\mathfrak{B}, \mathfrak{B}^*)$ for Banach algebras \mathfrak{A} and \mathfrak{B} . Our focus will be on $n = 1$ and Banach algebras of the type $\mathcal{A}(X)$.

First we make precise what is meant by ‘good factorisation’.

Definition 3.1. A Banach algebra \mathfrak{A} is self-induced if

$$\mathfrak{A} \cong \mathfrak{A} \underset{\mathfrak{A}}{\widehat{\otimes}} \mathfrak{A}.$$

The Banach algebra $\mathcal{A}(X)$ factors approximately through Y if

$$\mathcal{A}(X) \cong \mathcal{A}(Y, X) \underset{\mathcal{A}(Y)}{\widehat{\otimes}} \mathcal{A}(X, Y).$$

In both cases the isomorphisms are implemented by multiplication.

The usefulness of these factorisation properties is that one may, using the universal property of the tensor product, define linear maps in terms of balanced bilinear maps. For instance, each balanced bounded bilinear form $\phi: \mathfrak{A} \times \mathfrak{A} \rightarrow \mathbb{C}$ has the form $\phi(a, \alpha) = \langle a\alpha, f \rangle$ for a unique $f \in \mathfrak{A}^*$. A key example is the following.

Lemma 3.2. Suppose that \mathfrak{A} is self-induced. Let $D: \mathfrak{A} \rightarrow \mathfrak{A}^*$ be a bounded derivation, and let \mathfrak{B} be a Banach algebra that contains \mathfrak{A} as a closed 2-sided ideal. Then D may be extended to a bounded derivation $\tilde{D}: \mathfrak{B} \rightarrow \mathfrak{A}^*$.

PROOF. Let $T \in \mathfrak{B}$ and consider the bilinear map $\Phi_T: \mathfrak{A} \times \mathfrak{A} \rightarrow \mathbb{C}$ given by

$$\Phi_T(a, b) = \langle a, D(bT) \rangle - \langle Ta, D(b) \rangle, \quad a, b \in \mathfrak{A}.$$

Then Φ_T is balanced, i.e. $\Phi_T(ac, b) = \Phi_T(a, cb)$, so we may define

$$\langle ab, \tilde{D}(T) \rangle = \Phi_T(a, b), \quad a, b \in \mathfrak{A}.$$

We check that this defines a bounded derivation $\mathfrak{B} \rightarrow \mathfrak{A}^*$ extending D . ■

Remark 3.3. In the same way, D can be lifted to a derivation $\mathfrak{A} \rightarrow \mathfrak{B}^*$.

Example 3.4. Assume that the multiplication

$$\mathcal{A}(X) \widehat{\otimes} \mathcal{A}(X) \rightarrow \mathcal{A}(X)$$

is surjective. If X in addition has the approximation property, then $\mathcal{A}(X)$ is self-induced. To see this, suppose that

$$\sum_{n=1}^{\infty} A_n B_n = 0$$

with $A_n \rightarrow 0$ and $\sum \|B_n\| < \infty$, and let $\epsilon > 0$. Since X has the approximation property, we may choose $U \in \mathcal{A}(X)$ so that

$$\sup_n \|UA_n - A_n\| \leq \epsilon.$$

Then

$$\begin{aligned} \left\| \sum A_n \widehat{\otimes}_{\mathcal{A}(X)} B_n \right\| &\leq \left\| \sum UA_n \widehat{\otimes}_{\mathcal{A}(X)} B_n \right\| + \sum \|A_n - UA_n\| \|B_n\| \\ &\leq 0 + \epsilon \sum \|B_n\|. \end{aligned}$$

Thus, an important case occurs when X has the bounded approximation property, since the bounded left approximate identity in $\mathcal{A}(X)$ allows us to conclude, by means of Cohen's factorisation theorem, that multiplication is surjective.

The approximation property is not essential here as will become clear in the course of the paper. However, in the case of nuclear operators self-inducedness and the approximation property are one and the same thing.

Proposition 3.5. The Banach algebra $\mathcal{N}(X)$ is self-induced if and only if X has the approximation property.

PROOF. We define a bounded, balanced bilinear form on $\mathcal{N}(X)$ by

$$\phi(N, M) = \text{tr}(UV)$$

where $U, V \in X \widehat{\otimes} X^*$ with $\text{Tr}(U) = N$ and $\text{Tr}(V) = M$. This is well-defined since if $\text{Tr}(U) = 0$, then $U(X \widehat{\otimes} X^*) = (X \widehat{\otimes} X^*)U = \{0\}$. Suppose now that $\mathcal{N}(X)$ is self-induced. Then ϕ defines a bounded linear functional on $\mathcal{N}(X)$, which agrees with the standard trace on $\mathcal{F}(X)$. But then X must have the approximation property.

Conversely, if X has the approximation property, then $\mathcal{N}(X)$ is isometrically isomorphic to the tensor algebra $X \widehat{\otimes} X^*$ by [6, theorem VIII.3.4(v)]. To see that $X \widehat{\otimes} X^*$ is self-induced we shall use rank-1 projections. For brevity, let $\mathfrak{T} = X \widehat{\otimes} X^*$ and recall that the multiplication on \mathfrak{T} is given by $(x \otimes x^*)(y \otimes y^*) = \langle y, x^* \rangle x \otimes y^*$.

Let $\mu: \widehat{\mathfrak{A}} \otimes_{\widehat{\mathfrak{A}}} \mathfrak{A} \rightarrow \mathfrak{A}$ be the map induced by multiplication and let $\sum_{n=1}^{\infty} x_n \otimes x_n^*$ be an arbitrary element of $\widehat{\mathfrak{A}} \otimes_{\widehat{\mathfrak{A}}} \mathfrak{A}$. Choose $e \in X$ and $e^* \in X^*$ with $\langle e, e^* \rangle = 1$. Since $\mu((x \otimes e^*) \otimes (e \otimes x^*)) = x \otimes x^*$ for all $x \in X$ and $x^* \in X^*$ we have $\mu(\sum_{n=1}^{\infty} (x_n \otimes e^*) \otimes (e \otimes x_n^*)) = \sum_{n=1}^{\infty} x_n \otimes x_n^*$, proving that μ is surjective. Injectivity of μ is proved as in Example 3.4. ■

In order to investigate the relation between derivations and self-inducedness, we shall look at derivations of the following type.

Let \mathfrak{A} be a Banach algebra and let $\phi \in (\widehat{\mathfrak{A}} \otimes_{\mathfrak{A}} \mathfrak{A})^*$. For convenience we shall use the same symbol ϕ for the corresponding balanced bilinear functional. Then we check that

$$\langle a, D(b) \rangle := \phi(a, b) - \phi(b, a)$$

defines a bounded derivation $D: \mathfrak{A} \rightarrow \mathfrak{A}^*$.

How does this look in the setting of $\mathfrak{A} = \mathcal{A}(X)$? First we need to describe balanced bilinear functionals.

Lemma 3.6. *Let $\phi \in (\mathcal{A}(Y, X) \widehat{\otimes}_{\mathcal{A}(Y)} \mathcal{A}(X, Y))^*$. Then there is $T \in \mathcal{B}(X^*)$ such that*

$$\phi(F, G) = \text{tr}((FG)^*T) \quad F \in \mathcal{F}(Y, X), G \in \mathcal{F}(X, Y)$$

and

$$\sup\{|\text{tr}((FG)^*T)| \mid F \in \mathcal{F}(Y, X)_1, G \in \mathcal{F}(X, Y)_1\} < \infty.$$

If there is a constant $K > 0$ such that every $F \in \mathcal{F}(X)$ is a finite sum of products, $F = \sum_1^n U_i V_i$, with $U_i \in \mathcal{F}(Y, X)$, $V_i \in \mathcal{F}(X, Y)$ and $\sum_1^n \|U_i\| \|V_i\| \leq K \|F\|$, then $\mathcal{A}(X, Y) \widehat{\otimes}_{\mathcal{A}(Y)} \mathcal{A}(X, Y) \cong \mathcal{A}(X)$.

PROOF. To find $T \in \mathcal{B}(X^*)$ we apply ϕ to rank-1 operators. Choose $e \in X$ and $e^* \in X^*$ with $\langle e, e^* \rangle = 1$. Define $T \in \mathcal{B}(X^*)$ by

$$\langle x, T(x^*) \rangle = \phi(x \otimes e^*, e \otimes x^*), \quad x \in X, x^* \in X^*.$$

Then

$$\begin{aligned} \phi(x \otimes x^*, y \otimes y^*) &= \phi((x \otimes e^*)(e \otimes x^*), y \otimes y^*) \\ &= \phi(x \otimes e^*, (e \otimes x^*)(y \otimes y^*)) \\ &= \langle y, x^* \rangle \phi(x \otimes e^*, e \otimes y^*) \\ &= \langle y, x^* \rangle \langle x, T(y^*) \rangle \\ &= \text{tr}([(x \otimes x^*)(y \otimes y^*)]^*T). \end{aligned}$$

By linearity, ϕ is given by T as claimed. The norm estimate is just

$$\sup\{|\text{tr}((FG)^*T)| \mid F \in \mathcal{F}(Y, X)_1, G \in \mathcal{F}(X, Y)_1\} = \|\phi\| < \infty.$$

Now suppose that we have factorisation with $K > 0$ as described. Clearly then

the multiplication

$$\mu: \mathcal{A}(Y, X) \widehat{\otimes}_{\mathcal{A}(Y)} \mathcal{A}(X, Y) \rightarrow \mathcal{A}(X)$$

is surjective. Let $\phi \in (\mathcal{A}(Y, X) \widehat{\otimes}_{\mathcal{A}(Y)} \mathcal{A}(X, Y))^*$ with corresponding $T \in \mathcal{B}(X^*)$. The assumption ensures that

$$f: \sum U_i V_i \mapsto \text{tr}((\sum U_i V_i)^* T)$$

defines a bounded linear functional on $\mathcal{F}(X)$, and f extends to an element of $\mathcal{A}(X)^*$ so that the dual map

$$\mu^*: \mathcal{A}(X)^* \rightarrow (\mathcal{A}(Y, X) \widehat{\otimes}_{\mathcal{A}(Y)} \mathcal{A}(X, Y))^*$$

is also surjective, i.e. μ is an isomorphism. ■

In [2, section 4], Blanco discusses necessary conditions for weak amenability. He shows that, if $\mathcal{A}(X)$ is weakly amenable, then either X is indecomposable (i.e. X is not the direct sum of two infinite-dimensional Banach spaces) or the trace defines an unbounded bilinear map associated with a decomposition. These considerations are naturally further explored by means of self-inducedness. The Banach spaces of Pisier [19] for which $\mathcal{A}(X) = \mathcal{N}(X)$ are crucial in this. We now note some simple reformulations of this latter property. But first we need the following estimate of norms, which is essentially an elaboration of the proof of [6, theorem VIII.4.12]

Lemma 3.7. *Let X, Y, Z be Banach spaces, and let $T \in \mathcal{I}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$. If S is weakly compact, then $ST \in \mathcal{N}(X, Z)$ and*

$$|ST|_{\mathcal{N}} \leq \|S\| |T|_{\mathcal{I}}.$$

PROOF. That ST is nuclear is the statement of [6, theorem VIII.4.12.(i)]. We note from the proof of this that S being weakly compact, there is a reflexive space W and operators $A \in \mathcal{B}(Y, W)$ and $B \in \mathcal{B}(W, Z)$ such that $S = BA$ by [6, corollary VIII.4.9]. Furthermore, a close inspection of the proof shows that $\|S\| = \inf\{\|B\| \|A\|\}$, where the infimum is taken over such factorisations. Reasoning along with [6] we get

$$|AT|_{\mathcal{N}} = |AT|_{\mathcal{I}} \leq \|A\| |T|_{\mathcal{I}},$$

so that

$$|ST|_{\mathcal{N}} = |BAT|_{\mathcal{N}} \leq \|B\| |AT|_{\mathcal{N}} \leq \|B\| \|A\| |T|_{\mathcal{I}}.$$

Taking the infimum over $\|A\| \|B\|$ gives the required estimate. ■

Proposition 3.8. *Let X be a Banach space. Then $(i) \implies (ii) \iff (iii) \implies (iv)$, where:*

- (i) $\mathcal{A}(X) = \mathcal{N}(X)$;
- (ii) $\mathcal{A}(X) \subseteq \mathcal{I}(X)$;
- (iii) there is $C > 0$ such that

$$|\mathrm{tr}(AB)| \leq C\|A\|\|B\|, \quad A, B \in \mathcal{F}(X);$$

- (iv) the multiplication $\mathcal{A}(X) \widehat{\otimes} \mathcal{A}(X) \rightarrow \mathcal{A}(X)$ maps onto $\mathcal{N}(X)$.

In particular, if the multiplication $\mathcal{A}(X) \widehat{\otimes} \mathcal{A}(X) \rightarrow \mathcal{A}(X)$ is known to be surjective, then all four are equivalent.

PROOF. Since in general $\mathcal{N}(X) \subseteq \mathcal{I}(X)$, (i) \implies (ii) is obvious.

(ii) \implies (iii): An application of the closed graph theorem shows that the inclusion $(\mathcal{A}(X), \|\cdot\|) \hookrightarrow (\mathcal{I}(X), |\cdot|_{\mathcal{I}})$ is continuous, thus providing $C > 0$ so that $|A|_{\mathcal{I}} \leq C\|A\|$ and hence $|A^*|_{\mathcal{I}} \leq C\|A\|$ for all $A \in \mathcal{A}(X)$, since $|A|_{\mathcal{I}} = |A^*|_{\mathcal{I}}$ ([7, theorem 5.15]). This gives

$$|\mathrm{tr}(AB)| = |\mathrm{tr}(B^*A^*)| \leq |A^*|_{\mathcal{I}}\|B\| \leq C\|A\|\|B\|$$

for all $A, B \in \mathcal{F}(X)$.

(iii) \implies (ii): By one of the definitions [18, 6.4.1] of the integral norm, (iii) states that $|A^*|_{\mathcal{I}} \leq C\|A\|$ for each $A \in \mathcal{F}(X)$ from which (ii) follows, again using the fact that $|A|_{\mathcal{I}} = |A^*|_{\mathcal{I}}$.

(iii) \implies (iv): By Lemma 3.7, we have for $A, B \in \mathcal{F}(X)$ the inequality

$$|AB|_{\mathcal{N}} \leq \|A\|\|B\|_{\mathcal{I}} \leq C\|A\|\|B\|.$$

Hence, if $A = \sum A_n B_n$ with $A_n, B_n \in \mathcal{A}(X)$ and $\sum \|A_n\|\|B_n\| < \infty$ we have that

$$\sum |A_n B_n|_{\mathcal{N}} \leq C \sum \|A_n\|\|B_n\| < \infty,$$

so that the series is absolutely convergent in the nuclear norm, and thus $A \in \mathcal{N}(X)$. Since the multiplication $\mathcal{N}(X) \widehat{\otimes} \mathcal{N}(X) \rightarrow \mathcal{N}(X)$ is always surjective, we arrive at (iv). ■

We shall now show that the weak amenability of $\mathcal{A}(X)$ forces either $\mathcal{A}(X)$ to be self-induced or the underlying space X to be very peculiar, combining some of the pathological properties of the spaces of Pisier [19] and Gowers and Maurey [9].

Theorem 3.9. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is not self-induced. Then $\mathcal{A}(X)$ is weakly amenable if and only if both (a) and (b) hold, where*

- (a) $\mathcal{A}(X) = \mathcal{N}(X)$
- (b) The kernel, K , of the operator trace $\mathrm{Tr}: X \widehat{\otimes} X^* \rightarrow \mathcal{N}(X)$ is 1-dimensional.

Suppose that (a) and (b) hold. Then the following hold:

- (c) We have

$$\mathcal{B}(X) = \mathcal{I}(X) \oplus \mathbb{C}\mathbf{1}_X \text{ and } \mathcal{B}(X^*) = \mathcal{I}(X^*) \oplus \mathbb{C}\mathbf{1}_{X^*}.$$

(d) If $X = Y \oplus Z$ for some closed subspaces Y and Z of X , then either Y or Z is finite-dimensional (i.e. X is indecomposable). Similarly, X^* is indecomposable.

PROOF. Suppose that $\mathcal{A}(X)$ is weakly amenable. First we note that multiplication is surjective, since the map

$$\mathcal{A}(X) \widehat{\otimes} \mathcal{A}(X) \rightarrow \mathcal{A}(X): F \otimes G \mapsto FG - GF$$

has closed range, and $\mathcal{A}(X)$ has no bounded traces. Hence all four conditions of Proposition 3.8 are equivalent.

Now let $\phi \in (\mathcal{A}(X) \widehat{\otimes}_{\mathcal{A}(X)} \mathcal{A}(X))^*$, and let $T_\phi \in \mathcal{B}(X^*)$ be the corresponding continuous linear operator specified in Lemma 3.6. Since $\mathcal{A}(X)$ is weakly amenable, it follows from the paragraph preceding the lemma that there are an integral operator $T \in \mathcal{I}(X^*)$ and $\lambda \in \mathbb{C}$ so that

$$T_\phi = T + \lambda \mathbf{1}_{X^*}.$$

If $\mathcal{A}(X)$ is not self-induced, we may choose ϕ so that $\lambda \neq 0$. In this case we have

$$|\mathrm{tr}(AB)| = |\mathrm{tr}(AB)^*| \leq C \|A\| \|B\|, \quad A, B \in \mathcal{F}(X)$$

for appropriate $C > 0$. The statement (a) now follows from Proposition 3.8 (iii) \implies (i).

In general $\mathcal{N}(X)$ is weakly amenable if and only if $\dim K \leq 1$ ([10]). Noting that X does not have the approximation property (if it did have this property, $\mathcal{A}(X)$ would be self-induced, cf. Example 3.4) we arrive at (b). (Recall that $K = \{0\} \iff X$ has the approximation property.)

Setting $K = \mathbb{C}u$ with $u = \sum x_n \otimes x_n^*$ and $\mathrm{tr}u = \sum \langle x_n, x_n^* \rangle = 1$ the functional $\varphi: \mathcal{B}(X^*) \rightarrow \mathbb{C}$ given by

$$\varphi(T) = \sum \langle x_n, T(x_n^*) \rangle, \quad T \in \mathcal{B}(X^*),$$

is multiplicative with kernel $\mathcal{I}(X^*)$ (see the proof of [10, corollary 4]). Using the trace duality between $X \widehat{\otimes} X^*$ and $\mathcal{B}(X^*)$ we find that

$$(\mathcal{N}(X))^* = K^\perp = \ker \varphi \quad (*).$$

Hence, when (a) holds, we get $\ker \varphi = \mathcal{I}(X^*)$, thus proving the last equality in (c). The first equality follows by means of the multiplicative linear functional $T \mapsto \varphi(T^*)$ on $\mathcal{B}(X)$ and the fact that T is integral if and only if T^* is integral [7, theorem 5.15].

To prove (d) first note that, if $P \in \mathcal{B}(X^*)$ is a projection, then $\varphi(P)$ is either 1 or 0. A simple application of the closed graph theorem gives that the integral and uniform norms are equivalent on $\mathcal{I}(X^*)$. Accordingly there is a constant $C > 0$ so that

$$|\mathrm{tr}(A^*T) - \varphi(T)\mathrm{tr}A| \leq C \|A\|, \quad A \in \mathcal{F}(X), T \in \mathcal{B}(X^*). \quad (**)$$

If $P \in \mathcal{B}(X)$ is a projection with $\mathrm{rank} P = \infty$, we may choose for each $n \in \mathbb{N}$ a

projection $Q_n \in \mathcal{F}(X)$ with

$$\text{tr}Q_n = n, \|Q_n\| \leq n^{\frac{1}{2}}, \text{ and } PQ_n = Q_n,$$

cf [19, theorem 1.14]. In conjunction with (**) we must then have $\varphi(P^*) = 1$. Thus, if X were decomposable, we would have $2 = 1$. If $P' \in \mathcal{B}(X^*)$ is a projection of infinite rank, choose projections $Q'_n \in \mathcal{F}(X^*)$ as above. We may use local reflexivity to modify the Q'_n to obtain projections $Q_{*n} \in \mathcal{F}(X)$ such that

$$\text{tr}Q_{*n} = n, \|Q_{*n}\| \leq 2n^{\frac{1}{2}}, \text{ and } P'Q_{*n} = Q_{*n},$$

so that also X^* is indecomposable. ■

4. Weak amenability of self-induced Banach algebras

From now on we shall concentrate on self-induced Banach algebras. In order to compare the cohomology of such algebras we shall exploit the double complex of Waldhausen [5]. First we consider the lower left hand corner of a general double co-complex in the first quadrant:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & M^{03} & \longrightarrow & \\
 & & & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & M^{02} & \longrightarrow & M^{12} & \longrightarrow \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 (\mathcal{D}) & 0 & \longrightarrow & M^{01} & \longrightarrow & M^{11} & \longrightarrow & M^{21} & \longrightarrow \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & & 0 & \longrightarrow & M^{10} & \longrightarrow & M^{20} & \longrightarrow & M^{30} \\
 & & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & & 0 & & 0 & & 0 .
 \end{array}$$

The upper indices are meant as coordinates in the first quadrant. We shall assume that the diagram is commutative. On the horizontal axis we define the cohomology \mathcal{H}_h^n as the kernel modulo the image of the sequence

$$\longrightarrow M^{(n+1)0} \longrightarrow .$$

The cohomology on the vertical axis, \mathcal{H}_v^n , is defined analogously. We want to compare \mathcal{H}_h^1 and \mathcal{H}_v^1 . In essence this consists of a spectral sequence argument. However, we give a direct construction of a comparison map using an ad hoc diagram chase.

Lemma 4.1. *Consider the diagram (D). If there is vertical exactness at coordinates*

$(1,1)$, $(2,0)$, $(3,0)$, $(1,2)$, and $(2,1)$, then we may define a linear map

$$\mathfrak{D}: \mathcal{H}_v^1 \rightarrow \mathcal{H}_h^1$$

such that

- (1) If there is horizontal exactness at $(1,1)$ and $(0,2)$, then \mathfrak{D} is injective.
- (2) If there is horizontal exactness at $(1,2)$, $(2,1)$, and $(0,3)$, then \mathfrak{D} is surjective.

PROOF. First we describe a procedure to associate a cocycle at $(2,0)$ to each cocycle at $(0,2)$. We adopt the convention that indices on cochains indicate belonging, i.e. $m^{ij} \in M^{ij}$, $\mu^{ij} \in M^{ij}$ etc. Let m^{02} be a vertical cocycle. The numbers at the arrows show the progression in the diagram chase.

$$\begin{array}{ccccccc}
 & & 0 & \xrightarrow{2.} & 0 & & \\
 & & \uparrow 1. & & \uparrow 4. & & \\
 m^{02} & \xrightarrow{3.} & m^{12} & \xrightarrow{6.} & 0 & & \\
 & & \uparrow 5. & & \uparrow 8. & & \\
 & & m^{11} & \xrightarrow{7.} & m^{21} & \xrightarrow{10.} & 0 \\
 & & & & \uparrow 9. & & \uparrow 12. \\
 & & & & m^{20} & \xrightarrow{11.} & m^{30}.
 \end{array}$$

The existence of m^{11} and m^{20} is due to vertical exactness. Since the vertical map at $(3,0)$ is injective, $m^{30} = 0$ so that m^{20} is a horizontal cocycle.

We next show that the procedure $m^{02} \mapsto m^{20}$ implements a map into \mathcal{H}_h^1 . Suppose that, for the same m^{02} , we have made different choices m_*^{11}, m_*^{20} . Then

$$\begin{array}{ccc}
 & 0 & \\
 & \uparrow 1. & \\
 m^{11} - m_*^{11} & \xrightarrow{3.} & m^{21} - m_*^{21} \\
 & & \uparrow 5. \\
 \uparrow 2. & & \\
 \mu^{10} & \xrightarrow{4.} & \mu^{20} .
 \end{array}$$

Here μ^{10} exists by vertical exactness at $(1,1)$. Since we have vertical injectivity at $(2,0)$, we must have $\mu^{20} = m^{20} - m_*^{20}$, so m^{20} and m_*^{20} are cohomologous.

We next show that this map lifts to the desired map \mathfrak{D} . Hence assume that m^{02} cobounds vertically. Then the procedure gives

$$\begin{array}{ccccc}
 m^{02} & \xrightarrow{2.} & m^{12} & & \\
 1. \uparrow & & 4. \uparrow & & \\
 m^{01} & \xrightarrow{3.} & m^{11} & \xrightarrow{5.} & 0 \\
 & & & & 6. \uparrow \\
 & & & & 0,
 \end{array}$$

i.e. coboundaries go to coboundaries.

Now suppose that there is horizontal exactness at positions (1,1) and (0,2) and that the procedure $m^{02} \mapsto m^{20}$ has resulted in a coboundary. This is described in

$$\begin{array}{ccc}
 m_*^{11} & \xrightarrow{4.} & m^{21} \\
 3. \uparrow & & 1. \uparrow \\
 m_*^{10} & \xrightarrow{2.} & m^{20}.
 \end{array}$$

Elements from the procedure are unstarred m^{ij} 's. We next get

$$\begin{array}{ccccc}
 \mu^{02} & \xrightarrow{5.} & m^{12} & & \\
 4. \uparrow & & 1. \uparrow & & \\
 \mu^{01} & \xrightarrow{3.} & m^{11} - m_*^{11} & \xrightarrow{2.} & 0.
 \end{array}$$

The arrow 1. is valid because m_*^{11} is a vertical coboundary, and μ^{01} exists by horizontal exactness at (1,1). By horizontal injectivity at (0,2) we must have $\mu^{02} = m^{02}$, i.e. m^{02} is a coboundary.

Finally, suppose that there is horizontal exactness at (2,1) and (1,2) and let m^{20} be a horizontal cocycle. Then we get the diagram

$$\begin{array}{ccccccc}
m^{03} & \xrightarrow{12.} & & 0 & & & \\
11. \uparrow & & & 10. \uparrow & & & \\
m^{02} & \xrightarrow{9.} & m^{12} & \xrightarrow{8.} & 0 & & \\
& & 7. \uparrow & & 6. \uparrow & & \\
& & m^{11} & \xrightarrow{5.} & m^{21} & \xrightarrow{4.} & 0 \\
& & & & 3. \uparrow & & 2. \uparrow \\
& & & & m^{20} & \xrightarrow{1.} & 0.
\end{array}$$

Here m^{11} and m^{02} exist due to horizontal exactness at (2,1) and (1,2). Since we have horizontal injectivity at (0,3) we get $m^{03} = 0$, altogether showing that the cochain m^{02} which we found is a cocycle and that the given m^{20} is the result of the procedure applied to m^{02} , i.e. \mathfrak{D} is surjective. ■

Remark 4.2. *Note that in order to define the map \mathfrak{D} we did not use the assumptions of vertical exactness at position (1,2) in full. All that is needed is that the cocycle m^{12} corresponding to the cocycle m^{02} is actually a coboundary.*

We want to use Lemma 4.1 to establish instances of Morita invariance of Hochschild cohomology, essentially by refining the arguments in [11]. The definition of Morita equivalence is usually given in terms of functors between categories of modules. We only need a slightly weaker concept (which in the case of Banach algebras with bounded one-sided approximate identities coincides with the full version of Morita equivalence).

Definition 4.3. *Let \mathfrak{A} and \mathfrak{B} be self-induced Banach algebras. Then \mathfrak{A} and \mathfrak{B} are M-equivalent, in symbols $\mathfrak{A} \underset{M}{\sim} \mathfrak{B}$, if there are bimodules ${}_{\mathfrak{A}}P_{\mathfrak{B}}$ and ${}_{\mathfrak{B}}Q_{\mathfrak{A}}$ and balanced pairings*

$$[\cdot]: {}_{\mathfrak{A}}P_{\mathfrak{B}} \widehat{\otimes}_{\mathfrak{B}} Q_{\mathfrak{A}} \rightarrow \mathfrak{A} \text{ and } [\cdot]: {}_{\mathfrak{B}}Q_{\mathfrak{A}} \widehat{\otimes}_{\mathfrak{A}} P_{\mathfrak{B}} \rightarrow \mathfrak{B},$$

which are bimodule isomorphisms satisfying

$$[p \widehat{\otimes}_{\mathfrak{B}} q].p' = p.[q \widehat{\otimes}_{\mathfrak{A}} p'] \text{ and } [q \widehat{\otimes}_{\mathfrak{A}} p].q' = q.[p \widehat{\otimes}_{\mathfrak{B}} q'], \quad p, p' \in {}_{\mathfrak{A}}P_{\mathfrak{B}}, \quad q, q' \in {}_{\mathfrak{B}}Q_{\mathfrak{A}}.$$

The double complex to which we shall apply Lemma 4.1 is the dual complex of the Waldhausen double complex [5]. We shall write P and Q for short, instead of ${}_{\mathfrak{A}}P_{\mathfrak{B}}$ and ${}_{\mathfrak{B}}Q_{\mathfrak{A}}$. The lower left hand corner of the Waldhausen double complex is

$$\begin{array}{ccccccc}
 & & 0 & \longleftarrow & \mathfrak{B} \widehat{\otimes} \mathfrak{B} \widehat{\otimes} \mathfrak{B} & \longleftarrow & \\
 & & & & \downarrow & & \\
 & & 0 & \longleftarrow & \mathfrak{B} \widehat{\otimes} \mathfrak{B} & \longleftarrow & P \widehat{\otimes} \mathfrak{B} \widehat{\otimes} Q \longleftarrow \\
 & & & & \downarrow & & \downarrow \\
 (\mathcal{W}) & 0 & \longleftarrow & \mathfrak{B} & \longleftarrow & P \widehat{\otimes} Q & \longleftarrow & P \widehat{\otimes} Q \widehat{\otimes} \mathfrak{A} \longleftarrow \\
 & & & \downarrow & & \downarrow & & \downarrow \\
 & & & 0 & \longleftarrow & \mathfrak{A} & \longleftarrow & \mathfrak{A} \widehat{\otimes} \mathfrak{A} \longleftarrow & \mathfrak{A} \widehat{\otimes} \mathfrak{A} \widehat{\otimes} \mathfrak{A} \longleftarrow \\
 & & & & & \downarrow & & \downarrow & \downarrow \\
 & & & & & 0 & & 0 & 0
 \end{array}$$

The complexes on the axes are the usual Hochschild complexes. The n 'th column is the complex ${}_{\mathfrak{A}}P \widehat{\otimes} C_{\star}(\mathfrak{B}, Q \widehat{\otimes} \mathfrak{A}^{\widehat{\otimes}(n-1)})$, where $C_{\star}(\mathfrak{B}, Q \widehat{\otimes} \mathfrak{A}^{\widehat{\otimes}(n-1)})$ is the normalised bar resolution of the left \mathfrak{B} -module $Q \widehat{\otimes} \mathfrak{A}^{\widehat{\otimes}(n-1)}$. Similarly, the m 'th row is the complex ${}_{\mathfrak{B}}Q \widehat{\otimes} C_{\star}(\mathfrak{A}, P \widehat{\otimes} \mathfrak{B}^{\widehat{\otimes}(m-1)})$. For details, see [11]. Concerning exactness we have the following:

Lemma 4.4. *Let (\mathcal{W}^*) be the dual double co-complex of (\mathcal{W}) and suppose that $\mathfrak{A} \underset{M}{\sim} \mathfrak{B}$. Then there is vertical exactness at places (n, i) for $i = 0, 1$ and $n \geq 1$ and horizontal exactness at places (i, n) for $i = 0, 1$ and $n \geq 1$ in (\mathcal{W}^*) . If \mathfrak{A} has a bounded left approximate identity, then columns of (\mathcal{W}^*) are acyclic except possibly on the vertical edge.*

PROOF. With minor modifications the proofs of [11, lemma 3.1] and [12, theorem 4.6] can be adapted to the present situation, so the reader is referred to these references. ■

Applying this to algebras of approximable operators we get:

Theorem 4.5. *Suppose that $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are self-induced and that $\mathcal{A}(X) \underset{M}{\sim} \mathcal{A}(Y)$. If X has the bounded approximation property, then there is an injection*

$$\mathcal{H}^1(\mathcal{A}(Y), \mathcal{A}(Y)^*) \rightarrow \mathcal{H}^1(\mathcal{A}(X), \mathcal{A}(X)^*).$$

In particular, if X has the bounded approximation property and $\mathcal{A}(X)$ is weakly amenable, then $\mathcal{A}(Y)$ is weakly amenable.

PROOF. According to Lemma 4.4, the double co-complex (\mathcal{W}^*) corresponding to

$\mathfrak{A} = \mathcal{A}(X)$, $\mathfrak{B} = \mathcal{A}(Y)$ satisfies the hypothesis of Lemma 4.1 to conclude injectivity. ■

Remark 4.6. From [12, corollary 4.9] it follows that if X and Y both have the bounded approximation property, then $\mathcal{A}(X) \underset{M}{\sim} \mathcal{A}(Y)$ implies that

$$\mathcal{H}^n(\mathcal{A}(Y), \mathcal{A}(Y)^*) \cong \mathcal{H}^n(\mathcal{A}(X), \mathcal{A}(X)^*) \text{ for all } n \in \mathbb{N}.$$

5. Some illustrative applications

As mentioned in the introduction, our approach will be to establish weak amenability for some key examples and then conclude weak amenability for other Banach algebras by means of the relation $\underset{M}{\sim}$. Towards this end we start with

Theorem 5.1. *Let X be an arbitrary Banach space. Then for any $1 \leq p < \infty$ we have*

$$\mathcal{H}^1(\mathcal{B}(\ell_p(X)), \mathcal{A}(\ell_p(X)^*)) = \{0\}.$$

PROOF. First note that, in general, if I is a closed ideal in a Banach algebra \mathfrak{A} with $\text{cl}(I^2) = I$, then any bounded derivation $D: \mathfrak{A} \rightarrow I^*$ is determined by its restriction to I , since for $i, j \in I$ and $a \in \mathfrak{A}$ we have $\langle ij, D(a) \rangle = \langle i, D(ja) \rangle - \langle ai, D(j) \rangle$. Appealing to Proposition 2.2 we conclude that an operator $T \in \mathcal{B}(\ell_p(X)^*)$ corresponds to a bounded derivation $D: \mathcal{B}(\ell_p(X)) \rightarrow \mathcal{A}(\ell_p(X))^*$, such that

$$\langle A, D(S) \rangle = \text{tr}((SA - AS)^*T), \quad A \in \mathcal{F}(\ell_p(X)), S \in \mathcal{B}(\ell_p(X)).$$

We want to find $\lambda \in \mathbb{C}$ such that $T - \lambda 1$ is integral. We start by setting up some notation. We view $\mathcal{B}(\ell_p(X))$ as consisting of infinite matrices with each entry an operator from $\mathcal{B}(X)$. For $n \in \mathbb{N}$ we let V_n and H_n denote the left and right shifts by n places. We let $\mathfrak{M} = \{\mathbf{W} \in \mathcal{F}(\ell_p(X)) \mid \exists n \in \mathbb{N}: \mathbf{W}H_n = V_n\mathbf{W} = 0\}$, i.e. \mathfrak{M} is the dense subalgebra of $\mathcal{A}(\ell_p(X))$ consisting of matrices of the form

$$\mathbf{W} = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix},$$

where W denotes a finite square matrix with entries from $\mathcal{F}(X)$ and the 0s represent infinite 0-matrices of the appropriate size. Let $\mathbf{W} \in \mathfrak{M}$, and choose a $(d \times d)$ -matrix W to represent \mathbf{W} . For $N \in \mathbb{N}$ we write $\Delta_{N,d}(W)$ for the $(Nd \times Nd)$ -matrix obtained by repeating the matrix W along the diagonal N times, i.e.

$$\Delta_{N,d}(W) = \begin{pmatrix} W & & \\ & \ddots & \\ & & W \end{pmatrix}.$$

Note that

$$\left\| \begin{pmatrix} \Delta_{N,d}(W) & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|\mathbf{W}\|.$$

Note also that a given $\mathbf{W} \in \mathfrak{M}$ can be represented by different matrices W , since we may add 0-rows and 0-columns. In order to prove that $T - \lambda \mathbf{1}$ is integral, it suffices to prove that there is a constant $C > 0$ such that

$$|\operatorname{tr}(\mathbf{W}^*(T - \lambda \mathbf{1}))| \leq C \|\mathbf{W}\|, \quad \mathbf{W} \in \mathfrak{M}.$$

Let $\mathbf{W} \in \mathfrak{M}$. Then

$$|\operatorname{tr}((\mathbf{W} - H_n \mathbf{W} V_n)^* T)| = |\langle H_n \mathbf{W}, D(V_n) \rangle| \leq \|\mathbf{W}\| \|D\|.$$

It follows that the sequence $(\operatorname{tr}((H_n \mathbf{W} V_n)^* T))$ is bounded. Let LIM be a Banach limit, and define a linear functional (possibly unbounded) $f: \mathfrak{M} \rightarrow \mathbb{C}$ by

$$f(\mathbf{W}) = \operatorname{LIM}(\operatorname{tr}((H_n \mathbf{W} V_n)^* T)), \quad \mathbf{W} \in \mathfrak{M}.$$

We now prove that $f(\mathbf{U}\mathbf{W}) = f(\mathbf{W}\mathbf{U})$, $\mathbf{U}, \mathbf{W} \in \mathfrak{M}$. By including some 0-entries, if necessary, we may suppose that \mathbf{U} and \mathbf{W} are represented by matrices, U and V respectively, of equal size, say $d \times d$. For $n \in \mathbb{N}$, let the 0-matrix of size $n \times n$ be denoted 0_n and consider the operators in \mathfrak{M} given by the matrices

$$R_n(W) = \begin{pmatrix} 0_n & 0 & 0 & 0 \\ 0 & 0_{Nd} & \Delta_{N,d}(W) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_n(U) = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & 0_{Nd} & 0 \\ 0 & \Delta_{N,d}(U) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\|R_n(W)\| = \|\mathbf{W}\|$ and $\|S_n(U)\| = \|\mathbf{U}\|$. From the identity

$$\langle R_n(W), D(S_n(U)) \rangle = \operatorname{tr} \left(\begin{pmatrix} 0_n & 0 & 0 & 0 \\ 0 & \Delta_{N,d}(WU) & 0 & 0 \\ 0 & 0 & -\Delta_{N,d}(UW) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* T \right)$$

and from translation invariance of the Banach limit we get

$$\begin{aligned} |f(\mathbf{W}\mathbf{U} - \mathbf{U}\mathbf{W})| &= \frac{1}{N} |\operatorname{LIM} \langle R_n(W), D(S_n(U)) \rangle| \\ &\leq \frac{1}{N} \|D\| \|\mathbf{W}\| \|\mathbf{U}\|. \end{aligned}$$

Since N is arbitrary, we arrive at $f(\mathbf{W}\mathbf{U}) = f(\mathbf{U}\mathbf{W})$. It follows that there exists $\lambda \in \mathbb{C}$ such that $f(\mathbf{W}) = \lambda \operatorname{tr}(\mathbf{W})$. This is the λ we are looking for; indeed we have

$$\operatorname{tr} \left(\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}^* (T - \lambda \mathbf{1}) \right) =$$

$$\begin{aligned} & \operatorname{tr} \left(\left(\begin{pmatrix} W & 0 & 0 & 0 \\ 0 & 0_n & 0 & 0 \\ 0 & 0 & -W & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* \right) (T - \lambda \mathbf{1}) \right) + \operatorname{tr} \left(\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0_n & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* \right) (T - \lambda \mathbf{1}) \right) = \\ & \operatorname{tr} \left(\left(\begin{pmatrix} W & 0 & 0 & 0 \\ 0 & 0_n & 0 & 0 \\ 0 & 0 & -W & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* \right) T \right) + \operatorname{tr} \left(\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0_n & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* \right) T \right) - \lambda \operatorname{tr}(W). \end{aligned}$$

Since

$$\begin{pmatrix} W & 0 & 0 & 0 \\ 0 & 0_n & 0 & 0 \\ 0 & 0 & -W & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix} \right]$$

for an appropriate coordinate projection P , we get by taking LIM that

$$|\operatorname{tr}(\mathbf{W}^*(T - \lambda \mathbf{1}))| = \left| \operatorname{tr} \left(\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}^* (T - \lambda \mathbf{1}) \right) \right| \leq \|D\| \|\mathbf{W}\|,$$

which is what we wanted. ■

Corollary 5.2. *Let X be a Banach space. Then*

$$\mathcal{A}(\ell_p(X)) \text{ is weakly amenable} \iff \mathcal{A}(\ell_p(X)) \text{ is self-induced.}$$

PROOF. The Banach space $\ell_p(X)$ is decomposable, so if $\mathcal{A}(\ell_p(X))$ is weakly amenable, then it is self-induced by Theorem 3.9.

Conversely, if $\mathcal{A}(\ell_p(X))$ is self-induced, then every derivation

$$D: \mathcal{A}(\ell_p(X)) \rightarrow \mathcal{A}(\ell_p(X))^*$$

can be extended to a derivation $\tilde{D}: \mathcal{B}(\ell_p(X)) \rightarrow \mathcal{A}(\ell_p(X))^*$. By Theorem 5.1 \tilde{D} is inner, and this shows that $\mathcal{A}(\ell_p(X))$ is weakly amenable. ■

Remark 5.3. *In the proof of Theorem 5.1 the only properties of $\ell_p(X)$ we used were (i): there is a constant $C > 0$ with $\|\Delta_{N,d}(W)\| \leq C\|W\|$ for all W , (ii): \mathfrak{M} is dense in $\mathcal{A}(\ell_p(X))$. The latter is equivalent to $\lim_n H_n A V_n = 0$ for all $A \in \mathcal{A}(\ell_p(X))$. Hence there are many other Banach spaces of sequences from X for which the proof works, notably $c_0(X)$. However, in the present paper we shall only make use of the spaces $\ell_p(X)$, $1 \leq p < \infty$.*

The next result concerning weak amenability of $\mathcal{A}(L_p(\mu, X))$ strengthens [1, theorem 4.1] by weakening the hypothesis ‘ X^* has the bounded approximation

property' to 'X has the bounded approximation property'. The data of the space $L_p(\mu, X)$ are a measure space (Ω, Σ, μ) and a sequence (Ω_n) of pairwise disjoint sets in Σ with $0 < \mu(\Omega_n) < \infty$ (to avoid simply dealing with the case $X \oplus \cdots \oplus X$). Without loss of generality we may further suppose that μ is a probability measure, since every compact set in $L_p(\mu, X)$ has σ -finite support.

Theorem 5.4. *Let X be a Banach space with the bounded approximation property. Then $\mathcal{A}(L_p(\mu, X))$ is weakly amenable.*

PROOF. Since X has the bounded approximation property, the same is true for the spaces $\ell_p(X)$ and $L_p(\mu, X)$. In particular $\mathcal{A}(\ell_p(X))$ is self-induced, and therefore weakly amenable, by Corollary 5.2. Thus we may prove the theorem by showing $\mathcal{A}(L_p(\mu, X)) \underset{M}{\sim} \mathcal{A}(\ell_p(X))$.

First we give some notation and well-known facts. A *mesh* $\mathfrak{m} = \{E_n \mid n \in \mathbb{N}\}$ is a partition $\Omega = \bigcup_{n=1}^{\infty} E_n$ into pairwise disjoint measurable sets. A mesh \mathfrak{m} defines a norm-1 projection $P_{\mathfrak{m}} \in \mathcal{B}(L_p(\mu, X))$ by the rule

$$P_{\mathfrak{m}}(f) = \sum_{E \in \mathfrak{m}, \mu(E) \neq 0} \left(\frac{1}{\mu(E)} \int_E f d\mu \right) \xi_E.$$

The set $\{\text{meshes}\}$ is ordered by refinement, and then $\lim_{\mathfrak{m} \rightarrow \infty} P_{\mathfrak{m}} = \mathbf{1}_{\mathcal{A}(L_p(\mu, X))}$ uniformly on compacta. For a mesh-projection $P_{\mathfrak{m}}$ the range is isometrically isomorphic to $\ell_p^{\kappa}(X)$, where κ is the cardinality of $\{E \in \mathfrak{m} \mid \mu(E) > 0\}$. In particular $L_p(\mu, X)$ has a complemented subspace isometric to $\ell_p(X)$, so that

$$L_p(\mu, X) \cong L_p(\mu, X) \oplus \ell_p(X).$$

Since $\mathcal{A}(\ell_p(X))$ has a bounded left approximate identity, it follows that $\mathcal{A}(\ell_p(X))$ factors approximately through $\mathcal{A}(L_p(\mu, X))$. To show that $\mathcal{A}(L_p(\mu, X))$ factors approximately through $\mathcal{A}(\ell_p(X))$ first note that

$$\{P_{\mathfrak{m}}H \mid H \in \mathcal{A}(L_p(\mu, X))_1, \mathfrak{m} \text{ a mesh}\}$$

is dense in $\mathcal{A}(L_p(\mu, X))_1$. It follows that each $A \in \mathcal{A}(L_p(\mu, X))_1$ is the sum of a series

$$A = \sum_1^{\infty} 2^{-n} P_n H_n,$$

where the P_n s are mesh-projections and $\{H_n\} \subseteq \mathcal{A}(L_p(\mu, X))_1$. Identifying the ranges of mesh-projections with the appropriate $\ell_p(X)$ -spaces we get

$$A = \sum_1^{\infty} 2^{-n} \tilde{P}_n \tilde{H}_n,$$

with $\tilde{P}_n \in \mathcal{B}(\ell_p(X), L_p(\mu, X))_1$ and $\tilde{H}_n \in \mathcal{A}(L_p(\mu, X), \ell_p(X))_1$. Since $\mathcal{A}(L_p(\mu, X))$ has a bounded left approximate identity, we conclude that $\mathcal{A}(L_p(\mu, X))$ factors approximately through $\mathcal{A}(\ell_p(X))$. Altogether $\mathcal{A}(L_p(\mu, X)) \underset{M}{\sim} \mathcal{A}(\ell_p(X))$. ■

In order to facilitate the use of factorisation properties, the generalisation given by Blanco in [1] of Johnson's C_p -spaces is very useful. We quote it here:

Definition 5.5. A Banach space J is called a Johnson space if it has the form $(\oplus_1^\infty G_n)_p$, $p = 0, 1 \leq p < \infty$, where $(G_n)_n$ is a sequence of finite-dimensional Banach spaces such that for each $i \in \mathbb{N}$ the set $\{n \in \mathbb{N} \mid G_n \cong G_i \text{ isometrically}\}$ is infinite.

Let $J = (\oplus G_n)_p$ be a Johnson space. A Banach space X is called a J -space if there is $\lambda \geq 1$ such that for every finite-dimensional subspace E of X , there is a subspace G of X containing E such that the Banach-Mazur distance $d(G, G_i) \leq \lambda$ for some $i \in \mathbb{N}$.

The usefulness of these notions hinges on the following:

Proposition 5.6. Let $J = (\oplus_1^\infty G_n)_p$ be a Johnson space, and let X be a J -space. Then $\mathcal{A}(J)$ is weakly amenable, and $\mathcal{A}(X)$ factors approximately through $\mathcal{A}(J)$.

PROOF. It is an immediate consequence of Corollary 5.2 that $\mathcal{A}(J)$ is weakly amenable.

Let $A \in \mathcal{F}(X)$ and choose $\text{range}(A) \subseteq G$ and corresponding G_i in accordance with the definition of X being a J -space. This gives a factorisation

$$\begin{array}{ccc} G_i & \xrightarrow{V} & J \\ \uparrow V & & \downarrow U \\ X & \xrightarrow{A} & X \end{array}$$

with $\|U\| \|V\| \leq \lambda \|A\|$. The claim now follows from Lemma 3.6. ■

The next result is [1, proposition 3.3].

Proposition 5.7. (Blanco). Let $J = (\oplus_1^\infty G_n)_p$ be a Johnson space, and let X be a J -space. Then $\mathcal{A}(X \oplus J)$ is weakly amenable.

PROOF. Since J has the bounded approximation property we obviously have that $\mathcal{A}(J)$ factors through $\mathcal{A}(X \oplus J)$ and from Proposition 5.6 it follows that $X \oplus J$ factors approximately through J , i.e. $\mathcal{A}(X \oplus J) \underset{M}{\sim} \mathcal{A}(J)$. Since $\mathcal{A}(J)$ is weakly amenable by Proposition 5.6, the proof is concluded by an appeal to Theorem 4.5. ■

As a final illustration, we shall look at the James spaces \mathfrak{J}_p . Blanco [1] shows that $\mathcal{A}(\mathfrak{J}_p)$ is weakly amenable by showing that there is a Johnson space J_p such that \mathfrak{J}_p is a J_p -space and $\mathfrak{J}_p \cong \mathfrak{J}_p \oplus J_p$, whence the result follows from Proposition 5.7. Using the relation $\underset{M}{\sim}$ makes it possible to extend this result to vector-valued James spaces. We start by briefly recalling basic properties of the spaces \mathfrak{J}_p following the notation of [1]. Let $1 < p < \infty$ and let $(\alpha_n) \in \mathbb{C}^{\mathbb{N}}$. Define $\|\cdot\|_{\mathfrak{J}_p}$ by

$$\|(\alpha_n)\|_{\mathfrak{J}_p} = \sup\left\{\left(\sum_{n=1}^{m-1} |\alpha_{i_n} - \alpha_{i_{n+1}}|^p\right)^{\frac{1}{p}} \mid i_1 < \dots < i_m, m \geq 2\right\}.$$

Then

$$\mathfrak{J}_p = \{(\alpha_n) \in \mathbb{C}^{\mathbb{N}} \mid \|(\alpha_n)\|_{\mathfrak{J}_p} < \infty, \lim_n \alpha_n = 0\}.$$

With this norm \mathfrak{J}_p is a Banach space. The sequences $e_n = (\delta_{kn})_k$ form a normalised, 1-unconditional basis, \mathbf{e} , for \mathfrak{J}_p , the *canonical basis*. We now define vector-valued James spaces. But we start with a general setting that is a special case of the spaces described in [17].

Definition 5.8. *Let E be a Banach space with a normalised 1-unconditional basis $\mathbf{b} = \{b_1, \dots\}$, and let X be any Banach space. Then we define $E \underset{\mathbf{b}}{\widehat{\otimes}} X$ as*

$$E \underset{\mathbf{b}}{\widehat{\otimes}} X = \{(x_n) \in X^{\mathbb{N}} \mid \sum_1^{\infty} \|x_n\| b_n \in E\},$$

with the norm $\|(x_n)\|_{\mathbf{b}} = \|\sum_1^{\infty} \|x_n\| b_n\|$. The X -valued James p -space is $\mathfrak{J}_p(X) = \mathfrak{J}_p \underset{\mathbf{e}}{\widehat{\otimes}} X$, where \mathbf{e} is the canonical basis. In this case we use the notation $\|\cdot\|_{\mathfrak{J}_p(X)} = \|\cdot\|_{\mathbf{e}}$.

It is straightforward to verify that $(E \underset{\mathbf{b}}{\widehat{\otimes}} X, \|\cdot\|_{\mathbf{b}})$ is a Banach space, which may be viewed as a completion of the algebraic tensor product $E \otimes X$ if we identify $(\sum_1^{\infty} \alpha_n b_n) \otimes x \in E \otimes X$ with $(\alpha_n x) \in E \underset{\mathbf{b}}{\widehat{\otimes}} X$. Note that in this picture $\|\cdot\|_{\mathbf{b}}$ is a cross-norm: $\|(\sum_1^{\infty} \alpha_n b_n) \otimes x\|_{\mathbf{b}} = \|\sum_1^{\infty} \alpha_n b_n\| \|x\|$. In accordance with this picture consider $S \in \mathcal{B}(E)$ and $T \in \mathcal{B}(X)$. If the linear map $S \otimes T: E \otimes X \rightarrow E \otimes X$ extends to a bounded operator $E \underset{\mathbf{b}}{\widehat{\otimes}} X \rightarrow E \underset{\mathbf{b}}{\widehat{\otimes}} X$, the latter will be denoted $S \underset{\mathbf{b}}{\widehat{\otimes}} T$.

The above result by Blanco is the case $X = \mathbb{C}$ of the following theorem.

Theorem 5.9. *Let X be a Banach space. If X has the bounded approximation property, then the Banach algebra $\mathcal{A}(\mathfrak{J}_p(X))$ is weakly amenable for every $1 < p < \infty$.*

PROOF. For each $n \in \mathbb{N}$, define a closed subspace of $\mathfrak{J}_p(X)$ by $\mathfrak{J}_{p,n}(X) = \{x_1 e_1 + \dots + x_n e_n \mid x_1, \dots, x_n \in X\}$. Let (G_k) be a sequence of Banach spaces obtained by repeating each $\mathfrak{J}_{p,n}(X)$ infinitely. Define a Banach space by $J_p(X) =$

$(\oplus G_k)_p$. We prove that $\mathcal{A}(\mathfrak{J}_p(X)) \underset{M}{\sim} \mathcal{A}(J_p(X))$. Invoking Theorem 4.5 and Corollary 5.2, the claim follows, since both spaces $\mathfrak{J}_p(X)$ and $J_p(X)$ have the bounded approximation property and $\ell_p(J_p(X)) \cong J_p(X)$. Let $P_n: \mathfrak{J}_p(X) \rightarrow \mathfrak{J}_p(X)$ be the canonical projection onto $\mathfrak{J}_{p,n}(X)$. Then $P_n A \rightarrow A$ for all $A \in \mathcal{A}(\mathfrak{J}_p(X))$. Since each P_n has an obvious factorisation

$$P_n = \iota_n Q_n, \quad Q_n \in \mathcal{A}(\mathfrak{J}_p(X), J_p(X)), \iota_n \in \mathcal{A}(J_p(X), \mathfrak{J}_p(X)), \|Q_n\| = \|\iota_n\| = 1,$$

it follows that each $A \in \mathcal{A}(\mathfrak{J}_p(X))$ has a decomposition

$$A = \sum_1^\infty T_n S_n, \quad S_n \in \mathcal{B}(\mathfrak{J}_p(X), J_p(X)), T_n \in \mathcal{B}(J_p(X), \mathfrak{J}_p(X)),$$

with $\sum \|S_n\| \|T_n\| \leq 2\|A\|$. Since $\mathfrak{J}_p(X)$ has the bounded approximation property, we may write $A = A_1 A_2 A_3$, $A_1, A_2, A_3 \in \mathcal{A}(\mathfrak{J}_p(X))$. It follows that $\mathcal{A}(\mathfrak{J}_p(X))$ factors approximately through $\mathcal{A}(J_p(X))$.

To prove that $\mathcal{A}(J_p(X))$ factors approximately through $\mathcal{A}(\mathfrak{J}_p(X))$ we just note that Blanco's decomposition of \mathfrak{J}_p works equally well for $\mathfrak{J}_p(X)$ with the same proof, so that we have $\mathfrak{J}_p(X) \cong \mathfrak{J}_p(X) \oplus J_p(X)$. Since $J_p(X)$ has the bounded approximation property, we may factor as desired. ■

6. Conclusion

As demonstrated, many questions of the weak amenability of Banach algebras (notably of the type $\mathcal{A}(X)$) can be approached using factorisation of Morita equivalence type. This has been illustrated by giving a framework behind much of the reasoning in Blanco's papers [1] and [2]. We would like to raise some questions related to this.

Question 6.1. *In Proposition 4.2 of [2] Blanco shows that if P is a Banach space such that P and P^* both have cotype 2, then $\mathcal{A}(\ell_2(P))$ is weakly amenable. Part of his argument consists in using a factorisation theorem by Pisier ([19, theorem 4.1]), which combined with Lemma 3.6 shows that $\mathcal{A}(\ell_2(P))$ is self-induced. However, P can be chosen such that $\mathcal{A}(P)$ is not weakly amenable.*

In the same paper, Blanco constructs a reflexive space E with an unconditional basis such that $\mathcal{A}(E)$ is not weakly amenable. Since E has the bounded approximation property, $\mathcal{A}(\ell_p(E))$ is self-induced and hence weakly amenable for $1 \leq p < \infty$.

The spaces $\ell_p(X)$ have the form $\ell_p \widehat{\otimes}_e X$ and are tight tensor products in the sense of [13]. Thus we may view $\mathcal{A}(\ell_p(X))$ as a tensor product $\mathcal{A}(\ell_p) \widetilde{\otimes} \mathcal{A}(X)$. The preceding paragraphs can be phrased as a stabilising effect of the functor $\mathcal{A}(\ell_p) \widetilde{\otimes} -$, which is analogous to stabilising in the theory of C^ -algebras. This leads to the question:*

Is $\mathcal{A}(\ell_2(X))$ self-induced for all X ? Is $\mathcal{A}(\ell_p(X))$ self-induced for all X and all $1 \leq p < \infty$? A test case would be the space constructed in [20] for which multiplication is not surjective.

Question 6.2. *Our reasoning has relied on matrix-structures with a certain uni-*

formity loosely speaking enabling us to shift matrices around. Hence the Tsirelson space T , for which Blanco established weak amenability of $\mathcal{A}(T)$, is possibly inaccessible to our methods. But in proving that $\mathcal{A}(T)$ is weakly amenable, it suffices to prove that $\mathcal{A}(X)$ is weakly amenable for some ‘nice’ X such that $\mathcal{A}(T) \underset{M}{\sim} \mathcal{A}(X)$.

This leads to the question:

What are the spaces X such that $\mathcal{A}(X)$ and $\mathcal{A}(T)$ are Morita equivalent? Which among these have an unconditional basis?

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