

ON THE ROLE OF AREA IN ELEMENTARY GEOMETRY

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ABSTRACT

Treatment of area based on the SMSG approach is widespread at school and also occurs in university textbooks. It was based on assuming, in addition to sixteen pre-area postulates, four postulates specifically on area, the last of which is that the area of a rectangle is the product of the length and the breadth. We show here that this twentieth postulate can be *deduced* from the earlier material and so is redundant.

1. Historical context

Euclid's *Elements* were completed logically in the period 1882–99, from Pasch (1882) through Peano (1889) to Hilbert [8]. Hilbert's treatment of the geometrical quantities (length or distance, angle-measure and area), which he dealt with as separate, undefined terms, was to a considerable extent conservative, and he did not presuppose and use a knowledge of the real number system. One very significant aspect of his approach was that he dealt successfully with the material up to and including the ratio results for triangles, *before* introducing the concept of area. He was then able to *define* the measure of area of a triangle ([8, §20, p. 62]), and so the problem of showing that areas of polygonal regions are well-defined need start only at the level above triangles. His approach in this respect was given a more formal presentation in Forder [7]. A consequence of Hilbert's achievement is that area, with fundamental results based on it, is no longer an essential component of geometry; in fact area is omitted by some authors, e.g. Choquet [5] and Papy [12]. Of course most authors regard area as an important topic in its own right and include it, even though it need not be taken as fundamental.

In a strikingly new departure in 1932, G.D. Birkhoff [2] presupposed a knowledge of \mathbf{R} and assumed a Cartesian coordinate system on each line, and a polar coordinate system on the set of half lines emanating from any one point. His approach was laid out more elementarily in Birkhoff and Beatley [3]. Birkhoff's basic approach did not involve area, as is shown by the fact that none of his four postulates mentions area, and the two authors were quite aware that area of a triangle could be defined and area of more general regions then built up straightforwardly, as shown in [3, p. 222]. Instead they chose to take area as an undefined concept, adding some postulates for it: they stated (p. 222) that they would have preferred to define area but chose instead the other approach in view of pupils' earlier expe-

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rience with area; however, the authors had already dealt with the ratio results for triangles.

There followed in 1961 the SMSG approach [16], in which the postulates on area evidently stemmed from those of Birkhoff and Beatley. These can be found in Cederberg [4] or Tuller [15], where various sets of postulates are conveniently given in appendices. With this approach, similarity results are definitely based on a treatment of area as a primitive or undefined term. This type of treatment of area is widespread at school; see Jacobs [9] and Wheeler and Wheeler [17] for examples.

2. Mathematical context

To set the context we suppose that the following have been introduced and developed: the *basic geometrical sets* of plane geometry, viz. lines AB , segments $[A, B]$, half-lines $[A, B$, closed half-planes, interior and exterior regions for the arms of an angle, triangular regions $[A, B, C]$, convex quadrilaterals $[A, B, C, D]$, the *quantities* distance $|A, B|$ and degree-measure of angles $|\angle BAC|$, *congruence* \cong and *parallelism*. However, we stop short of dealing with the ratio results for triangles. There is also assumed a knowledge of the set \mathbf{R} of real numbers as an order-complete ordered field.

With this preparation, we now note that the SMSG postulates concerning area¹ are the following:

Postulate 17. *To every polygonal region there corresponds a unique positive number.*

Definition. The **area** of a polygonal region is the number assigned to it by Postulate 17.

Postulate 18. *If two triangles are congruent, then the triangular regions have the same area.*

Postulate 19. *Suppose that the region R is the union of two regions R_1 and R_2 . Suppose that R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2 .*

Postulate 20. *The area of a rectangle is the product of the length of its base and the length of its altitude.*

3. A new contribution

We use terminology in accordance with informal axiomatic theory within the context of set theory, as laid out, e.g. by Stoll [14, pp. 134–136].

To start with, Postulate 17 seems liable to misinterpretation, so we begin by expressing matters in more technical terms. We denote by \mathcal{PR} the set of polygonal regions in the Euclidean plane. We introduce as a primitive term, called *area* and denoted by Δ , a function $\Delta : \mathcal{PR} \rightarrow \mathbf{R}$. Then we introduce the modified postulate below:

Postulate 17*. **For each $R \in \mathcal{PR}$, we have $\Delta(R) > 0$.**

¹Reprinted with permission from School Mathematics Study Group, *Mathematics for High School: Geometry, Part II, Student's Text*, Yale University Press, 1961.

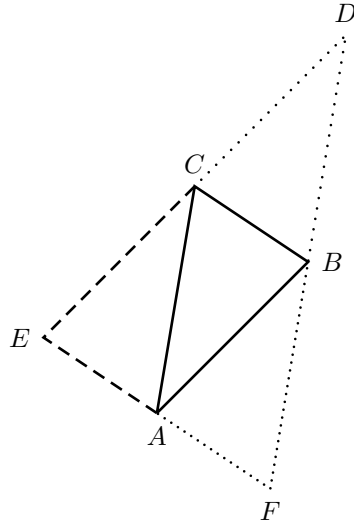


FIG. 1—A triangle as half a parallelogram

That is the start: we have a function that takes only positive values, but it is otherwise arbitrary. Our purpose is to assume properties for it that will narrow it to a certain degree of uniqueness. It might seem that so far things are not tied down to any extent, but it can be noted that equality of area induces an equivalence relation on \mathcal{PR} . For if we write $R_1 \sim R_2$ if $\Delta(R_1) = \Delta(R_2)$, then $R_1 \sim R_1$ as $\Delta(R_1) = \Delta(R_1)$; if $R_1 \sim R_2$ then $R_2 \sim R_1$ as if $\Delta(R_1) = \Delta(R_2)$ then $\Delta(R_2) = \Delta(R_1)$; if $R_1 \sim R_2$ and $R_2 \sim R_3$ then $R_1 \sim R_3$, as if $\Delta(R_1) = \Delta(R_2)$ and $\Delta(R_2) = \Delta(R_3)$, it follows that $\Delta(R_1) = \Delta(R_3)$. Any properties assumed for Δ must be compatible with this equivalence relation.

Using the symbol \cong for congruence, we assume Postulate 18 unchanged:

Postulate 18. **If T_1, T_2 are triangular regions such that $T_1 \cong T_2$ then $\Delta(T_1) = \Delta(T_2)$.**

We could assume Postulate 19 as it stands, but that would prolong the justification needed to be done, so we substitute for it the following equivalent property, which fits in more conveniently:

Postulate 19*. **If $R = \cup_{j=1}^n T_j$, where $R \in \mathcal{PR}$ and the T_j are non-overlapping triangular regions, then $\Delta(R) = \sum_{j=1}^n \Delta(T_j)$.**

We remark that two triangular regions are said to be non-overlapping if they are disjoint or have only boundary segments or boundary points in common.

We now proceed without assuming Postulate 20.

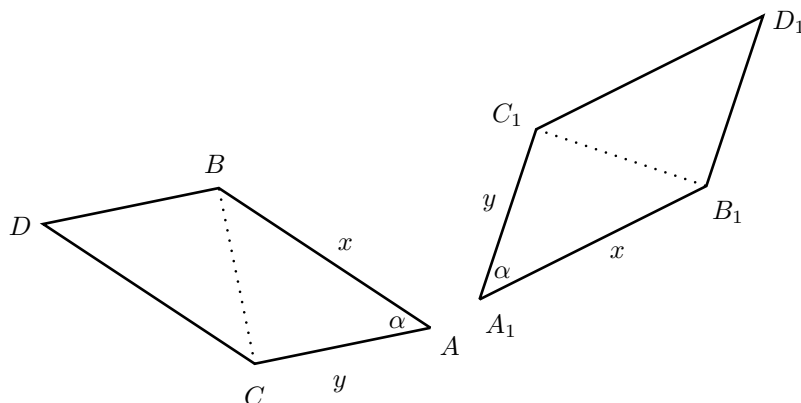


FIG. 2—Congruent parallelograms

We note that any triangle $[A, B, C]$ can be extended to three parallelograms $[A, B, D, C]$, $[B, C, E, A]$, $[C, A, F, B]$ as in Figure 1, and we can conclude at this stage that these parallelograms have equal areas, the area of each being equal to $2\Delta[A, B, C]$.

We use the notation $|P, Q|$ for the distance between points P and Q . As in Figure 2, if $[A, B, D, C]$ is a parallelogram with opposite vertices A and D , and if $[A_1, B_1, D_1, C_1]$ is a parallelogram with opposite vertices A_1 and D_1 , such that

$$|A, B| = |A_1, B_1|, |A, C| = |A_1, C_1|, |\angle BAC| = |\angle B_1A_1C_1|,$$

then $[A, B, C] \cong [A_1, B_1, C_1]$ and $[C, B, D] \cong [C_1, B_1, D_1]$. We can conclude that $[A, B, D, C]$ and $[A_1, B_1, D_1, C_1]$ have equal areas. We are thus justified in introducing a notation $A_\alpha(x, y)$ for the area of any parallelogram with adjacent sides of length x and y , respectively, and with contained angle equal to α .

By 'producing' one such parallelogram along opposite sides so as to make a larger one as in Figure 3, our postulates allow us to deduce that

$$A_\alpha(x + x', y) = A_\alpha(x, y) + A_\alpha(x', y), \quad (3.1)$$

for all positive real numbers x and x' , for each fixed positive real number y and each fixed angle α . For

$$\begin{aligned} \Delta[A, E, F, C] &= \Delta[A, B, D] + \Delta[A, C, D] + \Delta[B, E, D] + \Delta[F, E, D] \\ &= \Delta[A, B, D, C] + \Delta[B, E, F, D]. \end{aligned}$$

Now for each fixed y , (3.1) is an instance of Darboux's functional equation in x (see,

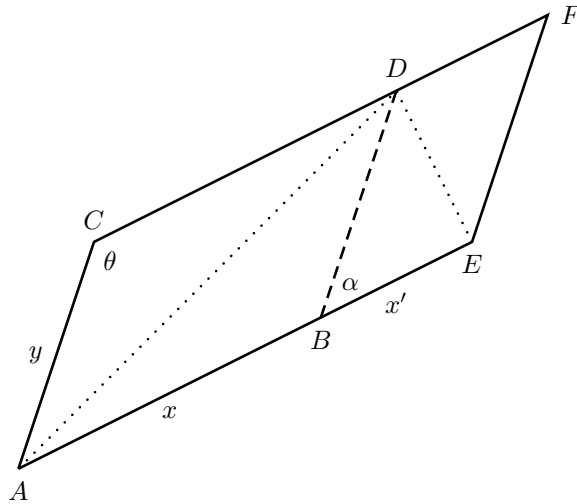


FIG. 3—Additive property of area function

for example, Enriques [6, pp. 226–9]). There is a well-known elementary argument for deducing that

$$A_\alpha(r, y) = rA_\alpha(1, y),$$

for each positive rational number r ; this is laid out in detail in Barry [1, p. 221]. It is also well-known that this can be extended from positive rational r to each positive irrational x , and so holds for all positive real x , under some extra condition. In this case an appropriate extra condition is the fact that $A_\alpha(x, y) > 0$, which renders $A_\alpha(x, y)$ monotonically increasing in x for fixed y ; this proof is laid out in detail in Barry [1, pp. 221–2]. The essence of the proof is as follows: as $A_\alpha(r, y) \rightarrow 0$ ($r \rightarrow 0+$) it follows that $A_\alpha(x, y) \rightarrow 0$ ($x \rightarrow 0+$). From this it follows that $A_\alpha(x, y)$ is continuous from the right, in x , at each positive number, and we can conclude that $A_\alpha(x, y) = xA_\alpha(1, y)$ for each real $x > 0$. As an analogue of (3.1) holds with respect to y , keeping x fixed, by a similar argument we have $A_\alpha(1, y) = yA_\alpha(1, 1)$, and so

$$A_\alpha(x, y) = xyA_\alpha(1, 1) \tag{3.2}$$

for all positive real numbers x and y .

If we now use standard notation for a triangle $[A, B, C]$ we have by (3.2) that

$$\Delta[A, B, C] = \frac{1}{2}bcA_\alpha(1, 1) = \frac{1}{2}caA_\beta(1, 1) = \frac{1}{2}abA_\gamma(1, 1),$$

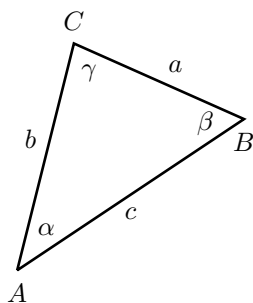


FIG. 4—A general triangle

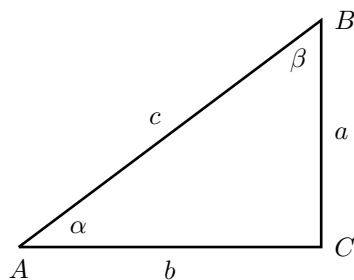


FIG. 5—A right-angled triangle

from which it follows that

$$\frac{A_{\alpha}(1, 1)}{a} = \frac{A_{\beta}(1, 1)}{b} = \frac{A_{\gamma}(1, 1)}{c}. \quad (3.3)$$

The well-known ratio result for similar triangles follows immediately from this.

If we apply (3.3) to a triangle that has a right angle at C we have that α and β are both acute, $|\alpha + \beta| = 90^\circ$ and

$$A_{\alpha}(1, 1) = \frac{a}{c} A_{90^\circ}(1, 1), \quad A_{\beta}(1, 1) = \frac{b}{c} A_{90^\circ}(1, 1). \quad (3.4)$$

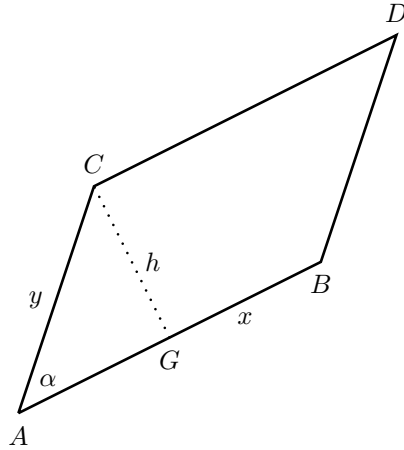


FIG. 6—Area in terms of base and altitude

As in Figure 3, if $\theta = \angle ACD$ we have $|\alpha + \theta| = 180^\circ$ and $A_\theta(1, 1) = A_\alpha(1, 1)$, so that the first formula in (3.4) extends to obtuse angles α .

If G is the foot of the perpendicular from C to AB and $h = |CG|$, then

$$\frac{A_\alpha(1, 1)}{h} = \frac{A_{90^\circ}(1, 1)}{y},$$

and so $A_\alpha(x, y) = xhA_{90^\circ}(1, 1)$ by (3.2).

By taking $\gamma = 90^\circ$ we are dealing with area of rectangles and have the specialised form of (3.2) $A_{90^\circ}(x, y) = xyA_{90^\circ}(1, 1)$, and we see that area is determined on them to within an arbitrary positive multiplying constant $A_{90^\circ}(1, 1)$. In specifying that $A_{90^\circ}(1, 1) = 1$ we are merely linking the unit of area to the unit of length, so that, e.g. a square of side of length 1cm has area 1cm^2 . We then have $A_{90^\circ}(x, y) = xy$.

Thus we have deduced Postulate 20 from the earlier postulates.

As an aside, to prove Pythagoras' theorem using Figure 7 from (3.4) we have that

$$\begin{aligned} A_\gamma(1, 1) &= \frac{x}{c} A_{90^\circ}(1, 1) = \frac{c}{a} A_{90^\circ}(1, 1), \\ A_\beta(1, 1) &= \frac{y}{b} A_{90^\circ}(1, 1) = \frac{b}{a} A_{90^\circ}(1, 1), \end{aligned}$$

so that we have

$$xa = c^2, \quad ya = b^2, \quad (x + y)a = b^2 + c^2.$$

As D is in the segment $[B, C]$, we then have $x + y = a$ and so $a^2 = b^2 + c^2$.

We note that, with our choice of $A_{90^\circ}(1, 1)$, we have in (3.4) the usual formulae

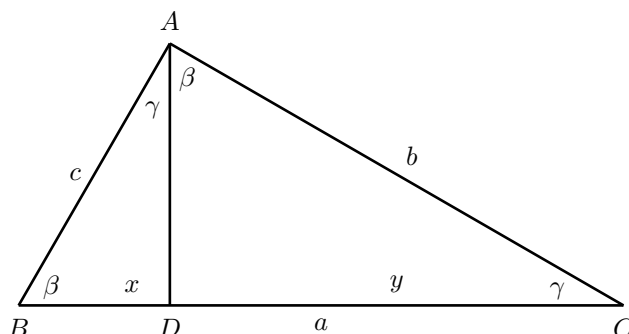


FIG. 7—Pythagoras' theorem

for sines of angles α and β in a right-angled triangle and so (3.3) reduces to the standard sine rule for a triangle.

4. Implementation

The fact that, by Postulate 17, each polygonal region has a unique area does not guarantee that the specification ensures that we can actually find that area in all cases. Dealing with this level of generalisation involves much demanding material in combinatorial topology, as shown by Hilbert. For instance to know what a polygonal region is, we need to have the Jordan curve theorem for a *polygonal* simple closed curve. It can be proved that such a region can be *triangulated*, that is expressed as a union of a finite number of non-overlapping triangular regions; such a result would enable the area of any polygonal region to be worked out, using Postulate 19 and the areas of triangles.

There is a considerable amount of discussion of area in Moise [11, pp. 184–211] and various treatments are considered there, including the MSG approach using a weaker version of Postulate 20, one dealing with squares instead of rectangles. There is also material of this type in Roe [13]. There is a very thorough approach in Millman and Parker [10, pp. 236–51], featuring the MSG characterisation of area, with squares rather than rectangles in the final postulate, but justification is provided for the existence of area.

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