EXPLICIT EXTENDED LOOP-LIKE SOLUTIONS OF THE LAPLACE–YOUNG CAPILLARY EQUATION USING A SYMBOLIC MANIPULATION PACKAGE

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Abstract

We obtain explicit asymptotic solutions as far as the first four loop-like mathematical solutions of the Laplace–Young capillary equation, which was originally derived to model liquid rise in capillary tubes and the shape of static liquid drops. The method of matched asymptotic expansions is employed extensively; most of the algebra is carried out using a symbolic manipulation package.

1. Introduction

Surface tension effects have been studied for centuries: Leonardo da Vinci attempted an ad hoc intuitive explanation of capillary effects, and even Newton, on studying the rise of liquid up a thin tube, gave a qualitative explanation based on the attraction of the liquid to the tube. A consistent theory of capillary surfaces originated in the eighteenth century; Young and Laplace then independently developed their capillary equation in the nineteenth century, and this has been the basis for all continuum based approaches since then. Laplace subsequently used an approximate boundary layer analysis on the wide capillary tube problem in order to approximate the capillary rise and the liquid surface shape. This was one of the first times that a boundary layer analysis was applied to a singular perturbation problem. Gauss later further developed the theory of surfaces in general. Capillary phenomena are enjoying new interest of late as a result of the demands of space-age technology and medicine.

The Laplace–Young capillary equation, which models a balance between gravity and surface tension effects, is non-linear. Due to this nonlinearity, the usual approach to capillary problems has been either numerical or asymptotic [4]. Here, we consider mathematical solutions of the pendant drop problem corresponding to a self-intersecting solution consisting of a series of loops [4]. It has previously been shown that matched asymptotic expansions can be used to resolve the structure of the solutions of the Laplace–Young equation. Here we extend previous results by explicitly calculating closed-form asymptotic solutions [12–16] describing the third and fourth loop (see Figure 1). The calculations are extremely tedious, and most of the algebra was carried out using Maple 8.

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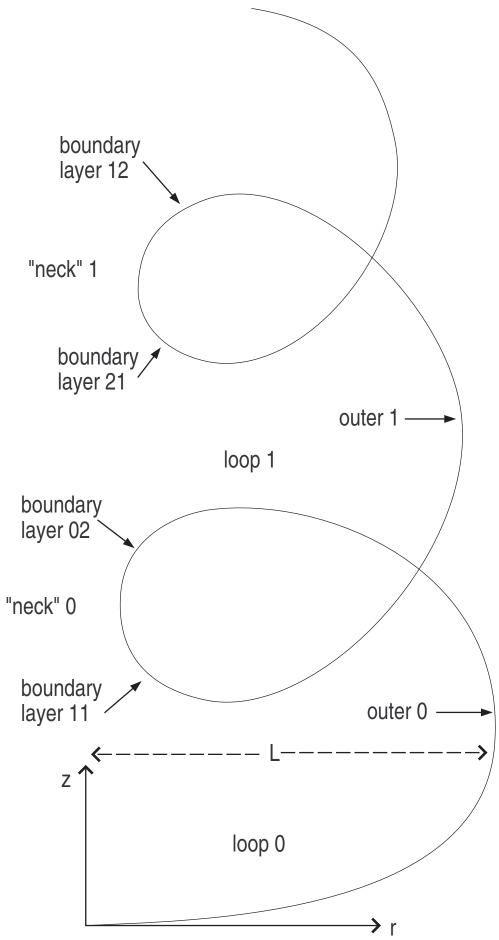

FIG. 1—Loop-like mathematical solutions of the Laplace–Young equation. (The loops are numbered from zero).

2. Governing equations

Consider a liquid with a surface tension γ and density ρ and hence an associated capillary length $a = \sqrt{\frac{\gamma}{\rho g}}$ where g is the acceleration due to gravity. Consider a (multiple) pendant drop (in theory) hanging from a horizontal substrate (Figure 1). We consider a configuration that is axisymmetric with profile $z^* = z^*(r^*)$ in polar coordinates. We first present the equation in a standard dimensionless form where both lengths have been made dimensionless, with the capillary length and the pressure jump at the bottom of the lowest (first) loop with $\rho g L$, where L is a characteristic length feature of the loop. We choose L [12–18] to be the loop half-width at the point of maximum radius in the largest loop.

Referring to Figure 1, the Laplace–Young equation:

$$\begin{aligned} \frac{dX}{d\phi} &= \frac{X\cos\phi}{XP + XY - \sin\phi} \\ \frac{dY}{d\phi} &= \frac{X\sin\phi}{XP + XY - \sin\phi} \end{aligned} \tag{2.1}$$

where X, Y are the dimensionless radial and axial coordinates, P is the dimensionless pressure jump at the apex of the loop and ϕ is the angle of inclination of the profile. In addition, we have three boundary conditions: one for each equation and a third to fix the constant P. These are:

$$X(\phi = 0) = Y(\phi = 0) = 0; X(\phi = \pi/2) = R \tag{2.2}$$

where $R = L/a = \sqrt{\frac{L^2 \rho g}{\gamma}}$ is the dimensionless half-width and a Bond number measuring the relative importance of gravity and surface tension effects. We will seek asymptotic solutions for the case $R << 1$, which physically corresponds to the loop half-width being much smaller than the capillary length, i.e. we consider the case where the surface tension is large or the drop is small.

The interested reader who wishes to learn more about the historical background to this problem should consult Concus and Finn, summarised in [6]. Concus and Finn derive some asymptotic limits and numerically determine the shape of some pendant drops, including loops.

2.1. Asymptotic solutions for small Bond number

The asymptotic structure has been outlined previously [12; 13]. To understand what follows, note that the first 'loop' (loop 0) has an outer nearly spherical solution, a boundary layer and a neck region. The outer and neck solutions are joined via the thin boundary layer. Each subsequent loop has a lower boundary layer matching into an outer solution, which in turn matches into an upper boundary layer, which then resolves into a neck region (see Figure 1).

2.2. The first loop: loop 0

We reproduce the solutions for the first two drops without much comment [12]: the notation is self explanatory on referring to Figure 1. The pressure term is just

$$P = 2R + \frac{R}{3}. \tag{2.3}$$

The outer solutions are

$$Xout0 = -\frac{R(-12+12(\cos\phi)^2+R^2\cos(3\phi)+3R^2\cos\phi-2R\cos(2\phi)-2R^2)}{12\sin\phi} \tag{2.4}$$

$$Yout0 = R - R\cos(\phi) + \frac{1}{3}R^3\ln|\cos(\phi)+1| - \frac{1}{6}R^3\cos(2\phi) - \frac{1}{6}R^3 + \frac{1}{3}R^3\cos\phi - \frac{1}{3}R^3\ln 2. \tag{2.5}$$

The boundary layer (numbered $_{02}$) valid in an $O(R)$ neighbourhood of $\phi = \pi$ has solutions

$$Xbl02 = R^2\left(\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2+8/3}\right) \tag{2.6}$$

$$Ybl02 = 2R + R^3\left(-\frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2+\frac{8}{3}} + \frac{2}{3}\ln|\Phi+\sqrt{\Phi^2+\frac{8}{3}}| - \frac{4}{3}\ln 2 - \frac{1}{3} + \frac{2}{3}\ln R\right). \tag{2.7}$$

For $\phi > \pi$ the neck solutions are valid up to within about $O(R)$ of $\phi = 2\pi$:

$$Xneck0 = -\frac{2}{3}\frac{R^3}{\sin(\phi)} \tag{2.8}$$

$$Yneck0 = 2R + R^3(\frac{2}{3}\ln|\csc(\phi)-\cot(\phi)| - \frac{2}{3}\ln 2 - \frac{2}{3}\ln 3 + \frac{4}{3}\ln R) \tag{2.9}$$

2.3. The second loop: loop 1

The new loop begins with a boundary layer (numbered $_{11}$) valid in an $O(R)$ neighbourhood of $\phi = 2\pi$. The solutions are:

$$Xbl11 = R^2\left(-\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2+\frac{8}{3}}\right) \tag{2.10}$$

$$Ybl11 = 2R + R^3\left(\frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2+\frac{8}{3}} + \frac{2}{3}\ln|\Phi+\sqrt{\Phi^2+\frac{8}{3}}|\right.$$

$$- 2\ln 2 + 2\ln R - \frac{2}{3}\ \ln R + \frac{1}{3}\Big). \tag{2.11}$$

As ϕ increases towards 3π a new outer solution emerges:

$$Xout1 = R\ \sin(\phi) + \frac{1}{12}\frac{R^3(8\cos(2\phi) - \cos(3\phi) - 3\cos(\phi) + 4)}{\sin(\phi)} \tag{2.12}$$

$$Yout1 = 2R + R(-\cos(\phi) + 1) + R^3\left(\frac{4}{3}\ \cos(\phi) - \ln|\frac{\sin(\phi)}{\cos(\phi) + 1}|\right.$$
$$\left. -\frac{1}{6}\ \cos(2\phi) - \frac{1}{2}\ + \frac{1}{3}\ \ln|\sin(\phi)| - \frac{4}{3}\ \ln R - \frac{5}{3}\ \ln 2 + \frac{8}{3}\ \ln R)\right). \tag{2.13}$$

The outer solutions become singular as $\phi \to 3\pi$, and a new boundary layer (numbered$_{12}$) occurs:

$$Xbl12 = R^2\left(\frac{1}{2}\ \Phi + \frac{1}{2}\ \sqrt{\Phi^2 + \frac{16}{3}}\right) \tag{2.14}$$

$$Ybl12 = 4R + R^3\left(-\frac{1}{4}\ \Phi^2 - \frac{1}{4}\ \Phi\sqrt{\Phi^2 + \frac{16}{3}} + \frac{4}{3}\ln|\Phi + \sqrt{\Phi^2 + \frac{16}{3}}| - \frac{4}{3}\right) \tag{2.15}$$

The boundary layer solutions are only valid in an $O(R)$ neighbourhood of $\phi = 3\pi$. As ϕ increases further new neck solutions arise:

$$Xneck1 = -\frac{4}{3}\ \frac{R^3}{\sin(\phi)} \tag{2.16}$$

$$Yneck1 = 4R + R^3\left(\frac{4}{3}\ln|\csc(\phi) - \cot(\phi)| - \frac{8}{3}\ln R\right.$$
$$\left. - \frac{4}{3}\ \ln 2 + \frac{16}{3}\ \ln R - \frac{2}{3}\right) \tag{2.17}$$

2.4. Loop 2 and 3

The structure of the solutions in the next two loops is similar to that in loop 1, but the computations are extremely tedious and could not have been carried out without the assistance of Maple 8. In general, matching was carried out using an intermediate variable. These results were then checked using modified van Dyke matching [2; 6; 9] as follows: if the outer solution is available to $O(\Delta)$ and the inner solution to $O(\delta)$, then the inner expansion to $O(\delta)$ of the outer expansion equals the outer expansion to $O(\Delta)$ of the inner solution.

Loop 2 comprises a boundary layer, outer solution, boundary layer and neck.

Loop 2 The boundary layer scales are: $X = R^2\xi$; $Y = 4R + R^3\zeta$; $\phi = 4\pi - R\Phi$, so the leading order equation is:

$$-\left(\frac{d}{d\Phi}\xi(\Phi)\right)(2\xi(\Phi) + \Phi) - \xi(\Phi) = 0, \tag{2.18}$$

with solution

$$\xi = -\frac{1}{2}\ \Phi + \frac{1}{2}\ \sqrt{\Phi^2 + 4C_{14}}, \tag{2.19}$$

and

$$-\frac{d}{d\Phi}\zeta(\Phi) + \frac{(-\frac{1}{2}\ \Phi + \frac{1}{2}\ \sqrt{\Phi^2 + 4C_{14}})\Phi}{\sqrt{\Phi^2 + 4C_{14}}} = 0, \tag{2.20}$$

with solution

$$\zeta(\Phi) = \frac{1}{4}\ \Phi^2 - \frac{1}{4}\ \Phi\sqrt{\Phi^2 + 4C_{14}} + C_{14}\ln|\Phi + \sqrt{\Phi^2 + 4C_{14}}| + C_{15}. \tag{2.21}$$

To match with the neck 1 solution, we first find that the inner expansion of the $Xneck1$ solution yields:

$$\frac{4}{3}\ \Phi^{-1}R^2 = O(R^4), \tag{2.22}$$

while the outer expansion of the boundary layer solution gives:

$$\left(-2\pi + \frac{1}{2}\ \phi + \frac{1}{2}\ \sqrt{(4\pi - \phi)^2}\right)R + \sqrt{(4\pi - \phi)^2}C_{14}(4\pi - \phi)^{-2}R^3 + O(R^5) \tag{2.23}$$

and matching requires that:

$$C_{14} = \frac{4}{3}, \tag{2.24}$$

and so

$$Xbl21 = R^2(-\frac{1}{2}\ \Phi + \frac{1}{2}\ \sqrt{\Phi^2 + \frac{16}{3}}). \tag{2.25}$$

The inner expansion of the $Yneck1$ solution is:

$$4R + \left(\frac{4}{3}\ \ln|\frac{1}{2}\ \Phi| + \frac{20}{3}\ln R - \frac{4}{3}\ \ln 2 - \frac{2}{3} - \frac{8}{3}\ \ln R\right)R^3 + O(R^5), \tag{2.26}$$

while the outer expansion of the boundary layer solution is:

$$\left(4+\frac{1}{4}\,(4\pi-\phi)^2-\frac{1}{4}\,(4\pi-\phi)\sqrt{(4\pi-\phi)^2}\right)R+\left(-\frac{2}{3}\,\frac{\sqrt{(4\pi-\phi)^2}}{4\pi-\phi}\right.$$
$$\left.+\,C_{15}+\frac{4}{3}\,\ln|4\pi-\phi+\sqrt{(4\pi-\phi)^2}|-\frac{4}{3}\,\ln R\right)R^3=O(R^5), \quad (2.27)$$

and matching requires that:

$$C_{15}=-4\ln 2+\frac{20}{3}\ln R-\frac{8}{3}\ln 3. \quad (2.28)$$

Switch-back terms [7] are added to resolve the $\ln R$ term; specifically, we write $Y=4R+R^3\ln R\,\zeta_{SW}+R^3\zeta$. The solution Y is:

$$Ybl21=4R+R^3\left(\frac{1}{4}\,\Phi^2-\frac{1}{4}\,\Phi\sqrt{\Phi^2+\frac{16}{3}}+\frac{4}{3}\,\ln|\Phi+\sqrt{\Phi^2+\frac{16}{3}}|\right.$$
$$\left.-\,4\ln 2+\frac{20}{3}\ln R-\frac{8}{3}\,\ln R\right). \quad (2.29)$$

Outside an $O(R)$ neighbourhood of $\phi=4\pi$ a new outer solution becomes necessary, with the rescaling $X=Rx=R\sigma_{20}+R^3\sigma_{21}; Y=4R+Ry=4R+R^2\tau_{20}+R^3\tau_{21}$. The relevant equations at leading order are now:

$$\left(\frac{d}{d\phi}\sigma_{20}(\phi)\right)(-\sin(\phi)+2\sigma_{20}(\phi))-\sigma_{20}(\phi)\cos(\phi)=0, \quad (2.30)$$

with solution:

$$\sigma_{20}=\frac{1}{2}(\sin\phi+\sqrt{\sin^2\phi+4C_2}). \quad (2.31)$$

We can simplify the resulting analysis considerably if we first match at the present order. Thus we immediately determine the constant C_2 by matching the outer 1 solution with the preceding boundary layer solution. Expanding the outer 1 solution in terms of the inner variable $\phi=4\pi-R\Phi,\Phi<0,R\to 0$, we obtain

$$\frac{R}{2}(\sin(2\pi-R\Phi+\sqrt{\sin^2(2\pi-R\Phi)+4C_2})\sim\sqrt{4C_2}-\frac{1}{2\sqrt{4C_2}}R\Phi \quad (2.32)$$

if $C_2\neq 0$, while for the case $C_2=0$ we obtain:

$$\frac{R}{2}(\sin(2\pi-R\Phi+\sqrt{\sin^2(2\pi-R\Phi)+4C_2})\sim -R^2\frac{\Phi}{2}. \quad (2.33)$$

Expanding the boundary layer solution $Xbl11$ in terms of the outer variable we obtain

$$\frac{R^2}{2}\left(-\frac{2\pi-\phi}{R+\sqrt{\frac{(\phi-2\pi)^2}{R^2}-\frac{8}{3}}}\right)\sim\frac{R}{2}(\phi-2\pi), \tag{2.34}$$

and matching requires that $C_2 = 0$ and so $\sigma_{20} = \sin\phi$. Proceeding to the Y equation we have:

$$\frac{d}{d\phi}\tau_{20}(\phi)-\sin(\phi)=0, \tag{2.35}$$

with solution

$$\tau_{20}=-\cos\phi+C_{16}. \tag{2.36}$$

Proceeding to second order we have

$$\cos(\phi)(\sin(\phi)(4-\cos(\phi)+C_{16})-\frac{1}{3}\ \sin(\phi)+2\sigma_{21}(\phi))+ \left(\frac{d}{d\phi}\sigma_{21}(\phi)\right)\sin(\phi)-\sigma_{21}(\phi)\cos(\phi)=0, \tag{2.37}$$

with solution

$$\sigma_{21}=\frac{1}{12}\ \frac{11\cos(2\phi)-\cos(3\phi)-3\cos(\phi)+3C_{16}\cos(2\phi)+12C_{17}}{\sin(\phi)}, \tag{2.38}$$

while the Y equation is

$$\frac{d}{d\phi}\tau_{21}(\phi)-\left(\frac{11}{12}\cos(2\phi)-\frac{1}{12}\cos(3\phi)-\frac{1}{4}\cos(\phi)+\frac{1}{4}C_{16}\cos(2\phi)+C_{17} -\sin(\phi)\left(\sin(\phi)(4\cos(\phi)+C_{16})-\frac{1}{3}\sin(\phi)+ \frac{1}{6}\ \frac{11\cos(2\phi)-\cos(3\phi)-3\cos(\phi)+3C_{16}\cos(2\phi)+12C_{17}}{\sin(\phi)}\right)\right)(\sin\phi)^{-1} =0, \tag{2.39}$$

with solution

$$\tau_{21}=\frac{1}{2}C_{16}\cos(\phi)-\frac{1}{4}C_{16}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}|+\frac{11}{6}\cos(\phi)-\frac{11}{12}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}| -\frac{1}{6}\ \cos(2\phi)-\frac{1}{6}\ +\frac{1}{3}\ \ln|\sin(\phi)|-C_{17}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}|+C_{18}. \tag{2.40}$$

The inner expansion of the X outer solution is:

$$\left(-\Phi - \frac{7}{12}\Phi^{-1} - \frac{1}{4}\frac{C_{16}}{\Phi} - \frac{C_{17}}{\Phi}\right) R^2 + O(R^3), \tag{2.41}$$

while the outer expansion of the X boundary layer solution is:

$$\left(-2\pi + \frac{1}{2}\phi + \frac{1}{2}\sqrt{(4\pi-\phi)^2}\right) R + \frac{4}{3}\frac{\sqrt{(4\pi-\phi)^2}}{(4\pi-\phi)^2} R^3 + O(R^5), \tag{2.42}$$

and matching now requires that:

$$C_{16} = 3 - 4C_{17}. \tag{2.43}$$

The inner expression of the Y outer solution is:

$$(6-4C_{17})R + \left(\frac{1}{2}\Phi^2 - 2C_{17} + 3 - \frac{4}{3}\ln R - \frac{5}{3}\ln|-\frac{1}{2}\Phi| + C_{18} + \frac{1}{3}\ \ln|-\Phi|\right) R^3 + O(R^5), \tag{2.44}$$

while the outer expansion of the Y inner solution is:

$$\left(4 + \frac{1}{4}(4\pi-\phi)^2 - \frac{1}{4}(4\pi-\phi)(-4\pi+\phi)\right) R + \left(-\frac{2}{3}\ \frac{-4\pi+\phi}{4\pi-\phi} - 4\ln 2 + \frac{4}{3}\ \ln|\frac{8}{3}\ \frac{-4\pi+\phi}{(4\pi-\phi)^2}| + 8\ln R - \frac{8}{3}\ln R\right) R^3 + O(R^5), \tag{2.45}$$

and matching yields the results:

$$C_{17} = \frac{1}{2} \tag{2.46}$$

and

$$C_{18} = -4\ln R - \frac{4}{3} - \frac{5}{3}\ln 2 + 8\ln R. \tag{2.47}$$

Switchback terms are included in the usual way by rescaling the outer solution as: $Y = 4R + R\tau_{21} + R^3\ln R\,\tau_{SW} + R^3\tau_{22}$. Hence the outer solutions are given by:

$$Xout2 = R\sin(\phi) + \frac{1}{12}\frac{R^3(14\cos(2\phi) - \cos(3\phi) - 3\cos(\phi) + 6)}{\sin\phi} \tag{2.48}$$

$$Yout2 = 4R + R(-\cos(\phi)+1) + R^3\left(\frac{7}{3}\cos(\phi) - \frac{5}{3}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}|\right.$$

$$-\frac{1}{6}\ \cos(2\phi) - \frac{3}{2} + \frac{1}{3}\ \ln|\sin(\phi)| - 4\ln R - \frac{5}{3}\ln 2 + 8\ln R\Big). \qquad (2.49)$$

As $\phi \to 5\pi$ another boundary layer emerges with $X = R^2\xi$; $Y = 6R + R^3\zeta$; $\phi = 5\pi - R\Phi$. The leading order equations are

$$-\left(\frac{d}{d\phi}\xi(\Phi)\right)(2\xi(\Phi) - \Phi) + \xi(\Phi) = 0, \qquad (2.50)$$

with solution

$$\xi = \frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2 - 4C_{19}}, \qquad (2.51)$$

and

$$-\frac{d}{d\phi}\zeta(\Phi) - \frac{(\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2 - 4C_{19}})\Phi}{\sqrt{\Phi^2 - 4C_{19}}}, \qquad (2.52)$$

with solution

$$\zeta = -\frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2 - 4C_{19}} - C_{19}\ln|\Phi + \sqrt{\Phi^2 - 4C_{19}}| + C_{20}. \qquad (2.53)$$

Proceeding to the X matching, the inner expansion of the outer solution is

$$(\Phi + 2\Phi^{-1})R^2 + O(R^4), \qquad (2.54)$$

while the outer expansion of the boundary layer is

$$\left(\frac{5}{2}\pi - \frac{1}{2}\phi + \frac{1}{2}\sqrt{(5\pi - \phi)^2}\right)R - \sqrt{(5\pi - \phi)^2}C_{19}(5\pi - \phi)^{-2}R^3 + O(R^5), \qquad (2.55)$$

and matching requires that

$$C_{19} = -2. \qquad (2.56)$$

For the Y matching, the outer expansion of the boundary layer solution is:

$$\left(6 - \frac{1}{4}(5\pi - \phi)^2 - \frac{1}{4}(5\pi - \phi)\sqrt{(5\pi - \phi)^2}\right)R + \Big(-\sqrt{5\pi - \phi)^2}(5\pi - \phi)^{-1} + C_{20} + 2\ln|5\pi - \phi + \sqrt{(5\pi - \phi)^2}| - 2\ln R\Big)R^3 + O(R^5), \qquad (2.57)$$

while the inner expansion of the outer solution is:

$$6R + \Big(-\frac{1}{2}\Phi^2 - 4 - \frac{5}{3}\ln 2 + 10\ln R - \frac{5}{3}\ln|2\Phi^{-1}| - 4\ln R + \frac{1}{3}\ \ln|\Phi|\Big)R^3 + O(R^5), \qquad (2.58)$$

and matching yields the results:

$$C_{20} = -\frac{16}{3}\ln 2 + 10\ln R - 4\ln R - 3. \tag{2.59}$$

So the boundary layers are:

$$Xbl22 = R^2\left(\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2+8}\right) \tag{2.60}$$

$$\begin{aligned} Ybl22 = 6R + R^3(-\frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2+8} + 2\ln|\Phi + \sqrt{\Phi+8}| \\ -\frac{16}{3}\ln 2 + 10\ln R - 4\ln R - 3) \end{aligned} \tag{2.61}$$

Outside the boundary layer a neck rescaling is necessary: $X = R^3\mu$; $Y = 6R + R^3\nu$ and the leading order equations are:

$$-\left(\frac{d}{d\phi}\mu(\phi)\right)\sin(\phi) - \mu(\phi)\cos(\phi) = 0, \tag{2.62}$$

with solution:

$$\mu = \frac{C_{21}}{\sin\phi}, \tag{2.63}$$

and

$$-\left(\frac{d}{d\phi}\nu(\phi)\right)\sin(\phi) - C_{21} = 0, \tag{2.64}$$

with solution

$$\nu = -C_{21}\ln|\csc(\phi) - \cot(\phi)| + C_{22}. \tag{2.65}$$

For matching the X solution, the inner expansion of the neck solution is

$$\frac{C_{21}}{\Phi}R^2 + O(R^4) \tag{2.66}$$

while the outer expansion of the boundary layer is:

$$\left(\frac{5}{2}\pi - \frac{1}{2}\phi + \frac{1}{2}\sqrt{(5\pi-\phi)^2}\right)R + 2\frac{\sqrt{(5\pi-\phi)^2}}{(5\pi-\phi)^2}R^3 + O(R^5), \tag{2.67}$$

and matching requires that:

$$C_{21} = -2. \tag{2.68}$$

In the Y matching, the inner expansion of the neck solution is:

$$2R + (2\ln|-2\Phi^{-1}| - 2\ln R + C_{22})R^3 + O(R^5) \tag{2.69}$$

and the outer expansion of the boundary layer solution is:

$$\begin{aligned} &\left(6 - \frac{1}{4}(5\pi-\phi)^2 - \frac{1}{4}(5\pi-\phi)(-5\pi+\phi)\right) R \\ &+ \left(-\frac{-5\pi+\phi}{5\pi-\phi} - 3 - \frac{16}{3}\ln 2 + 2\ln|4\frac{-5\pi+\phi}{(5\pi-\phi)^2}| + 12\ln R - 4\ln R\right) R^3 \\ &+O(R^5), \end{aligned} \tag{2.70}$$

and matching now yields

$$C_{22} = -2 - \frac{10}{3}\ln 2 + 12\ln R - 4\ln 3. \tag{2.71}$$

The $\ln R$ switchback terms are resolved by rescaling: $Y = 6R + R^3 \ln R\, \nu_{SW} + R^3\nu$. So the neck solutions are:

$$Xneck2 = -2\frac{R^3}{\sin(\phi)} \tag{2.72}$$

and

$$\begin{aligned} Yneck2 = 6R + R^3(2\ln|\csc(\phi) - \cot(\phi)| - 2 - \frac{10}{3}\ln 2 + 12\ln R \\ -4\ln R) \end{aligned} \tag{2.73}$$

Loop 3 Loop 3 again comprises a boundary layer, outer solution, boundary layer and a neck. The boundary layer scales are: $X = R^2\xi$; $Y = 6R + R^3\zeta$; $\phi = 6\pi - R\Phi$ so the leading order equations are:

$$\left(\frac{d}{d\Phi}\xi(\Phi)\right)(2\xi(\Phi) + \Phi) - \xi(\Phi) = 0, \tag{2.74}$$

with solution

$$\xi = -\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2 + 4C_{23}}, \tag{2.75}$$

and

$$-\frac{d}{d\Phi}\zeta(\Phi) + \frac{(-\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2 + 4C_{23}})\Phi}{\sqrt{\Phi^2 + 4C_{23}}} = 0, \tag{2.76}$$

with solution

$$\zeta = \frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2 + 4C_{23}} + C_{23}\ln|\Phi + \sqrt{\Phi^2 + 4C_{23}}| + C_{24}. \tag{2.77}$$

To match with the neck 2 solution, we first find that the inner expansion of the $Xneck2$ solution yields:

$$2\Phi^{-1}R^2 + O(R^4), \tag{2.78}$$

while the outer expansion of the boundary layer solution gives:

$$\left(-3\pi + \frac{1}{2}\phi + \frac{1}{2}\sqrt{(6\pi-\phi)^2}\right) R + \sqrt{(6\pi-\phi)^2}C_{23}(6\pi-\phi)^{-2}R^3 + O(R^5), \tag{2.79}$$

and matching requires that

$$C_{23} = 2, \tag{2.80}$$

and so

$$Xbl31 = R^2\left(-\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2+8}\right). \tag{2.81}$$

The inner expansion of the $Yneck2$ solution is:

$$2R + (2\ln|\frac{\Phi}{2}| + 14\ln R - 4\ln R - 2 - \frac{10}{3}\ln 2)R^3 + O(R^5), \tag{2.82}$$

while the outer expansion of the boundary layer solution is

$$\left(4 + \frac{1}{4}6\pi-\phi)^2 - \frac{1}{4}(6\pi-\phi)\sqrt{(6\pi-\phi)^2}\right)R + \left(-\sqrt{(6\pi-\phi)^2}(6\pi-\phi)^{-1} + C_{24} + 2\ln|6\pi-\phi+\sqrt{(6\pi-\phi)^2}| - 2\ln R\right)R^3 + O(R^5), \tag{2.83}$$

whence matching requires that:

$$C_{24} = 14\ln R - 4\ln 3 - 1 - \frac{22}{3}\ln 2. \tag{2.84}$$

The $\ln R$ term is resolved using the rescaling: $Y = 6R + R^3\ln R\,\zeta_{SW} + R^3\zeta$ and the Y solution is

$$Ybl31 = 6R + R^3\left(\frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2+8} + 2\ln|\Phi+\sqrt{\Phi^2+8}| + 14\ln R - 4\ln 3 - 1 - \frac{22}{3}\ln 2\right). \tag{2.85}$$

Outside an $O(R)$ neighbourhood of $\phi = 6\pi$ a new outer solution becomes necessary, with the rescaling $X = Rx = R\sigma_{30} + R^3\sigma_{31}$; $Y = 6R + Ry = 6R + R^2\tau_{30} + R^3\tau_{31}$

and the relevant equations at leading order are now

$$\left(\frac{d}{d\phi}\sigma_{30}(\phi)\right)(-\sin(\phi)+2\sigma_{30}(\phi))-\sigma_{30}(\phi)\cos(\phi)=0, \tag{2.86}$$

with leading order solution

$$\sigma_{30}=\frac{1}{2}(\sin\phi+\sqrt{\sin^2\phi+4C_3}). \tag{2.87}$$

For preliminary matching we expand the outer 1 solution in terms of the inner variable $\phi=6\pi-R\Phi, \Phi<0, R\to 0$ to obtain

$$\frac{R}{2}(\sin(2\pi-R\Phi+\sqrt{\sin^2(2\pi-R\Phi)+4C_3})\sim\sqrt{4C_3}-\frac{1}{2\sqrt{4C_3}}R\Phi \tag{2.88}$$

if $C_3\neq 0$, while for the case $C_3=0$ we obtain:

$$\frac{R}{2}\ (\sin(2\pi-R\Phi+\sqrt{\sin^2(2\pi-R\Phi)+4C_3})\sim -R^2\Phi/2. \tag{2.89}$$

Expanding the boundary layer solution $Xbl11$ in terms of the outer variable we obtain

$$\frac{R^2}{2}\left(-\frac{2\pi-\phi}{R+\sqrt{\frac{(\phi-2\pi)^2}{R^2}-\frac{8}{3}}}\right)\sim\frac{R}{2}(\phi-2\pi), \tag{2.90}$$

and matching requires that $C_3=0$, and so $\sigma_{30}=\sin\phi$.

The Y equation gives

$$\frac{d}{d\phi}\tau_{30}(\phi)-\sin(\phi)=0, \tag{2.91}$$

with solution

$$\tau 30=C_{25}-\cos\phi. \tag{2.92}$$

Proceeding to the second order we have

$$\cos(\phi)(\sin(\phi)(6-\cos(\phi)+C_{25})-1/3\ \sin(\phi)+2\sigma_{31}(\phi))+\left(\frac{d}{d\phi}\sigma_{31}(\phi)\right)\sin(\phi)-\sigma_{31}(\phi)\cos(\phi)=0, \tag{2.93}$$

with solution

$$\sigma_{31}=\frac{1}{12}\frac{17\cos(2\phi)-\cos(3\phi)-3\cos(\phi)+35C_{25}\cos(2\phi)+12C_{26}}{\sin(\phi)}, \tag{2.94}$$

while the Y equation is:

$$\frac{d}{d\phi}\tau_{31}(\phi) - \left(\frac{17}{12}\cos(2\phi) - \frac{1}{12}\cos(3\phi) - \frac{1}{4}\cos(\phi) + \frac{1}{4}C_{25}\cos(2\phi) + C_{26}\right.$$
$$- \sin(\phi)\left(\sin(\phi)(6-\cos(\phi)+C_{25}) - \frac{1}{3}\sin(\phi)\right.$$
$$\left.\left. + \frac{1}{6}\ \frac{17\cos(2\phi) - \cos(3\phi) - 3\cos(\phi) + 3C_{25}\cos(2\phi) + 12C_{26}}{\sin(\phi)}\right)\right)(\sin(\phi))^{-1}$$
$$= 0, \qquad (2.95)$$

with solution

$$\tau_{31} = \frac{1}{2}C_{25}\cos(\phi) - \frac{1}{4}C_{25}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}| + \frac{17}{6}\cos(\phi) - \frac{17}{12}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}|$$
$$-\frac{1}{6}\ \cos(2\phi) - \frac{1}{6}\ + \frac{1}{3}\ \ln|\sin(\phi)| - C_{26}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}| + C_{27}. \qquad (2.96)$$

The inner expansion of the X outer solution is:

$$\left(-\Phi - \frac{13}{12}\Phi^{-1} - \frac{1}{4}\ \frac{C_{25}}{\Phi} - \frac{C_{26}}{\Phi}\right)R^2 + O(R^3), \qquad (2.97)$$

while the outer expansion of the X boundary layer solution is:

$$\left(-3\pi + \frac{1}{2}\Phi + \frac{1}{2}\sqrt{(6\pi-\phi)^2}\right)R + 2\frac{\sqrt{(6\pi-\phi)^2}}{(6\pi-\phi)^2}R^3 + O(R^5), \qquad (2.98)$$

and matching now requires that:

$$C_{25} = \frac{11}{3}\ - 4C_{26}. \qquad (2.99)$$

The inner expansion of the Y solution is:

$$\left(\frac{26}{3} - 4C_{26}\right)R + \left(\frac{1}{2}\Phi^2 - 2C_{26} + \frac{13}{3} - 2\ln R\right.$$
$$\left.-\frac{7}{3}\ln|-\frac{1}{2}\Phi| + C_{27} + \frac{1}{3}\ln|-\Phi|\right)R^3 + O(R^5), \qquad (2.100)$$

while the outer expansion of the Y inner solution is:

$$\left(4 + \frac{1}{4}(6\pi-\phi)^2 - \frac{1}{4}(6\pi-\phi)(-6\pi+\phi)\right)R + \left(-\frac{(-6\pi+\phi)}{(6\pi-\phi)}\right.$$
$$\left.-\frac{22}{3}\ln 2 + 16\ln R + 2\ln|4\frac{(-6\pi+\phi)}{(6\pi-\phi)^2}| - 1 - 4\ln R\right)R^3$$
$$+O(R^5), \qquad (2.101)$$

and matching now yields the results:

$$C_{26} = \frac{7}{6}, \tag{2.102}$$

and

$$C_{27} = -\frac{17}{3}\ln 2 + 16\ln R - 4\ln 3 - 2. \tag{2.103}$$

Switchback terms are included in the usual way by rescaling the outer solution as: $Y = 6R + R\tau_{31} + R^3 \ln R\,\tau_{SW} + R^3\tau_{32}$. Hence the outer solutions are given by

$$Xout3 = R\sin(\phi) + \frac{1}{12}\frac{R^3(20\cos(2\phi) - \cos(3\phi) - 3\cos(\phi) + 8)}{\sin(\phi)} \tag{2.104}$$

$$Yout3 = 6R + R(-\cos(\phi) + 1) + R^3\left(\frac{10}{3}\cos(\phi) - \frac{7}{3}\ln|\frac{\sin(\phi)}{\cos(\phi)+1}| \right.$$
$$\left. -\frac{1}{6}\ \cos(2\phi) - \frac{19}{6} + \frac{1}{3}\ \ln|\sin(\phi)| - \frac{17}{3}\ln 2 + 16\ln R - 4\ln 3)\right). \tag{2.105}$$

As $\phi \to 7\pi$ another boundary layer emerges with $X = R^2\xi;\ \ Y = 8R + R^3\zeta$. The leading order equations are

$$-\left(\frac{d}{d\Phi}\xi(\Phi)\right)(2\xi(\Phi) - \Phi) + \xi(\Phi) = 0, \tag{2.106}$$

with solution

$$\xi = \frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2 - 4C_{28}}, \tag{2.107}$$

and

$$-\frac{d}{d\Phi}\zeta(\Phi) - \frac{(\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2 - 4C_{28}})\Phi}{\sqrt{\Phi^2 - 4C_{28}}} = 0, \tag{2.108}$$

with solution

$$\zeta = -\frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2 - 4C_{28}} - C_{28}\ln|\Phi + \sqrt{\Phi^2 - 4C_{28}}| + C_{29}. \tag{2.109}$$

Proceeding to the X matching, the inner expansion of the outer solution is

$$(\Phi + \frac{8}{3}\Phi^{-1})R^2 + O(R^4), \tag{2.110}$$

while the outer expansion of the boundary layer solution is:

$$\left(\frac{7}{2}\pi - \frac{1}{2}\phi + \frac{1}{2}\sqrt{(7\pi - \phi)^2}\right)R - \sqrt{(7\pi - \phi)^2}C_{28}(7\pi - \phi)^{-2}R^3$$

$$+O(R^5), \qquad (2.111)$$

and matching requires that:

$$C_{28} = -\frac{8}{3}. \qquad (2.112)$$

For the Y matching, the outer expansion of the boundary layer solution is:

$$\left(8 - \frac{1}{4}(7\pi-\phi)^2 - \frac{1}{4}(7\pi-\phi)\sqrt{(7\pi-\phi)^2}\right) R +$$
$$\left(-\frac{4}{3}\frac{\sqrt{(7\pi-\phi)^2}}{(7\pi-\phi)} + C_{29} + \frac{8}{3}\ln|7\pi-\phi+\sqrt{(7\pi-\phi)^2}| - \frac{8}{3}\ln R\right) R^3$$
$$+O(R^5), \qquad (2.113)$$

while the inner expansion of the outer solution is:

$$6R + \left(-\frac{1}{2}\Phi^2 - \frac{14}{3} + \frac{56}{3}\ln R - 4\ln 3 - \frac{7}{3}\ln|2\Phi^{-1}| - \frac{17}{3}\ln 2 + \frac{1}{3}\ \ln|\Phi|\right) R^3$$
$$+O(R^5), \qquad (2.114)$$

and matching yields the results:

$$C_{29} = \frac{56}{3}\ln R - 4\ln R - \frac{32}{3}\ln 2 - \frac{10}{3}. \qquad (2.115)$$

So the boundary layer solutions are

$$Xbl32 = R^2\left(\frac{1}{2}\Phi + \frac{1}{2}\sqrt{\Phi^2 + \frac{32}{3}}\right) \qquad (2.116)$$

$$Ybl32 = 8R + R^3\left(-\frac{1}{4}\Phi^2 - \frac{1}{4}\Phi\sqrt{\Phi^2+\frac{32}{3}} + \frac{8}{3}\ln|\Phi + \sqrt{\Phi^2+\frac{32}{3}}|\right.$$
$$\left. + \frac{56}{3}\ln R - 4\ln R - \frac{32}{3}\ln 2 - 16/3\right). \qquad (2.117)$$

Outside the boundary layer a neck rescaling is necessary: $X = R^3\mu$; $Y = 8R + R^3\nu$ and the leading order equations are:

$$-\left(\frac{d}{d\phi}\mu(\phi)\right)\sin(\phi) - \mu(\phi)\cos(\phi) = 0, \qquad (2.118)$$

with solution:

$$\mu = \frac{C_{30}}{\sin(\phi)}, \qquad (2.119)$$

and

$$-\left(\frac{d}{d\phi}\nu(\phi)\right)\sin(\phi) - C_{30} = 0, \tag{2.120}$$

with solution

$$\nu = -C_{30}\ln|\csc(\phi) - \cot(\phi)| + C_{31}. \tag{2.121}$$

For matching the X solution, the inner expansion of the neck solution is

$$\frac{C_{30}}{\Phi}R^2 + O(R^4), \tag{2.122}$$

while the outer expansion of the boundary layer solution is:

$$\left(\frac{7}{2}\pi - \frac{1}{2}\phi + \frac{1}{2}\sqrt{(7\pi-\phi)^2}\right)R + \frac{8}{3}\frac{\sqrt{(7\pi-\phi)^2}}{(7\pi-\phi)^2}R^3 + O(R^5), \tag{2.123}$$

and matching requires that:

$$C_{30} = -\frac{8}{3}. \tag{2.124}$$

In the Y matching the inner expansion of the neck solution is:

$$2R + (\frac{8}{3}\ln|-2\Phi^{-1}| - \frac{8}{3}\ln R + C_{31})R^3 + O(R^5), \tag{2.125}$$

and the outer expansion of the boundary layer is:

$$\begin{aligned}&\left(8 - \frac{1}{4}(7\pi-\phi)^2 - \frac{1}{4}(7\pi-\phi)(-7\pi+\phi)\right)R + \left(-\frac{4}{3}\frac{-7\pi+\phi}{7\pi-\phi}\right.\\ &\left.+\frac{64}{3}\ln R - \frac{10}{3} + \frac{8}{3}\ln|\frac{16}{3}\frac{-7\pi+\phi}{(7\pi-\phi)^2}| - 4\ln R - \frac{32}{3}\ln 2\right)R^3\\ &+O(R^5),\end{aligned} \tag{2.126}$$

and matching now yields:

$$C_{31} = \frac{64}{3}\ln R - \frac{8}{3}\ln 2 - \frac{20}{3}\ln R - 2. \tag{2.127}$$

The $\ln R$ terms are now resolved via the rescaling $Y = 8R + R^3\ln R\nu_{SW} + R^3\nu$, and so the neck solutions are:

$$Xneck3 = -\frac{8}{3}\frac{R^3}{\sin(\phi)} \tag{2.128}$$

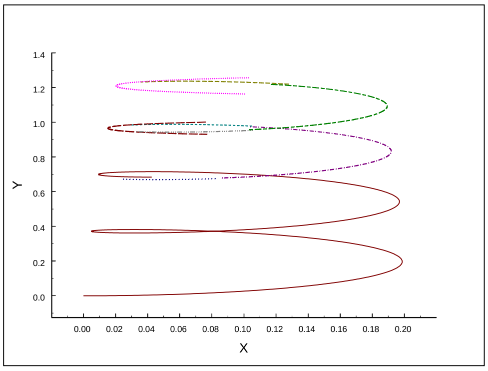

Fig. 2—For $\phi \leq 2\pi$ (the first two loops) the shape is computed numerically. For $\phi > 2\pi$ the constituent asymptotic solutions are shown for the third and fourth loops numbered 2 and 3. (First boundary layer, outer, second boundary layer,neck).

and

$$Yneck3 = 8R + R^3\left(\frac{8}{3}\ln|\csc(\phi) - \cot(\phi)| - 4 + \frac{64}{3}\ln R - \frac{8}{3}\ln 2 - \frac{20}{3}\ln 3)\right) \qquad (2.129)$$

3. Numerical Solutions

In order to have an independent check on the asymptotic solutions, we numerically integrate the equations (1) and compare the results. There are extensive tabulated solutions available for pendant and sessile solutions [5] but not for loop solutions. It is simplest to fix the unknown constant and to compute the Bond number as we then only have to deal with an initial value problem that we can solve using a Runge-Kutta technique. We note first that the systems (1) and (2) are singular at $\phi = 0$, so if we assume that for $\phi << 1$ we can set $X = A\phi^{\alpha}, Y = B\phi^{\beta}, \alpha, \beta > 0$. Thus we can neglect XY at leading order, and a small ϕ solution is easily found to be:

$$X \sim \frac{2\phi}{P} - \frac{\phi^3(P^2+3)}{3P^3}, Y \sim \frac{\phi^2}{P} - \frac{\phi^4(48P^2+3)}{4P^3}.$$

We can thus use this to start up the numerical solution for some small value of ϕ .

Some typical computations are shown in Figures 2–4. In Figure 2 we compute the first two loops numerically and then continue the plot for the next two loops using the asymptotic solutions. In Figures 3 and 4 we show plots of X, Y versus ϕ illustrating how the method of matched asymptotic expansions works in this particular case. It is clear that the nature of the solutions is quite different in the

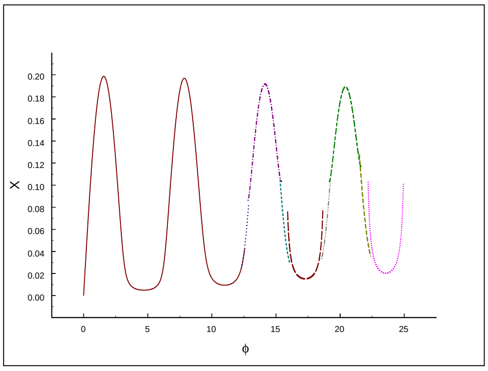

FIG. 3—X versus ϕ . For $\phi \leq 2\pi$ (the first two loops) is computed numerically. For $\phi > 2\pi$ the constituent asymptotic solutions are shown for the third and fourth loops numbered 2 and 3. (First boundary layer outer, second boundary layer, neck).

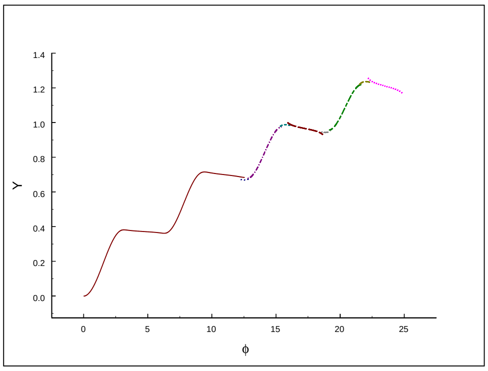

FIG. 4—Y versus ϕ . For $\phi \leq 2\pi$ (the first two loops) is computed numerically. For $\phi > 2\pi$ the constituent asymptotic solutions are shown for the third and fourth loops numbered 3 and 4. (First boundary layer, outer, second boundary layer, neck).

outer and neck regions: the boundary layers in effect are transition layers allowing for the change from outer-type solution to neck-type solution.

4. Closing Remarks

We have undertaken (successfully) the task of computing closed-form asymptotic solutions for several loops (self-intersecting mathematical solutions) of the Laplace–Young capillary equation. Of their nature, the computations tend to be repetitive in that each new loop has a similar asymptotic structure to the previous one, but the details of the computations are tedious and would not have been attempted without recourse to the symbolic manipulation package Maple. The method of matched asymptotic expansions is well suited to problems of this nature, and the solutions obtained are in agreement with numerical results.

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