UNSTEADY EXACT SOLUTION FOR FLOWS OF FLUIDS WITH PRESSURE-DEPENDENT VISCOSITIES

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Abstract

Stokes recognized that the viscosity of fluids could generally depend on their pressure, and this has been confirmed by numerous experiments. In this paper we consider several unsteady flows of fluids with pressure-dependent viscosity, and we establish explicit exact solutions for these problems that can serve as benchmarks for numerical solutions for flows in complicated geometries.

1. Introduction

There are many applications, elasto-hydrodynamics being one, where the fluid can be modelled as an incompressible fluid with a viscosity that depends on the pressure (see [15]). The justification for such an assumption stems from the fact that while the density changes by merely a few percent, the pressure can change significantly and the viscosity can change by several orders of magnitude. Of course, there is the possibility that the dependence of viscosity on density is such that even a small change in density causes this change. Experiments clearly suggest that viscosity varies exponentially with pressure and that it is the relationship between the viscosity and the pressure that causes the tremendous change that occurs in the viscosity. That the viscosity of liquids could depend upon the pressure was known to the pioneers of the field. Stokes [14] is in fact very careful to delineate the special class of flows, those in channels and pipes at moderate pressures, when viscosity could be assumed a constant.

There is also a considerable amount of literature even prior to 1930 concerning the variation of viscosity with pressure (see Bridgman [4] on the physics of high pressures for a detailed discussion of the same). Bridgman [4] makes it abundantly clear that he devoted a great deal of attention to determining the variation

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of viscosity with pressure for numerous liquids. Andrade [1] suggested that viscosity varied exponentially with pressure, a none too moderate variation. There has been a substantial addition to the early experimental literature concerning fluids with pressure-dependent viscosities (see Cutler *et al.* [6], Griest *et al.* [7], Johnson and Cameron [9] and Johnson and Greenwood [10]). Also, numerical studies related to approximate equations for elasto-hydrodynamics have been carried out (see [15]). (It should be noted, however, as Rajagopal and Szeri [13] have pointed out, that there is a serious error in these studies, due to an inconsistency in how the elasto-hydrodynamic approximation for fluids with pressure-dependent viscosities has usually been carried out.) Some rigorous theoretical studies concerning the existence and uniqueness of the flows of such fluids have also been conducted recently (see Malek *et al.* [11]), and a few exact solutions have been established (see Hron *et al.* [8]).

The determination of exact solutions plays a key role in the development of the subject. In addition to providing solutions to problems that have some technical relevance, they provide a means for checking complicated numerical schemes that are developed to study the flows of such fluids in complex geometries. The importance of this cannot be overemphasized, as there is a great deal to be gained from an understanding of special solutions, and both the classical theories for linearized elastic solids and the linearly viscous fluid (Navier–Stokes fluid) bear testimony to the same.

In this paper, we generate several explicit exact solutions that are admissible for the flows of a fluid with pressure-dependent viscosity, adding to the few that are available. The plan of the paper is as follows: In the next section we introduce the basic equations and the model; in Section 3 we introduce the exact solutions. We shall consider special separable pseudo-planar flows with respect to space and time, circularly polarized waves and some unsteady rectilinear shearing. In this framework we shall derive some exact analytical solutions to some simple boundary value problems. These solutions enable us to understand some interesting features of the various functional forms that may be considered to model the pressure dependence of the viscosity. Moreover, these solutions point out the fundamental role that external forces (such as gravity) play in the mechanical response of the fluid. The last section of the paper is devoted to a more general constitutive framework, namely an incompressible, homogeneous second-grade fluid, to show that once again in pressure-dependent viscous fluids external forces can play a significant role with regard to their mechanical response.

2. The model and the governing equations

We are interested in flows of an incompressible fluid whose Cauchy stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p)\mathbf{D}, \tag{2.1}$$

where $-p\mathbf{I}$ is the *constraint stress* due to the requirement of incompressibility, $\mu(p)$ is the pressure dependent viscosity, and

$$\mathbf{D} = \frac{1}{2}\left[\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\right], \tag{2.2}$$

where $\mathbf{v}$ denotes the velocity field.

As the fluid is incompressible it can only undergo isochoric motion, and thus

$$tr(\mathbf{D}) = div(\mathbf{v}) = 0. \tag{2.3}$$

Notice that unlike the classical Navier–Stokes constitutive equation, where the stress $\mathbf{T}$ is given explicitly in terms of the kinematical variable $\mathbf{D}$, here in (2.1) the stress is not expressed explicitly in terms of $\mathbf{D}$, as (2.3) implies

$$p = -\frac{1}{3} tr(\mathbf{T}), \tag{2.4}$$

and thus the viscosity μ is a function of the stress.

On substituting (2.1) into the balance of linear momentum

$$\rho \frac{d\mathbf{v}}{dt} = div(\mathbf{T}) + \rho \mathbf{b}, \tag{2.5}$$

we immediately obtain, on using (2.3),

$$\rho \frac{d\mathbf{v}}{dt} = -grad(p) + \mu(p)\nabla^2 \mathbf{v} + 2\mathbf{D}grad(\mu) + \rho \mathbf{b}. \tag{2.6}$$

Here ρ denotes the density, $\mathbf{b}$ the specific body force and ∇^2 the usual Laplace operator.

We shall now proceed to find a velocity–pressure relationship for $(\mathbf{v}, p)$ that satisfies (2.3) and (2.6). In our study we shall assume a variety of forms for the dependence of viscosity on the pressure:

$$\begin{aligned} &(i) && \mu(p) = \alpha p, && \alpha > 0; \\ &(ii) && \mu(p) = A\exp(\lambda p), && A > 0, \lambda > 0; \\ &(iii) && \mu(p) = Bp^n, && B > 0, n > 0. \end{aligned} \tag{2.7}$$

The above forms for the viscosity imply that $\mu(p) \to \infty$ as $p \to \infty$, a feature that has been verified experimentally [4]. We also note that except for (ii), the viscosity tends to zero as $p \to 0$. While we are not concerned with flows where $p \to 0$, it would be reasonable to also consider the possibilities:

$$\begin{aligned} &(i)' && \mu(p) = \alpha_0 + \alpha p, && \alpha_0, \alpha > 0; \\ &(iii)' && \mu(p) = B_0 + Bp^n, && B_0, B > 0, n > 0. \end{aligned} \tag{2.8}$$

In the case of the models $(i)'$ and $(iii)'$, $\mu(p) \to \infty$ as $p \to \infty$, and $\mu(p) \to$ constant as $p \to 0$. In the next section, important differences between models (i), $(i)'$ and (iii), $(iii)'$ will be discussed.

3. Unsteady plane shearing flows

With a view towards highlighting the difference in the response characteristics of the classical incompressible Navier–Stokes fluid and a fluid with pressure-dependent viscosity, we shall consider a class of unsteady plane shearing flows.

Let us consider the motion $(X, Y, Z) \to (x, y, z)$ of the form

$$x = X + f(t)\phi(Z), \quad y = Y + f(t)\psi(Z), \quad z = Z, \tag{3.1}$$

where (X, Y, Z) and (x, y, z) are rectangular Cartesian co-ordinates associated with a reference configuration and the current configuration of the body, respectively. The isochoric Eulerian velocity field associated with (3.1) is given by

$$u(z,t) = f_t(t)\phi(z), \quad v(z,t) = f_t(t)\psi(z), \quad w \equiv 0. \tag{3.2}$$

This motion corresponds to a flow in which points move on the Z = constant plane, with the velocity along the x and y co-ordinate directions differing from one Z = constant plane to another. The flows under consideration are special cases of pseudo-planar flows studied by Berker [3]. It is also worth observing that each point on a Z = constant plane has the same velocity, i.e. these planes are moving as rigid planes. However, both the direction and the speed of these rigid sheets vary as Z varies.

It follows from (3.1) that the deformation gradient and its derivative have the following matrix representation:

$$(\mathbf{F})_{ij} = \begin{pmatrix} 1 & 0 & f\phi_Z \\ 0 & 1 & f\psi_Z \\ 0 & 0 & 1 \end{pmatrix}, \quad (\mathbf{F}^{-1})_{ij} = \begin{pmatrix} 1 & 0 & -f\phi_Z \\ 0 & 1 & -f\psi_Z \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

Here, the suffix denotes the derivative with respect to that variable. It immediately follows that the velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ has the matrix representation

$$(\mathbf{L})_{ij} = \begin{pmatrix} 0 & 0 & f_t\phi_Z \\ 0 & 0 & f_t\psi_Z \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

and thus

$$(\mathbf{D})_{ij} = \frac{1}{2}\begin{pmatrix} 0 & 0 & f_t\phi_Z \\ 0 & 0 & f_t\psi_Z \\ f_t\phi_Z & f_t\psi_Z & 0 \end{pmatrix}. \tag{3.5}$$

A simple calculation then shows that the flow under consideration is not viscometric, unless $f(t)$ is a linear function of time.

On substituting (3.5) into (2.1) and the resulting expression into (2.6), we obtain

$$-grad(p) + div\,[2\mu(p)\mathbf{D}] - \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt}. \tag{3.6}$$

We now look for special solutions in which the pressure field p is of the form

$$p = p(z). \tag{3.7}$$

It follows from (3.1) and (3.7) that (3.6) reduces to the following system of differential equations:

$$\begin{aligned} [\mu(p) f_t \phi_z]_z &= \rho f_{tt} \phi, \\ [\mu(p) f_t \psi_z]_z &= \rho f_{tt} \psi, \\ -\tfrac{dp}{dz} - \rho g &= 0, \end{aligned} \tag{3.8}$$

where we have assumed that $\mathbf{b} = g\mathbf{k}$, $\mathbf{k}$ being the unit vector in the Z-direction.

It follows from $(3.8)_3$ that

$$p = -\rho g z + k_1, \tag{3.9}$$

where k_1 is a constant, and thus immediately, by virtue of $(3.8)_1$, $(3.8)_2$, we find that

$$f_{tt} = K f_t, \tag{3.10}$$

where K is an arbitrary constant.

Introducing (3.10) into $(3.8)_1$, $(3.8)_2$, we obtain a set of *uncoupled* ordinary differential equations with variable coefficients:

$$\begin{aligned} [\mu(p)\phi_z]_z &= \rho K \phi, \\ [\mu(p)\psi_z]_z &= \rho K \psi. \end{aligned} \tag{3.11}$$

In the absence of body forces, the pressure is simply a constant and therefore we find the well known, locally viscometric motions for isotropic fluid bodies discovered by Carroll [5].

Let us consider the classical problem of the flow between two infinite boundaries such that: the boundary at $z(\equiv Z) = 0$ oscillates with velocity $U\cos(\omega t)$ and the upper boundary is free. This is a situation that may be of interest in several geological applications and that may be solved using the class of flows indicated by (3.1).

For the sake of simplicity we consider the boundary conditions

$$u(0,t) = U\cos(\omega t), \quad \frac{\partial u}{\partial z}(d,t) = 0, \tag{3.12}$$

clearly compatible with the following ansatz

$$u(z,t) = f_t(t)\phi(z), \quad \psi(z) \equiv 0. \tag{3.13}$$

Because $\cos(\omega t)$ is the real part of $\exp(i\omega t)$, if we let $f_t(t) = \exp(i\omega t)$ a solution of (3.10) is possible when (3.12) is in force, if and only if

$$K = i\omega, \tag{3.14}$$

and therefore from (3.11) we obtain the equation

$$\rho i\omega\phi = [\mu(p)\phi_Z]_Z \tag{3.15}$$

that may be supplemented by boundary conditions

$$\phi(0) = U, \quad \phi_Z(d) = 0. \tag{3.16}$$

Moreover, without lack of generality, we rewrite (3.9) as

$$p(Z) = \rho g d\left(1 - \frac{Z}{d}\right). \tag{3.17}$$

We introduce the quantities

$$\begin{aligned} &\widetilde{Z} = \tfrac{Z}{d}, \quad \widetilde{t} = \omega t, \\ &\widetilde{p} = \tfrac{p}{\rho g d}, \quad \widetilde{\phi} = \tfrac{\phi}{U}, \quad \widetilde{\mu} = \tfrac{\mu}{\mu_0}, \end{aligned} \tag{3.18}$$

and we recast equations (3.15), (3.16) and (3.17) in the *dimensionless* form

$$\left.\begin{aligned} &\left[\widetilde{\mu}(\widetilde{p})\phi_{\widetilde{Z}}\right]_{\widetilde{Z}} - i\varepsilon^2\phi = 0; \\ &\widetilde{\phi}(0) = 1, \quad \widetilde{\phi}_{\widetilde{Z}}(1) = 0; \\ &\widetilde{p}(\widetilde{Z}) = \left(1 - \widetilde{Z}\right), \end{aligned}\right\} \tag{3.19}$$

where $\varepsilon^2 = \rho\omega/\left(\mu_0 d^2\right)$ and the range of interest is $\widetilde{Z} \in [0,1]$. We point out that the various models in (2.7) and (2.8) using (3.18) can be rewritten in the form

$$\left.\begin{aligned} &(A) \quad \widetilde{\mu} = \widetilde{\alpha}_0 + \widetilde{p}, \qquad \mu_0 = \alpha > 0, \widetilde{\alpha}_0 = \tfrac{\alpha_0}{\alpha}; \\ &(B) \quad \widetilde{\mu} = \exp\left[\widetilde{\lambda}\widetilde{p}\right], \quad \mu_0 = A > 0, \widetilde{\lambda} = \lambda\rho g d > 0; \\ &(C) \quad \widetilde{\mu} = \widetilde{B}_0 + \widetilde{p}^n, \qquad \mu_0 = B\left(\rho g d\right)^n > 0, \widetilde{B}_0 = \tfrac{B_0}{B(\rho g d)^n} n > 0. \end{aligned}\right\} \tag{3.20}$$

In what follows, for the sake of convenience, we drop the tilde.

The solution of (3.19) for the classical Navier–Stokes equations is obtained by solving the simple differential equation

$$\phi_{ZZ} - i\varepsilon^2\phi = 0. \tag{3.21}$$

Taking into account $(3.19)_2$, the solution of (3.21) is

$$\phi(Z) = \frac{\cosh\left((1+i)\,\frac{\sqrt{2}}{2}\varepsilon(1-Z)\right)}{\cosh\left((1+i)\,\frac{\sqrt{2}}{2}\varepsilon\right)}\,, \tag{3.22}$$

and therefore

$$u(y,t) = Re\left(\exp(it)\frac{\cosh\left((1+i)\,\frac{\sqrt{2}}{2}\varepsilon(1-Z)\right)}{\cosh\left((1+i)\,\frac{\sqrt{2}}{2}\varepsilon\right)}\right). \tag{3.23}$$

In the case of a pressure-dependent fluid, the equations (3.11) for the various models of viscosity reported in (3.20) are given in Table 1.

Table 1—Equations for the viscosity models reported in (3.20)

(A)	$[\alpha_0 + (1-Z)]\,\phi'' - \phi' - i\varepsilon^2\phi = 0;$
(B)	$\phi'' - \lambda\phi' - i\varepsilon^2 \exp[\lambda\,(Z-1)]\,\phi = 0;$
(C)	$[B_0 + (1-Z)^n]\,\phi'' - n\,(1-Z)^{n-1}\,\phi' - i\varepsilon^2\phi = 0.$

The general solution of equation (A) is obtained by first considering the transformation

$$\xi = \alpha_0 + (1-Z) \to d\xi = -dZ, \tag{3.24}$$

and then the equation

$$\xi\phi_{\xi\xi} + \phi_\xi - i\varepsilon^2\phi = 0; \tag{3.25}$$

whose solution is given by

$$\phi\,(\xi) = k_2 J_0\left(\frac{2\varepsilon\sqrt{\xi}}{\sqrt{i}}\right) + k_3 Y_0\left(-\frac{2\varepsilon\sqrt{\xi}}{\sqrt{i}}\right). \tag{3.26}$$

The general solution of (B) is given by

$$\phi(Z) = \exp\left[\frac{\lambda}{2}\,(Z-1)\right](k_2 Y_1\,(-\eta) - k_3 J_1(\eta))\,, \tag{3.27}$$

with

$$\eta = 2\frac{\varepsilon\exp\left[\frac{\lambda}{2}\,(Z-1)\right]}{\lambda\sqrt{i}}. \tag{3.28}$$

In (3.26) and (3.27) J_0, J_1, Y_0, Y_1 are real 0 and 1 index Bessel functions of the first and second kind (see [12]), whereas k_2 and k_3 are constants of integration.

The general solution for equation (C) in Table 1 (obviously $n \neq 1$) is quite complicated. Simple solutions may be obtained only in some special case as, for example, when $n = 2$ and $B_0 = 0$. Indeed, in this case the general solution is readily obtained as

$$\phi(Z) = k_2 (Z-1)^{\gamma_-} + k_3 (Z-1)^{-\gamma_+}, \tag{3.29}$$

where

$$\gamma_\pm = \frac{\sqrt{4\varepsilon^2 i + 1}}{2} \pm \frac{1}{2}. \tag{3.30}$$

When $B_0 = 0$ it is convenient to use the *linear* relationship $(3.19)_3$ to express ϕ as a function of p instead of z. In so doing, we have that

$$p^n \phi_{pp} + np^{n-1}\phi_p - i\varepsilon^2\phi = 0, \tag{3.31}$$

an equation that under the independent variable transformation

$$q = \frac{p^{1-n}}{1-n}, \tag{3.32}$$

$(n \neq 1)$ becomes

$$\phi_{qq} - i\Gamma q^m \phi = 0, \tag{3.33}$$

where

$$\Gamma = \frac{\varepsilon^2}{(m+1)^m}, \quad m = \frac{n}{1-n}. \tag{3.34}$$

In the case $m = -2$ (i.e. $n = 2$), which has been discussed already, or $m = -4$ (i.e., $n = 4/3$), where another transformation of both dependent and independent variables ($u = 1/q$ and $\Phi = \phi/q$) transforms the equation (3.33) into a constant coefficient, ordinary differential equation whose general solution is given by

$$\phi(q) = q\left(k_2 \exp\left(\frac{\sqrt{i\Omega}}{q}\right) + k_3 \exp\left(-\frac{\sqrt{i\Omega}}{q}\right)\right), \tag{3.35}$$

or going back to original dimensionless co-ordinates, we obtain that

$$\phi(Z) = \frac{k_2 \exp\left(3\varepsilon\sqrt{i}\,(Z-1)^{1/3}\right) + k_3 \exp\left(-3\varepsilon\sqrt{i}\,(Z-1)^{1/3}\right)}{(Z-1)^{1/3}}. \tag{3.36}$$

These solutions are not relevant to the specific boundary value problem (3.19).

On the other hand, the case $B_0 \neq 0$ is extremely complex. For example when $n = 2$, the equation (C) in Table 1 is written down explicitly as

$$\left(z^2 - 2z + B_0 + 1\right)\phi'' + 2(z-1)\phi' - i\varepsilon^2\phi = 0, \tag{3.37}$$

then the solutions of the quadratic equation $z^2 - 2z + B_0 + 1 = 0$ are

$$z_1 = 1 + i\sqrt{B_0}, \quad z_2 = 1 - i\sqrt{B_0}, \tag{3.38}$$

so that the substitution

$$\xi = -\frac{z - \left(1 + i\sqrt{B_0}\right)}{2i\sqrt{B_0}}, \tag{3.39}$$

transforms the equation (3.37) to a canonical form

$$\xi(1-\xi)\phi_{\xi\xi} - (2\xi - 1)\,\phi_\xi + i\varepsilon^2\phi = 0. \tag{3.40}$$

This last equation is an hypergeometric equation that may be solved using Legendre polynomials. In this case the analytical solution is so complex that it is more convenient to use a numerical method, and therefore we shall not pursue further the investigation of $B_0 \neq 0$.

To obtain the solution of the boundary value problem (3.19) in case (A) of Table 1, we introduce the notation

$$\gamma_1 = \frac{4}{\sqrt{2}}\left(\frac{\varepsilon\sqrt{\alpha_0}}{1+i}\right), \quad \gamma_2 = \frac{4}{\sqrt{2}}\left(\frac{\varepsilon\sqrt{\alpha_0+1}}{1+i}\right), \tag{3.41}$$

and imposing the boundary conditions $(3.19)_2$ we find that the integration constants in (3.26) are given by

$$\begin{aligned} k_2 &= \frac{Y_1(-\gamma_1)}{Y_1(-\gamma_1)J_0(\gamma_2)+J_1(\gamma_1)Y_0(-\gamma_2)}, \\ k_3 &= \frac{J_1(\gamma_1)}{Y_1(-\gamma_1)J_0(\gamma_2)+J_1(\gamma_1)Y_0(-\gamma_2)}. \end{aligned} \tag{3.42}$$

We point out that when $\alpha_0 \to 0$ this solution is no longer meaningful, because the function Y_1 in (3.42) blows up. This means that in the case of model (i) in (2.7), the solution of our boundary problem is not given by a rectilinear shear as in (3.1).

In Figure 1, we compare for $\varepsilon = 1$, plot (a), and $\varepsilon = 5$, plot (b), the classical solution (dotted line) with the solutions obtained when $\alpha_0 = 1$, (solid line), and when $\alpha_0 = 1/4$, (dashed line). In Figure 1 we are considering the solution of the original time-dependent boundary value problem at $t = 0$, and this comparison shows that in both situations (a) and (b) the difference between the classical and pressure-dependent viscosity models is mainly quantitative.

In case (B) it is possible to obtain the solution of the boundary value problem (3.19) considering the following expression for the integration constants in (3.27):

$$\begin{aligned} k_2 &= \frac{J_0(\gamma_1)\exp(\lambda/2)}{Y_0(-\gamma_1)J_1(\gamma_2)+Y_1(-\gamma_2)J_0(\gamma_1)}, \\ k_3 &= \frac{Y_0(-\gamma_1)\exp(\lambda/2)}{Y_0(-\gamma_1)J_1(\gamma_2)+Y_1(-\gamma_2)J_0(\gamma_1)}, \end{aligned} \tag{3.43}$$

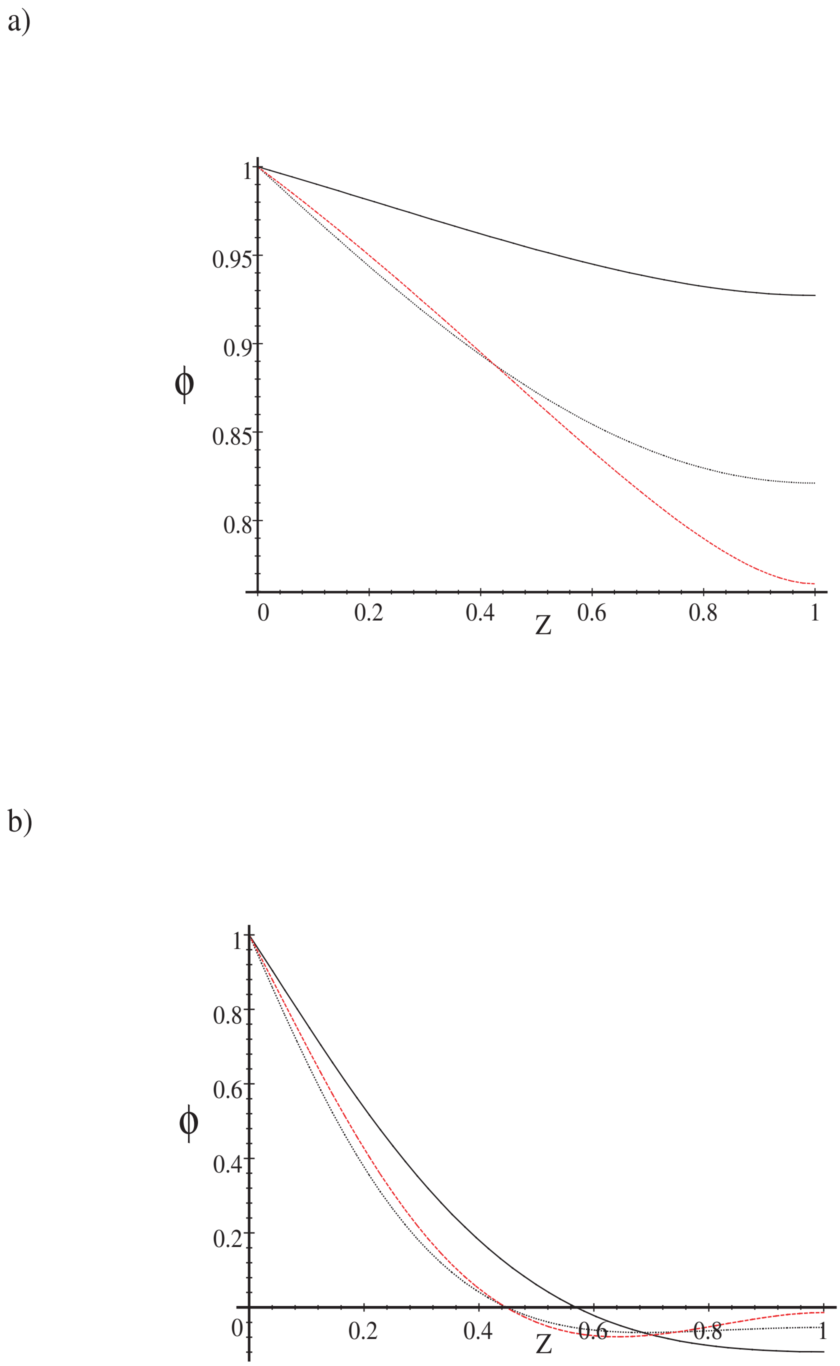

Fig. 1—The comparison between the solutions of (3.26) and the classical Navier–Stokes solution (3.22). For the numerical values used in this plot, we refer readers to the text.

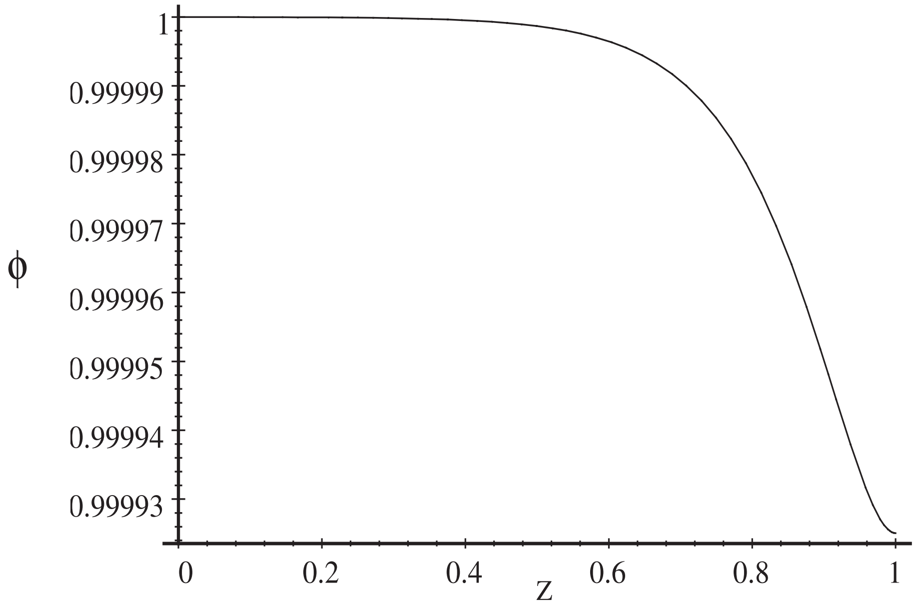

FIG. 2—Solution of (3.27) for $\lambda = 10$ and $\varepsilon = 1$.

where

$$\gamma_1 = \frac{4\varepsilon}{\sqrt{2}\lambda\,(1+i)}, \quad \gamma_2 = \frac{4\varepsilon\exp(-\lambda/2)}{\sqrt{2}\lambda\,(1+i)}. \tag{3.44}$$

In Figure 2, we plot the solution of this case for $\lambda = 10$ and $\varepsilon = 1$. We can see very clearly the appearance of a strong boundary layer which, moreover, is enhanced with increasing λ. This is an important *qualitative* difference from classical Navier–Stokes equations that may be worth exploring in detail, because it may be an explanation of real life phenomena such as shear bands in some special classes of fluids [2].

In case (C) it may be shown that when $B_0 = 0$ it is not possible to obtain solutions to our boundary value problem, because all the solutions blow-up when we impose the free boundary condition and therefore this boundary condition cannot be satisfied. This is exactly the same phenomenon we have in case (A) when $\alpha_0 \to 0$. To see this in a direct and simple way, solution (3.36) may be used. The blow up of this solution when we consider $\lim_{z\to 1}\phi_z = 0$, makes it impossible to satisfy the free surface boundary conditions for any choice of the integration constants.

This fact shows an important feature of the the models (i) and (iii) in (2.7) for which, we recall once again, $\mu(p) \to 0$ as $p \to 0$. Therefore the oscillatory motions we are considering are possible only in the presence of suitable external stress distribution.

3.1. Circularly polarized waves

Another class of solutions may be obtained by considering a generalization of the usual exponentially damped plane, circularly polarized waves characteristic of materials with linear shear response. This class of motions is defined as

$$\begin{aligned} x &= X + \phi(Z)\cos\omega t + \psi(Z)\sin\omega t, \\ y &= X + \phi(Z)\sin\omega t - \psi(Z)\cos\omega t, \\ z &= Z. \end{aligned} \tag{3.45}$$

In this case, it is convenient to carrying out the analysis by rephrasing everything in terms of the current co-ordinates. Indeed, the associated components for the velocity and the acceleration field may be written as

$$\begin{aligned} x &= -\omega\phi(z)\sin\omega t + \omega\psi(z)\cos\omega t, \\ y &= \omega\phi(z)\cos\omega t + \omega\psi(z)\sin\omega t, \\ z &= 0, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} x &= -\omega^2\phi(z)\cos\omega t - \omega^2\psi(z)\sin\omega t, \\ y &= -\omega^2\phi(z)\sin\omega t + \omega^2\psi(z)\cos\omega t, \\ z &= 0. \end{aligned} \tag{3.47}$$

The only non zero components of the stretching tensor are

$$2D_{13}(\equiv 2D_{31}) = -\omega\phi_z\sin\omega t + \omega\psi_z\cos\omega t, \tag{3.48}$$

and

$$2D_{13}(\equiv 2D_{31}) = \omega\phi_z\cos\omega t + \omega\psi_z\sin\omega t. \tag{3.49}$$

Therefore the balance equations (always with the restriction $p = p(z)$) reduce to

$$\begin{aligned} 0 &= \rho\left(\omega^2\phi(z)\cos\omega t + \omega^2\psi(z)\sin\omega t\right) \\ &+ \left[\mu(p)\left(-\omega\phi_z\sin\omega t + \omega\psi_z\cos\omega t\right)\right]_z, \end{aligned} \tag{3.50}$$

$$\begin{aligned} 0 &= \rho\left(\omega^2\phi(z)\sin\omega t - \omega^2\psi(z)\cos\omega t,\right) \\ &+ \left[\mu(p)\left(\omega\phi_z\cos\omega t + \omega\psi_z\sin\omega t\right)\right]_z, \end{aligned} \tag{3.51}$$

and

$$-p_z - \rho g = 0. \tag{3.52}$$

Once again, the presence of the external force is fundamental. This is because in this case we have the pressure being non constant, i.e. as in (3.9) $p = -\rho g z + k_1$, the solutions considered HERE are different from the ones obtained in the case of the standard Navier–Stokes model.

Therefore (3.50) and (3.52) reduce to the linear nonautonomous system

$$[\mu(p)\phi_z]_z = \rho\omega\psi(z), \quad [\mu(p)\psi_z]_z = -\rho\omega\phi(z). \tag{3.53}$$

This system may be rewritten introducing the complex function

$$\Omega = \phi + i\psi, \tag{3.54}$$

$$[\mu(p)\Omega_z]_z = -\rho\omega i\Omega. \tag{3.55}$$

The case of the classical Navier–Stokes equation has been solved by Carroll [5]. Explicit solutions of (3.55) may be found using the methods of the previous section.

3.2. Rectilinear shear

The exact solutions provided in the above analysis are only special examples of rectilinear shear. A more general ansatz than the one considered in (3.1) may be necessary to solve slightly different boundary value problems.

To give an example of this situation let us consider again the case of unsteady unidirectional motions, where $u = u(y,t)$, and $v = 0$, $w = 0$. In this case, the balance equations under the assumption of $p = p(z,t)$ reduce to

$$\rho u_t = [\mu(p)u_z]_z, \quad p = -\rho g z + p_0(t). \tag{3.56}$$

If we consider the case of p_0 being constant in time, we have an alternative ansatz to the one already considered, namely

$$u(z,t) = V(z) + U(z)T(t). \tag{3.57}$$

In this case we see $(3.56)_1$ reduces to

$$\rho U T_t = [\mu(p)(V_z + U_z T)]_z. \tag{3.58}$$

If we consider the case where

$$\mu(p)V_z = C_1, \tag{3.59}$$

it is always possible to solve (3.59) subject to the boundary conditions $V(0) = U$ and $V(d) = 0$. This means that the ansatz (3.57) may be used, for example, to solve the problem of the flow between two rigid boundaries (at distance d), one of which is suddenly started. The solution of this problem may be given in the form

$$u(z,t) = V(z) + \sum \exp(K_n t)U_n(z), \tag{3.60}$$

where

$$\rho U_n K_n = \left[\mu(p) U_{n,z}\right]_z\,, \tag{3.61}$$

and the boundary conditions $U_n(0) = U_n(d) = 0$, are imposed.

Another example is obtained when the pressure is given in the special form

$$p = -\rho g(z + ct). \tag{3.62}$$

In this situation, some solutions of (3.56) may be obtained requiring

$$u = F(\xi), \quad \xi = z + ct. \tag{3.63}$$

Introducing (3.63) in $(3.56)_1$ we obtain the following ordinary differential equation

$$\rho F' = \left[\mu(\xi) F'\right]'\,, \tag{3.64}$$

where the prime denotes differentiation with respect to ξ. This equation may be integrated after first obtaining a simple first-order, linear, ordinary differential equation

$$\mu(\xi) F' = \rho F + k_2, \tag{3.65}$$

(here k_2 is an integration constant).

4. Non-Newtonian fluid with pressure dependent viscosity

Finally, we will consider the possibility of a generalization to a non-Newtonian fluid with a pressure-dependent viscosity, namely the homogeneous incompressible fluid of second grade, by allowing its viscosity to depend on the Lagrange multiplier. It should be observed that unlike the case of an incompressible Navier–Stokes fluid, the Lagrange multiplier p is not the mean normal stress. This becomes obvious when one considers the Cauchy stress tensor, $\mathbf{T}$, for such a fluid, which is given by

$$\mathbf{T} = -p\mathbf{I} + \mu(p)\mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2. \tag{4.1}$$

Since $\mathrm{tr}(\mathbf{A}_1^2)$and $\mathrm{tr}(\mathbf{A}_2)$ are not zero, we immediately recognize that $p \neq -\frac{1}{3}\mathrm{tr}(\mathbf{T})$. Here we use the following notation: $\mathbf{A}_1 = 2\mathbf{D}$ and

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T \mathbf{A}_{n-1}. \tag{4.2}$$

Let us consider steady flows of the form $u = u(y), \quad v = 0, \quad w = 0$, where u, v and w are the x, y and z components of the velocity field, respectively. Let us further suppose that $p = p(y)$. As a specific example, let us study the flow down an inclined plane that makes an angle θ with the horizontal $x-$axis. Then, it follows that

$$(\mathbf{A}_1)_{ij} = \begin{pmatrix} 0 & u' & 0 \\ u' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left(\mathbf{A}_1^2\right)_{ij} = \begin{pmatrix} (u')^2 & 0 & 0 \\ 0 & (u')^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.3}$$

$$(\mathbf{A}_2)_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2u'^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the general case of a second-grade fluid it is not possible to reduce our problem to a quadrature, but in an any case it is possible to reduce the problem to the solution of a simple second-order, ordinary differential equation. Indeed, the non-trivial balance equations (considering $p = p(y)$) are

$$\begin{aligned} [\mu(p)u']' + \rho g \sin\theta &= 0, \\ -p_y + \left[(2\alpha_1 + \alpha_2)\, u'^2\right]' - \rho g \cos\theta &= 0. \end{aligned} \tag{4.4}$$

From $(4.4)_2$ we recover

$$p(y) = \left[(2\alpha_1 + \alpha_2)\, u'^2\right] - \rho g \cos\theta y + C_1, \tag{4.5}$$

and therefore from $(4.4)_1$

$$\mu(p)u' = -\rho g \sin\theta y + C_2. \tag{4.6}$$

The boundary conditions are given by no slip at the wall, i.e. $u(0) = 0$, and the free surface conditions $T_{yy}|_{y=h} = p_0$, $T_{yx}|_{y=h} = 0$, $T_{yz}|_{y=h} = 0$, where p_0 is the atmospheric pressure. Thus in the case of a fluid of second grade, we have

$$u(0) = 0, \quad p(h) - \left[(2\alpha_1 + \alpha_2)\, u'^2\right]_{y=h} = p_0, \quad \mu(p)u'|_{y=h} = 0. \tag{4.7}$$

Because $\mu(p) \neq 0$ for any value of p, from $(4.7)_3$ we obtain $u'(h) = 0$ and therefore we have that

$$\begin{aligned} p(y) &= (2\alpha_1 + \alpha_2)\, u'^2 - \rho g \cos\theta(y - h) + p_0, \\ \mu(p)u' &= -\rho g \sin\theta(y - h). \end{aligned} \tag{4.8}$$

The equation (4.8) is not in normal form and therefore it is not possible to reduce this equation to a quadrature; but it is possible to compute numerically the (real positive) value of $u'(0)$, say u_0', and therefore to solve by usual numerical methods the initial value problem (in normal form)

$$u'' = \frac{\rho g \cos\theta \frac{d\mu}{dp} u' - \rho g \sin\theta}{\mu(p) + 2(2\alpha_1 + \alpha_2)\frac{d\mu}{dp} u'^2},$$
$$u(0) = 0, \quad u'(0) = u'_0. \tag{4.9}$$

It is important to point out that this example also shows the crucial role played by the external force in determining the response of the material.

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