ON THE OCCURRENCE OF SOME FINITE GROUPS IN THE CENTRAL AUTOMORPHISM GROUP OF FINITE GROUPS

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Abstract

In this paper we first provide a number of results on the central automorphism group $\mathrm{Aut}_c(G)$ of a finite group G, which enable us to study the occurrence of some finite groups in the central automorphism group of finite groups. We then completely determine the finite solutions of the equation $\mathrm{Aut}_c(X) \cong J$, where $J \in \{Q_n, D_n, L(pq)\}$.

1. Introduction

Let G be a group. We let $\mathrm{Aut}(G)$ denote the group of all automorphisms of G. An automorphism σ of G is said to be *central* if σ commutes with every inner automorphism of G, or equivalently $x^{-1}\sigma(x)$ lies in the centre $Z(G)$ of G for every $x \in G$. The central automorphisms form a normal subgroup, denoted by $\mathrm{Aut}_c(G)$, of the full automorphism group $\mathrm{Aut}(G)$.

The central automorphism group of a finite group is of great importance in studying the full automorphism group of the group. There have been a number of results on the central automorphism group of a finite group, for examples see [1; 2; 3; 5; 6; 8].

In [4], Flannery and MacHale investigated finite groups which can occur as the automorphism group of a finite group. Inspired by this work, it is natural to ask whether or not there exists a finite group G whose central automorphism group is isomorphic to a given group J. The case when J is elementary abelian, or more generally homocyclic, was treated in [5; 6].

In this paper, we first produce a number of general results on the central automorphism group of a finite group to be used in our study. We then investigate the equation $\mathrm{Aut}_c(X) \cong J$, when $J \in \{Q_n, L(pq), D_n\}$, where Q_n and D_n are the dicyclic group of order $4n$ and the dihedral group of order $2n$,

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*Corresponding author, e-mail: jamali@saba.tmu.ac.ir respectively, and $L(pq)$ is the non-abelian group of order pq, where p and q are primes with $p > q$ such that q divides $p-1$.

In particular, we shall show that the following equations have no finite solutions:

$$\begin{aligned}&\mathrm{Aut}_c(X) \cong Q_n \quad (n \geq 2),\\ &\mathrm{Aut}_c(X) \cong D_n \quad (n \geq 5, n \neq 6),\\ &\mathrm{Aut}_c(X) \cong L(pq) \quad ((p,q) \neq (3,2)).\end{aligned}$$

Throughout the paper, all groups are supposed to be finite. All unexplained notation is standard and follows that of Gorenstein [7]. If $G = H_1 \times \ldots \times H_n$, we let π_i denote the projection of G on the i-th direct factor of G. The order of x is denoted by $o(x)$. Finally, for a finite abelian group G, and a prime p, Gp denotes the unique Sylow p-subgroup of G.

2. Preliminary and basic results

Let σ be a central automorphism of G. Clearly the map $f_\sigma : x \mapsto x^{-1}\sigma(x)$ defines a homomorphism from G into $Z(G)$. On the other hand, the map $\sigma_f : x \mapsto xf(x)$ defines an endomorphism of G for all f in $\mathrm{Hom}(G, Z(G))$. This endomorphism is a central automorphism if and only if $f(x) \neq x^{-1}$ for every x in $G - \{1\}$. In [1] Adney and Yen investigated conditions under which the mappings $\sigma \mapsto f_\sigma$ and $f \mapsto \sigma_f$ are inverses of each other, so that a bijection between $\mathrm{Aut}_c(G)$ and $\mathrm{Hom}(G, Z(G))$ exists. It was shown in [1, theorem 1] that a necessary and sufficient condition for a finite group G to have such a bijection is that G has no non-trivial abelian direct factors. Such groups are called *purely non-abelian* groups.

As any homomorphism $f : G \longrightarrow Z(G)$ induces a homomorphism $\bar{f} : G/G' \longrightarrow Z(G)$, and vice versa, we see that $|\mathrm{Hom}(G, Z(G))| = |\mathrm{Hom}(G/G'), Z(G)|$. Accordingly, if G is a purely non-abelian p-group, then $\mathrm{Aut}_c(G)$ is also a p-group, by using [3, lemma C].

Let G be a purely non-abelian group and $f, g \in \mathrm{Hom}(G, Z(G))$, then $\sigma_f\sigma_g = \sigma_g\sigma_f$ if and only if $f \circ g = g \circ f$. In particular, if $\mathrm{Im} f \leq G'$ for all f in $\mathrm{Hom}(G, Z(G))$, then $\mathrm{Aut}_c(G)$ is abelian.

We recall that if $H \leq Z(G)$ and K/G' is a direct factor of G/G', then any element f of $\mathrm{Hom}(K/G', H)$ induces an element $\bar{f}$ of $\mathrm{Hom}(G/G', H)$ which is trivial on the complement of K/G' in G/G'. To simplify the notation we shall identify f with the corresponding homomorphism from G into H which is induced by $\bar{f}$. Throughout the paper, we shall make use of this convention without any further explanation.

We give below some basic results that are needed for our investigation of the equation $\mathrm{Aut}_c(X) \cong J$.

Lemma 2.1. *Let G be a finite purely non-abelian group and $H \leq Z(G)$. Suppose that $G/G' = K/G' \times L/G'$.*

(i) $\mathrm{Aut}_c(G)$ *has a subgroup A of order* $|\mathrm{Hom}(K/G', H)|$.

(ii) *If* $H \leq L$, $\mathrm{Aut}_c(G)$ *has a subgroup isomorphic to* $\mathrm{Hom}(K/G', H)$.

PROOF. (i) We simply set $A = \{\sigma_f | f \in \mathrm{Hom}(K/G', H)\}$.
(ii) As H is contained in the kernel of each homomorphism in $\mathrm{Hom}(K/G', H)$, we see that $f \circ g = 0$ for all f, g in $\mathrm{Hom}(K/G', H)$. Hence, A is abelian and the mapping $\psi : f \mapsto \sigma_f$ is an isomorphism from $\mathrm{Hom}(K/G', H)$ onto A, so the result follows. ■

Corollary 2.2. *Let* G *be a purely non-abelian group. Then* $\mathrm{Aut}_c(G)$ *has an abelian subgroup isomorphic to* $\mathrm{Hom}(G/G', G' \cap Z(G))$.

Corollary 2.3. *Let* G *be a finite purely non-abelian group and let* A *and* B *be central subgroups of* G. *Suppose that* $G/G' = H/G' \times K/G'$. *If* $A \cap B = 1$, *then*

(i) $\mathrm{Aut}_c(G)$ *has subgroups of orders* $|\mathrm{Hom}(G/G', A)|$ *and* $|\mathrm{Hom}(G/G', B)|$ *intersecting trivially,*

(ii) $\mathrm{Aut}_c(G)$ *has subgroups of orders* $|\mathrm{Hom}(H/G', A)|$ *and* $|\mathrm{Hom}(K/G', A)|$ *intersecting trivially.*

Lemma 2.4. *Let* G *be a finite group and let* $G/G' = H/G' \times K/G'$. *Suppose that* A *and* B *are central subgroups of* G *with* $A \leq H$, $B \leq K$ *and* $A \cap B = 1$. *Set*

$R = \{\sigma_f | f \in \mathrm{Hom}(H/G', A), f(x) \neq x^{-1} \text{ for every } x \in H - \{1\}\}$,
$S = \{\sigma_f | f \in \mathrm{Hom}(K/G', B), f(x) \neq x^{-1} \text{ for every } x \in K - \{1\}\}$,
$T = \{\sigma_f | f \in \mathrm{Hom}(H/G', B)\}$,
$U = \{\sigma_f | f \in \mathrm{Hom}(K/G', A)\}$.

Then

(i) R, S, T, *and* U *are all subgroups of* $\mathrm{Aut}_c(G)$ *having mutually trivial intersections,*

(ii) T *and* U *are abelian and* $[R, S] = 1$,

(iii) $R \cup S \leq \mathcal{N}_{\mathcal{A}}(T) \cap \mathcal{N}_{\mathcal{A}}(U)$, *where* $\mathcal{A} = \mathrm{Aut}_c(G)$.

PROOF. (i)–(ii) These are obvious.
(iii) Let $\sigma_g \in R$, $\sigma_h \in T$, and set $\tau = \sigma_g^{-1}\sigma_h\sigma_g$. We then have $K \leq \mathrm{Ker} f_\tau$ because σ_g and σ_h fix K elementwise. Now since $\tau(x) = f_\tau(x)x = h(\sigma_g(x))x$ for all x in G, $f_\tau(x) \in B$. Thus $f_\tau \in \mathrm{Hom}(H, B)$ and so R normalizes T. Next, we let $\sigma_g \in R$ and $\sigma_h \in U$. Since σ_h fixes H elementwise, on setting $\tau = \sigma_g\sigma_h\sigma_g^{-1}$, we find that $\tau(y) = y$ for all $y \in H$ because $\sigma_g(H) = H$. Therefore $H \leq \mathrm{Ker} f_\tau$. Now $f_\tau(x)x = \tau(x) = g(h(\sigma_g^{-1}(x)))h(\sigma_g^{-1}(x))x$, for all x in G. It follows that $f_\tau(x) \in A$, and hence $f_\tau \in \mathrm{Hom}(K/G', A)$, which completes the proof. ■

Remark. In Lemma 2.4 above, if $G = H \times K$, such that H is abelian and K is purely non-abelian, then $G/G' = H \times K/K'$, and hence R, S, T, and U are defined simply by substituting H and K/K' for HG'/G' and KG'/G', respectively. In this case, if we take $A = H$ and $B = Z(K)$ then clearly $R = \mathrm{Aut}(H)$ and $S = \mathrm{Aut}_c(K)$.

Lemma 2.5. *With the notation and assumption of Lemma 2.4, if* $(|T|, |U|) = 1$, *then* $[T, U] = 1$.

PROOF. Suppose that $f \in \mathrm{Hom}(H/G', B)$ and $g \in \mathrm{Hom}(K/G', A)$. Let $y \in H$ and $d = o(g \circ f(yG'))$. Clearly d divides both $|H/G'|$ and $|B|$; so $d\big||T|$. Similarly $d\big||U|$, and hence $d = 1$. Now $f \circ g(yG') = 1 = g \circ f(yG')$. Using the same argument, we observe that $f \circ g(kG') = g \circ f(kG') = 1$, for all k in K. Therefore $f \circ g = g \circ f$ since $G = HK$. ■

We shall need the following key result in the next section.

Theorem 2.6. *If G is a finite, purely non-abelian group, then* $\mathrm{Aut}_c(G)$ *is nilpotent.*

PROOF. Let p be a prime divisor of $|G|$. Write $G/G' = P/G' \times K/G'$ and $Z(G) = Q \times B$, where P/G' and Q are Sylow p-subgroups of G/G' and $Z(G)$, respectively. Clearly $Q \leq P$ and $B \leq K$. On setting $H = P$, $A = Q$ and using Lemma 2.4, we have $[R, S] = 1$. Since $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(P/G', Q)||\mathrm{Hom}(K/G', B)| = |R||S|$ and $(|R|, |S|) = 1$, it follows that R is a normal Sylow p-subgroup of $\mathrm{Aut}_c(G)$. ■

Remark. In the above theorem, the condition on G is sufficient but not necessary for the nilpotency of $\mathrm{Aut}_c(G)$. For example, the non-abelian group $D_8 \times \mathbb{Z}_2$ has a nilpotent central automorphism group of order 32. This can be easily checked by GAP or Proposition 2.9(i) below.

Lemma 2.7. *Let G be a purely non-abelian p-group and let $G/G' = \langle xG' \rangle \times K/G'$. Assume that $z \in Z(G)$ and $o(z)\big|o(xG')$.*

(i) *The mapping $f : x^i kG' \mapsto z^i$, where $k \in K$, is a homomorphism from G/G' into $Z(G)$.*
(ii) *The set $\{\sigma_f | f \in \mathrm{Hom}(\langle xG' \rangle, \langle z \rangle)\}$ is an abelian subgroup of* $\mathrm{Aut}_c(G)$.
(iii) *If $zG' = x^s kG'$, where $k \in K$ and s is an integer, then $p\big|s$.*

PROOF. (i)–(ii) These are easily proved.
(iii) Suppose p does not divide s. Therefore $o(xG') = o(x^s G')$, and we have

$$o(xG') \geq o(z) \geq o(zG') \geq o(x^s G') = o(xG').$$

It follows that $o(z) = o(zG') = o(xG')$ and $\langle z \rangle \cap G' = 1$. So $G = \langle z \rangle \times K$, as $x \in \langle z \rangle K$, a contradiction. ■

The homomorphism f introduced in Lemma 2.7 above will be denoted by $f_{x,z}$ throughout the remainder of the paper.

Lemma 2.8. *Let $G = H \times K$, where $H, K \leq G$. If H is abelian and K is* *purely non-abelian, then* $\pi_1\sigma|_H \in \mathrm{Aut}(H)$ *and* $\pi_2\sigma|_K \in \mathrm{Aut}_c(K)$ *for all* σ *in* $\mathrm{Aut}_c(G)$.

PROOF. Let $\sigma \in \mathrm{Aut}_c(G)$ and $k \in K$. We may write $\sigma(k) = k'h$, where $k' \in K$ and $h \in H$. Then $k^{-1}k' \in Z(K)$. On setting $\tau = \pi_2\sigma|_K$, we have $\tau \in \mathrm{Hom}(K,K)$ and $k^{-1}\tau(k) \in Z(K)$. Now $f : k \mapsto k^{-1}\tau(k)$ defines a homomorphism from K into $Z(K)$, and hence $\tau = \sigma_f$ lies in $\mathrm{Aut}_c(K)$. Next let $\sigma \in \mathrm{Aut}_c(G)$ and $\eta = \pi_1\sigma|_H$. Suppose that $\mathrm{Ker}\eta \neq 1$; so $\eta(h) = 1$, for some $h \in H - \{1\}$. Therefore there is a non-trivial element k in K, such that $\sigma(h) = k$. It follows that k belongs to the kernel of $\pi_2\sigma^{-1}|_K$ contrary to the first part of the proof. ■

The following proposition is used in [8] implicitly without a proof.

Proposition 2.9. *Let* G *satisfy the conditions of Lemma 2.8, and let* $A = H$, $B = Z(K)$. *Using the notation of Lemma 2.4 we have*

(i) $|\mathrm{Aut}_c(G)| = |R||S||T||U|$,

(ii) $\mathrm{Aut}_c(G) = RSTU$.

PROOF. (i) It is easy to see that the map $\varphi : R \times S \times T \times U \longrightarrow RSTU$ defined by $(\sigma_f, \sigma_g, \sigma_h, \sigma_k) \mapsto \sigma_f\sigma_g\sigma_h\sigma_k$ is one-to-one, so that $|R||S||T||U| \leq |\mathrm{Aut}_c(G)|$. Next we consider the map $\psi : \mathrm{Aut}_c(G) \longrightarrow R \times \mathrm{Hom}(H, Z(K)) \times \mathrm{Hom}(K, H) \times S$ defined by $\sigma \mapsto (\pi_1\sigma_1, \pi_2\sigma_1, \pi_1\sigma_2, \pi_2\sigma_2)$, where σ_1 and σ_2 are the restrictions of σ to H and K, respectively. By Lemma 2.8, ψ is well-defined and one-to-one. Since $|U| = |\mathrm{Hom}(K, H)|$ and $|T| = |\mathrm{Hom}(H, Z(K)|$, we deduce that $|R||S||T||U| \geq |\mathrm{Aut}_c(G)|$ as required.
(ii) This is obvious from (i). ■

3. The equations $\mathrm{Aut}_c(X) \cong J$ where $J \cong Q_n$, $L(pq)$, or D_n

In this section, for the group $J \in \{Q_n, L(pq), D_n\}$ we produce all finite groups G such that $\mathrm{Aut}_c(G) \cong J$.

Before proceeding further we collect some basic properties of the groups D_n and Q_n to be used in the rest of the paper. Consider the groups

$$Q_n = \langle a, b | a^2 = b^n, a^{-1}ba = b^{-1}\rangle \quad (n \geq 2),$$
$$D_n = \langle a, b | a^2 = b^n = (ab)^2 = 1\rangle \quad (n > 2),$$

then we have the following elementary lemma:

Lemma 3.1. *We have:*

(i) Q_n *has a single element of order 2, and hence any two 2-subgroups of* Q_n *have a non-trivial intersection;*

(ii) *any subgroup of* Q_n *containing a Sylow 2-subgroup of* Q_n *is either cyclic or dicyclic;*

(iii) *if* x *is an involution in* D_n*, then* x *is central if and only if* $|\mathcal{C}_{D_n}(x)| > 4$*;*

(iv) *any two 2-subgroups of* D_n *with orders at least 4 have non-trivial intersection;*

(v) D_n *has a non-trivial direct factor if and only if* n *is even and* $n/2$ *is odd; in this case* $D_n \cong \mathbb{Z}_2 \times D_{n/2}$*; and*

(vi) *if* $n \neq 4$ *then each non-cyclic abelian subgroup of* D_n *is non-normal elementary abelian of order 4.*

We now begin with Q_n, and show that Q_n cannot occur as the central automorphism group of a finite group.

Theorem 3.2. *There is no finite group* G *such that* $\mathrm{Aut}_c(G) \cong Q_n$.

PROOF. It is convenient to consider three cases:

(a) *G is abelian.*

So $\mathrm{Aut}_c(G) = \mathrm{Aut}(G)$, and there is no finite group G with $\mathrm{Aut}(G) \cong Q_n$ by [4, theorem 4].

(b) *G is purely non-abelian.*

According to Theorem 2.6 above, $\mathrm{Aut}_c(G)$ is nilpotent and hence n is a power of 2. Now by Corollary 2.3 and Lemma 3.1(i), it follows that G/G' and $Z(G)$ have cyclic Sylow 2-subgroups. Hence $\mathrm{Aut}_c(G)$ is abelian by Lemma 2.7(ii), which is impossible.

(c) *G is neither abelian nor purely non-abelian.*

We may write $G = H \times K$, where H is abelian and K is purely non-abelian. Using the notation of Proposition 2.9, $|\mathrm{Aut}_c(G)| = |R||S||T||U|$, where by the remark after Lemma 2.4, $R = \mathrm{Aut}(H)$ and $S = \mathrm{Aut}_c(K)$. Since R, S, T, and U have mutually trivial intersections, one of them contains a Sylow 2-subgroup of $\mathrm{Aut}_c(G) (\cong Q_n)$. First, suppose that $|R|$ is even. Now R must be abelian by the case (a) and consequently $\mathrm{Aut}(H) = R \cong \mathbb{Z}_4$ by Lemma 3.1(ii), which is a contradiction. Next, assume that 2 divides either $|T|$ or $|U|$. Since U and T are abelian, H must have an element of order 4 by Lemma 3.1(ii), from which we conclude that $|R|$ is even, again a contradiction. Finally, if $|S|$ is even then $|H|$ is odd since $|T| = |\mathrm{Hom}(H, Z(K))|$. Hence $R = \mathrm{Aut}(H)$ has an element of order 2, which is impossible. ■

We now study the occurrence of the group $L(pq)$ in the central automorphism group of finite groups.

Proposition 3.3. *Let* G *be a finite group having no non-trivial direct factor* N *such that* $\mathrm{Aut}_c(G) \cong \mathrm{Aut}_c(G/N)$*. If* $\mathrm{Aut}_c(G) \cong L(pq)$*, then* G *is isomorphic to either* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_6 \times K$*, where* K *satisfies the following conditions:*

(i) *K is a purely non-abelian group with* $\mathrm{Aut}_c(K) = 1$*,*

(ii) K/K' *and* $Z(K)$ *are of odd order and one of them has a cyclic Sylow 3-subgroup.*

PROOF. We first observe, by Theorem 2.6, that G is not purely non-abelian.

Suppose that G is abelian. In view of [4, theorem 3], $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and hence $\mathrm{Aut}(G) \cong S_3$. Now we assume that G is neither abelian nor purely non-abelian. We may write $G = H \times K$, with H and K being abelian and purely non-abelian, respectively. By Lemma 2.4, $\mathrm{Aut}_c(H) \times \mathrm{Aut}_c(K) \leq L(pq)$. However, any non-trivial subgroup of $L(pq)$ is cyclic, so that $\mathrm{Aut}(H) = 1$ or $\mathrm{Aut}_c(K) = 1$. By the hypothesis, $\mathrm{Aut}(H) \ncong L(pq)$. Therefore $\mathrm{Aut}(H)$ is either trivial or cyclic of prime order. According to [4, theorem 1(b)], H is isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, or $\mathbb{Z}_6$. If $H \ncong \mathbb{Z}_2$, then $p = 3$ and $|\mathrm{Hom}(H, Z(K))| = 3$ or $|\mathrm{Hom}(K/K', H)| = 3$. Consequently, $\mathrm{Aut}_c(G) \cong S_3$ and hence $G \cong \mathbb{Z}_6 \times K$, where $\mathrm{Aut}_c(K) = 1$ and K/K', $Z(K)$ have cyclic Sylow 3-subgroups, and either $(|Z(K)|, 6) = 3$ or $(|K/K'|, 6) = 3$. Now we suppose that $|H| = 2$. Thus $|\mathrm{Hom}(H, Z(K)||\mathrm{Hom}(K/K', H)| = 2$. By Lemma 2.4 (iii), $\mathrm{Aut}_c(G)$ has a normal subgroup of order 2 which is central; this is again a contradiction. ■

Using the above proposition we arrive at the following theorem:

Theorem 3.4. *If* $(p, q) \neq (3, 2)$*, then there is no finite group* G *such that* $\mathrm{Aut}_c(G) \cong L(pq)$.

The following discussion is devoted to the equation $\mathrm{Aut}_c(X) \cong D_n$, where $n > 2$.

Lemma 3.5. *Let* G *be a finite abelian group such that* $\mathrm{Aut}(G) \cong D_n$. *Then* $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathrm{Aut}(G) \cong S_3$ *or* $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathrm{Aut}(G) \cong D_4$, *or* $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathrm{Aut}(G) \cong D_6$.

Proof. We distinguish two cases:

(a) *G is a p-group.*

Let $G = H_1 \times \ldots \times H_t$ be a direct decomposition of G into cyclic factors. Clearly $t \geq 2$. We first suppose that $p > 2$. Then $\mathrm{Aut}(H_i)$ has an element of order 2 for all i. Suppose σ is an element of order 2 in some $\mathrm{Aut}(H_i)$ which is non-central in D_n. We then have $\Pi\mathrm{Aut}(H_i) \leq C_{D_n}(\sigma)$. As $|C_{D_n}(\sigma)| \leq 4$ by Lemma 3.1(iii), we find that $t = 2$ and $H_1 = H_2 \cong \mathbb{Z}_3$. Thus $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathrm{Aut}(G) \cong GL(2, 3)$, a contradiction. Next, we assume that $p = 2$ and $G = H \times K$, where H and K are non-trivial. By Lemma 2.4, we see that $\mathrm{Aut}(G)$ has subgroups of orders $|\mathrm{Hom}(H, K)|$ and $|\mathrm{Hom}(K, H)|$, which have trivial intersection. Thus $|\mathrm{Hom}(H, K)| = |\mathrm{Hom}(K, H)| = 2$ by Lemma 3.1(iv). Let $H \cong \mathbb{Z}_2$ and $K \cong \mathbb{Z}_{2^j}$. If $j \geq 3$ then on setting $A = H$, $B = K$ and using the notation of Lemma 2.4, $U \lhd SU$, $T \lhd ST$. Since $|U| = |T| = 2$, we have $SU \leq \mathcal{C}(U)$, $ST \leq \mathcal{C}(T)$ and hence U, T are central by Lemma 3.1(iii); but this is impossible. Whence $j \leq 2$ and $K \cong \mathbb{Z}_2$ or $K \cong \mathbb{Z}_4$. Now

$$\mathrm{Aut}(G) \cong \begin{cases} S_3 & if\ G \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \\ D_4 & if\ G \cong \mathbb{Z}_2 \times \mathbb{Z}_4. \end{cases}$$

(b) *G is not a p-group.*

In this case $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ by Lemma 3.1(v) and part (a) of Lemma 3.5. Now evidently $\mathrm{Aut}_c(G) \cong D_6$. ■

Proposition 3.6. *Let G be a finite purely non-abelian group such that $\mathrm{Aut}_c(G) \cong D_n$. Then $n = 4$ and either $(G/G')_2 = \langle aG' \rangle \cong \mathbb{Z}_4$, $Z(G)_2 = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^m}$, where $xG' = a^2G'$, or $(G/G')_2 = \langle aG' \rangle \times \langle bG' \rangle \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_2$, $Z(G)_2 = \langle y \rangle \cong \mathbb{Z}_4$, where $yG' = a^r bG'$ for some even integer r.*

Proof. By Theorem 2.6, $\mathrm{Aut}_c(G)$ is nilpotent and hence n is a power of 2. Let H/G' and K be Sylow 2-subgroups of G/G' and $Z(G)$ respectively. Clearly $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(H/G', K)|$. It follows, by Corollary 2.3 and Lemma 3.1(iv), that H/G' is cyclic and $K \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^m}$, or K is cyclic and $H/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^m}$. We give a proof for the former case. The latter case is treated similarly. Suppose that $H/G' = \langle aG' \rangle \cong \mathbb{Z}_{2^t}$ and $K = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^m}$. We choose an element z in $\langle y \rangle$ such that $o(z) = (o(y), o(aG'))$. Assume that $zG' = a^r G'$ and $xG' = a^{2^{t-1}s}G'$. By Lemma 2.7(iii), r is even. Taking $f = f_{a,x}$ and $g = f_{a,z}$, we have

$$(f \circ g)(aG') = f(zG') = f(aG')^r = 1,$$
$$(g \circ f)(aG') = g(xG') = g(a^{2^{t-1}s}G') = z^{2^{t-1}s}.$$

Since $f \circ g \neq g \circ f$, we must have $z^{2^{t-1}s} \neq 1$. It follows that s is odd, $t \leq m$, and $o(z) = o(aG') = 2^t$. Now if $t \geq 3$, then by considering $h_i = f_{a,z^{2i}}$, where $1 \leq i \leq 2^{t-1}$, we get $f \circ h_i(aG') = f(z^{2i}G') = (z^{2^{t-1}s})^{2i} = 1$ and $h_i \circ f(aG') = h_i(a^r G') = 1$. Consequently, $[\sigma_f, \sigma_{h_i}] = 1$ and $\sigma_f \in Z(D_n)$ by Lemma 3.1(iii). We now write $D_n \cong \langle \sigma_f \rangle \times T$, where $T = \{\sigma_h | h \in \mathrm{Hom}(H/G', \langle y \rangle)\}$. By Lemma 2.7(ii), T is abelian, which is a contradiction. Hence $t \leq 2$ and $H/G' \cong \langle aG' \rangle \cong \mathbb{Z}_4$, $K \cong \langle x \rangle \times \langle y \rangle$, where $o(y) \geq 4$. ■

Proposition 3.7. *Let G be a non-abelian finite group having no non-trivial direct factor N such that $\mathrm{Aut}_c(G) \cong \mathrm{Aut}_c(G/N)$. If $\mathrm{Aut}_c(G) \cong D_n$ and G has a non-trivial abelian direct factor, then G is isomorphic to one of the following groups:*

(i) *$\mathbb{Z}_{12} \times K$, where the Sylow 3-subgroups of K/K' and $Z(K)$ are cyclic, $(|Z(K)|, |K/K'|) = 1$ and $|Z(K)||K/K'|$, being odd, is divisible by 3. In this case $n = 6$.*

(ii) *$\mathbb{Z}_2 \times K$, where $\mathrm{Aut}_c(K) \cong \mathbb{Z}_2$ and K is a purely non-abelian group. In this case $n = 4$.*

(iii) *$\mathbb{Z}_6 \times K$, where $\mathrm{Aut}_c(K) = 1$, $(6, |Z(K)||K/K'|) = 3$ and the Sylow 3-subgroups of K/K' and $Z(K)$ are cyclic. In this case $n = 3$.*

(iv) *$\mathbb{Z}_3 \times K$, where K is a purely non-abelian group with $\mathrm{Aut}_c(K) \cong \mathbb{Z}_2$, $(3, |Z(K)||K/K'|) = 3$ and the Sylow 3-subgroups of K/K' and $Z(K)$ are cyclic. In this case $n = 6$.*

Proof. Let $G = H \times K$ with H and K being abelian and purely non-abelian, respectively. We use the notation of the remark after Lemma 2.4, and distinguish three cases:

(a) $\mathrm{Aut}(H)$ *is not abelian.*

Since any non-abelian subgroup of D_n is dihedral, $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $H = \mathbb{Z}_2 \times \mathbb{Z}_4$ or $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ by Lemma 3.5. In either case, $\mathrm{Aut}(H)$ has two distinct elements of order 2 and hence $\mathrm{Aut}(H)$ has a non-central element σ of order 2. However, in view of Lemma 2.4(ii), $\mathrm{Aut}_c(K) \leq \mathcal{C}_{D_n}(\mathrm{Aut}(H)) \leq \mathcal{C}_{D_n}(\sigma)$, which implies that $|\mathrm{Aut}_c(K)| \leq 2$. If $|\mathrm{Aut}_c(K)|$ is even, then T and U both contain subgroups of order 4 with a non-trivial intersection, which is impossible. It follows that $S = \mathrm{Aut}_c(K) = 1$ and hence, by Lemma 2.5, $[T, U] = 1$. Now according to Lemma 2.4(iii), $R = \mathrm{Aut}(H) \leq \mathcal{N}(T) \cap \mathcal{N}(U)$. So $TU \lhd RTUS = \mathrm{Aut}_c(G)$. If $TU = 1$ then $\mathrm{Aut}_c(G) = \mathrm{Aut}(H) = \mathrm{Aut}_c(G/K)$, contradicting the hypothesis. Now if $|T||U|$ is even, then on setting $M = \{\tau \in TU | \tau^2 = 1\}$, we conclude that $|M| = 4$ by Lemma 3.1(vi). On the other hand, $TU \lhd D_n$ and M is a characteristic subgroup of TU. Thus $M \lhd D_n$, which is impossible by Lemma 3.1(vi). Therefore $|TU| = 3$, and $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, from which we conclude that $\mathrm{Aut}_c(G)$ has at least four elements of order 3, again a contradiction.

(b) $\mathrm{Aut}(H)$ *is non-cyclic abelian.*

As $\mathrm{Aut}(H) \leq D_n$, we have $\mathrm{Aut}(H) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and in view of [4], H is either isomorphic to $\mathbb{Z}_{12}$ or $\mathbb{Z}_8$. We let $\sigma \in \mathrm{Aut}(H)$ be a non-central involution in $\mathrm{Aut}_c(G)$. We then have $\mathcal{C}_{D_n}(\sigma) = \mathrm{Aut}(H)$ by Lemma 3.1(iii). Now, since $\mathrm{Aut}_c(K) \leq \mathcal{C}_{D_n}(\sigma)$, we see that $|\mathrm{Aut}_c(K)| = 1$. Again if $|TU|$ is even, then by an argument similar to the one given in the first case, $M \lhd D_n$, where M is defined as before. By Lemma 3.1(vi), $|M| = 2$ and consequently $M \leq Z(D_n)$. Hence $M \times \mathrm{Aut}(H) \leq D_n$ which is a contradiction. Hence $|TU|$ is odd and we have $|TU| = 3$, $H \cong \mathbb{Z}_{12}$, and $\mathrm{Aut}_c(G) \cong D_6$. Hence the Sylow 3-subgroups of K/K' and $Z(K)$ are cyclic, $(|Z(K)|, |K/K'|) = 1$, and $|Z(K)||K/K'|$ is odd.

(c) $\mathrm{Aut}(H)$ *is cyclic.*

In this case, if $|\mathrm{Aut}(H)| > 2$ and σ is an element of order 2 in $\mathrm{Aut}(H)$, then σ lies in the centre of D_n. Since $T\langle\sigma\rangle$ and $U\langle\sigma\rangle$ are abelian, T and U contain at most one element of order 2. Then $|T||U|$ is odd; for otherwise the set $T \cup S$ would contain a central element of order 2, since $T \lhd \mathrm{Aut}(H)T$ and $U \lhd \mathrm{Aut}(H)U$, which is a contradiction. On the other hand, if τ is an element of order 2 in $\mathrm{Aut}_c(K)$, then $\mathrm{Aut}(H) \leq \mathcal{C}_{D_n}(\tau)$ and hence $\tau \in Z(D_n)$, again a contradiction. Therefore $|\mathrm{Aut}_c(K)|$ is odd and a Sylow 2-subgroup of D_n is contained in $\mathrm{Aut}(H)$, which is impossible. So $\mathrm{Aut}(H)$ is either trivial or of order 2. If $\mathrm{Aut}(H) = 1$, then $|\mathrm{Aut}_c(K)| \leq 2$. To see this, we observe that if $|\mathrm{Aut}_c(K)| > 2$, then by Lemma 3.1 (iv),(vi), $|T| \leq 2$ and $|U| \leq 2$. Arguing as above, $T \leq Z(D_n)$ and $U \leq Z(D_n)$, and accordingly $\mathrm{Aut}_c(G) = T \times U \times \mathrm{Aut}_c(K)$, which is again a contradiction, using Proposition 3.6 and Lemma 3.1(v). We therefore have $|\mathrm{Aut}_c(K)| = 2$, $H \cong \mathbb{Z}_2$, $(|K/K'|, |Z(K)|) = 2$, and the Sylow 2-subgroups of K/K' and $Z(K)$ are cyclic. In this case $\mathrm{Aut}_c(G) = D_4$. Next we consider the case when $\mathrm{Aut}(H) = \langle\sigma\rangle$, where σ is an element of order 2. If $|\mathrm{Aut}_c(K)| > 2$, then σ is central and, as above, $|TU|$ is odd. Therefore $TU \lhd \mathrm{Aut}_c(G)(\cong D_n)$, and so $\mathrm{Aut}_c(G) \cong \langle\sigma\rangle \times (TU\mathrm{Aut}_c(K))$. It follows that $|\mathrm{Aut}_c(K)|$ is even and hence $\mathrm{Aut}_c(K)$ has a central element of order 2 by Theorem 2.6, which is impossible. If $|\mathrm{Aut}_c(K)| = 2$, then $H \cong \mathbb{Z}_3$ and $K/K', Z(K)$ have cyclic Sylow 3-subgroups and $(3, |K/K'||Z(K)|) = 3$. In this case $\mathrm{Aut}_c(G) \cong D_6$. Now, suppose that $\mathrm{Aut}_c(K) = 1$. Hence $H \cong \mathbb{Z}_6$

and K/K', $Z(K)$ have cyclic Sylow 3-subgroups and $(6, |Z(K)||K/K'|) = 3$, $(|K/K'|, |Z(K)|) = 1$. In this case $\mathrm{Aut}_c(G) \cong S_3$. ■

From the above results we arrive at the following conclusion:

Theorem 3.8. *There is no finite group* G *such that* $\mathrm{Aut}_c(G) \cong D_n$*, where* $n \geq 5$ *and* $n \neq 6$.

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