ON THE PRODUCT XY FOR THE ELLIPTICALLY SYMMETRIC PEARSON TYPE VII DISTRIBUTION

By
Saralees Nadarajah*
School of Mathematics, University of Manchester, Manchester M60 1QD, UK
and
Samuel Kotz
Department of Engineering Management and Systems Engineering, The George Washington University, Washington, D.C., 20052

[Accepted 18 August 2005. Published 8 September 2006.]

Abstract

The distribution of products of random variables is of interest in many areas of the sciences. This has increased the need to have available the widest possible range of statistical results on products of random variables. In this paper, the distribution of the product XY is derived when (X, Y) has the elliptically symmetric Pearson type VII distribution. Extensive tabulations of the associated percentage points are also given.

1. Introduction

For given random variables X and Y, the distribution of the product XY is of interest in many areas of the sciences. We discuss some examples from econometrics and the social sciences.

In traditional portfolio selection models certain cases involve the product of random variables. The best examples of this are in the case of investment in a number of different overseas markets. In portfolio diversification models (see, for example, Grubel [11]), not only are prices of shares in local markets uncertain, but also the exchange rates are uncertain so that the value of the portfolio in domestic currency is related to a product of random variables. Similarly, in models of diversified production by multinationals (see, for example, Rugman [24]), there is local production uncertainty and exchange rate uncertainty, so that profits in home currency are again related to a product of random variables. An entirely different example is drawn from the econometric literature. In making a forecast from an estimated equation, Feldstein [7] pointed out that both the parameter and the value of the exogenous variable in the forecast period could be considered as random variables. Hence, the forecast was proportional to a product of random variables.

Products of random variables constitute an important class of variables in the social sciences. They are most important because of the regularity with which they

*Corresponding author, e-mail: saralees.nadarajah@manchester.ac.uk

are used to capture interaction effects in regression contexts. But they are significant also in several theoretical statements in the social sciences. Perhaps the best-known statement of this type is the Cobb-Douglas production function, which is used regularly in economics to relate outputs to inputs. In psychology, the best known example is the Atkinson [2] model for the prediction of behavior. In this model, a behavioral response is multiplicatively related to drive strength, habit strength, the incentive value of the anticipated reinforcement and the intensity of the stimulus. In theorizing about the relationship between attitudes and behavior, social psychologists Rokeach and Kliejunas [23] argue that behavior can best be predicted from the weighted product of one's attitude toward an object and one's attitude toward a situation. Sociologists Palmore and Hammond [19] argue that deviation from norms is equal to the product of barriers to legitimate opportunities and the degree of exposure to illegitimate ones.

Researchers studying job satisfaction have used product of variables in two ways in their research. First, studies by Schaffer [26], Decker [4], Ewen [5] and Waters [32] have all examined the importance of building overall satisfaction measures as the sum of separate aspects of job satisfaction multiplied by the importance of this aspect of the job for the individual. That is, the measure is a sum of products of variables. Second, job satisfaction is seen as a function of both the properties the worker perceives in a job and the value he or she attaches to each of these properties (Goldthorpe *et al.* [9]). Kalleberg [14; 15] tests several versions of this general theoretical statement; some of the models explored view job satisfaction as the sum of weighted products of variables. The variables were formed by multiplying one's perception of whether or not a job has a given property by the value one attaches to that property.

The distribution of XY has been studied by several authors, especially when X and Y are independent random variables and come from the same family. For instance, see Sakamoto [25] for uniform family; Harter [12] and Wallgren [31] for Student's t family; Springer and Thompson [27] for normal family; Stuart [29] and Podolski [20] for gamma family,; Steece [28], Bhargava and Khatri [3] and Tang and Gupta [30] for beta family; Abu-Salih [1] for power function family; and Malik and Trudel [17] for exponential family (see also Rathie and Rohrer [22] for a comprehensive review of known results). However, there is relatively little work of this kind when X and Y are correlated random variables. The only work known to the authors is that by Garg *et al.* [8] for Dirichlet family and Nadarajah [18] for Lomax family.

In this paper, we study the distribution of the product XY when (X, Y) has the elliptically symmetric Pearson type VII distribution given by the joint pdf

$$f(x,y) = \frac{N-1}{\pi m\sqrt{1-\rho^2}} \left(1 + \frac{x^2+y^2-2\rho xy}{m\left(1-\rho^2\right)}\right)^{-N} \tag{1.1}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, $N > 1$, $m > 0$, and $-1 < \rho < 1$. The bivariate t-distribution and the bivariate Cauchy distribution are special cases of (1.1) for $N = \frac{(m+2)}{2}$ and $m = 1$, $N = \frac{3}{2}$, respectively. The parameter ρ is the correlation coefficient between the x and y components. Applications of (1.1) are of increasing importance in classical as well as in Bayesian statistical modeling. It is becoming a competitor to the bivariate normal distribution for several reasons:

- (1.1) is a generalization of the classical univariate Student t distribution, which is of central importance in statistical inference.
- Application of (1.1) is a very promising approach in multivariate analysis. Classical multivariate analysis is soundly and rigidly tilted toward the multivariate normal distribution, while (1.1) offers a more viable alternative with respect to real-world data, particularly because its tails are more realistic. We have seen recently some unexpected applications in novel areas such as cluster analysis, discriminant analysis, multiple regression, robust projection indices and missing data imputation.
- (1.1) has, for the past 20 to 30 years, played a crucial role in Bayesian analysis of multivariate data. It serves by now as the most popular prior distribution (because elicitation of prior information in various physical, engineering and financial phenomena is closely associated with multivariate t distributions) and generates meaningful posterior distributions.

For details on properties of (1.1) see Johnson [13], Fang *et al.* [6], and Kotz and Nadarajah [16].

The aim of this paper is to calculate the distribution of the product XY when (X, Y) has the joint pdf (1.1). Extensive tabulations of the associated percentage points are also provided. The calculations of this paper involve the Gauss hypergeometric function defined by

$$G(\alpha; \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},$$

where $(c)_k = c(c+1)\cdots(c+k-1)$ denotes the ascending factorial. We also need the following lemmas:

Lemma 1. (See Prudnikov *et al.* [21, vol. 1, equation (2.2.9.7)].) *For $a > 0$, $b^2 < ac$ and $0 < p < 2\rho$,*

$$\int_0^{\infty} \frac{x^{p-1}}{\left(ax^2 + 2bx + c\right)^{\rho}} dx = a^{-\frac{p}{2}} c^{\frac{p}{2-\rho}} B(p, 2\rho - p) G\left(\frac{p}{2}, \rho - \frac{p}{2}; \rho + \frac{1}{2}; 1 - \frac{b^2}{ac}\right).$$

Lemma 2. (See Prudnikov *et al.* [21, vol. 1, equation (2.2.9.10)].) *For $a > 0$ and $c > 0$,*

$$\int_0^{\infty} \frac{x^{n-\frac{1}{2}}}{\left(ax^2 + 2bx + c\right)^{n}} dx = \frac{\pi (2n-3)!! \left(b + \sqrt{ac}\right)^{\frac{1}{2}-n}}{\sqrt{a} 2^{2n-\frac{3}{2}} (n-1)!},$$

provided that $b^2 < ac$ for $n = 1, 2, 3, \ldots$ or that $b^2 > ac$ for $n = 1, 3, 5, \ldots$.

Lemma 3. (See Prudnikov *et al.* [21, vol. 1, equation (2.2.9.20)].) *For $a > 0$, $b > 0$*

and $c > 0$,

$$\int_0^\infty \frac{x^n}{\left(ax^2+2bx+c\right)^{n+\frac{1}{2}}} dx = -\frac{2^{n-2}m!}{(2n-1)!!} \frac{\partial}{\partial a} \left(\frac{1}{\sqrt{c}\left(\sqrt{ac}+b\right)^{m+1}} \right).$$

Lemma 4. (See Prudnikov *et al.* [21, vol. 1, equation (2.2.9.8)].) *For* $b < a$ *and* $0 < p < 2\rho$,

$$\int_0^\infty \frac{x^{p-1}}{\left\{(x+a)^2-b^2\right\}^\rho} dx$$
$$= B(\alpha, 2\rho-\alpha)\left(a^2-b^2\right)^{\frac{\alpha}{2-\rho}} G\left(\alpha, 2\rho-\alpha; \rho+\frac{1}{2}; \frac{1}{2}\left(1-\frac{a}{\sqrt{a^2-b^2}}\right)\right).$$

Further properties of the Gauss hypergeometric function can be found in Prudnikov *et al.* [21] and Gradshteyn and Ryzhik [10].

2. PDF

Theorem 1 derives an explicit expression for the pdf of $Z = XY$ in terms of the Gauss hypergeometric function.

Theorem 1. *Suppose* X *and* Y *are jointly distributed according to (1.1). Then, the pdf of* $Z = XY$ *can be expressed as*

$$f(z) = \frac{N-1}{\pi} m^{N-1} B\left(N+\frac{1}{2}, N-\frac{1}{2}\right)\left(1-\rho^2\right)^{N-\frac{1}{2}} z^{\frac{1}{2}-N} h(z), \tag{2.1}$$

where

$$h(z) = \begin{cases} \left\{\frac{1}{2} + \frac{m\left(1-\rho^2\right)}{4 \mid z \mid} - \frac{\rho z}{2 \mid z \mid}\right\}^{\frac{1}{2}-N}, \\ \qquad \text{if } \left\{m\left(1-\rho^2\right)-2\rho z\right\}^2 \geq 4z^2, \\ G\left(\frac{N}{2}+\frac{1}{4}, \frac{N}{2}-\frac{1}{4}; N+\frac{1}{2}; 1-\frac{\left\{m\left(1-\rho^2\right)-2\rho z\right\}^2}{4z^2}\right), \\ \qquad \text{if } \left\{m\left(1-\rho^2\right)-2\rho z\right\}^2 < 4z^2 \end{cases}$$

for $-\infty < z < \infty$.

Proof. Set $(X, Y) = (X, Z/X)$. Under this transformation, the Jacobian is $\frac{1}{|X|}$ and so one can express the joint pdf of (X, Z) as

$$f(x, z) = \frac{N-1}{\pi m \sqrt{1-\rho^2} \mid x \mid} \left(1 + \frac{x^2 + z^2/x^2 - 2\rho z}{m\left(1-\rho^2\right)}\right)^{-N}$$

$$= \frac{(N-1)m^{N-1}\left(1-\rho^2\right)^{N-\frac{1}{2}} \mid x \mid^{2N-1}}{\pi\left[x^4+\left\{m\left(1-\rho^2\right)-2\rho z\right\}x^2+z^2\right]^N}. \tag{2.2}$$

Thus, the marginal pdf of Z can be written as

$$\begin{aligned}
f(z) &= \frac{(N-1)m^{N-1}\left(1-\rho^2\right)^{N-\frac{1}{2}}}{\pi}\int_{-\infty}^{\infty}\frac{\mid x\mid^{2N-1}}{\left[x^4+\left\{m\left(1-\rho^2\right)-2\rho z\right\}x^2+z^2\right]^N}dx \\
&= \frac{2(N-1)m^{N-1}\left(1-\rho^2\right)^{N-\frac{1}{2}}}{\pi}\int_{0}^{\infty}\frac{x^{2N-1}}{\left[x^4+\left\{m\left(1-\rho^2\right)-2\rho z\right\}x^2+z^2\right]^N}dx \\
&= \frac{(N-1)m^{N-1}\left(1-\rho^2\right)^{N-\frac{1}{2}}}{\pi}\int_{0}^{\infty}\frac{y^{N-\frac{1}{2}}}{\left[y^2+\left\{m\left(1-\rho^2\right)-2\rho z\right\}y+z^2\right]^N}dy \\
&= \frac{(N-1)m^{N-1}\left(1-\rho^2\right)^{N-\frac{1}{2}}}{\pi}I(z),
\end{aligned} \tag{2.3}$$

after substituting $y=x^2$. If $\{m\left(1-\rho^2\right)-2\rho z\}^2<4z^2$, then application of Lemma 1 shows that

$$\begin{aligned}
I(z) &= B\left(N+\frac{1}{2},N-\frac{1}{2}\right)z^{\frac{1}{2}-N} \\
&\quad\times G\left(\frac{N}{2}+\frac{1}{4},\frac{N}{2}-\frac{1}{4};N+\frac{1}{2};1-\frac{\left\{m\left(1-\rho^2\right)-2\rho z\right\}^2}{4z^2}\right).
\end{aligned} \tag{2.4}$$

On the other hand, if $\{m\left(1-\rho^2\right)-2\rho z\}^2\geq 4z^2$, then application of Lemma 4 shows that

$$\begin{aligned}
I(z) &= B\left(N+\frac{1}{2},N-\frac{1}{2}\right)z^{\frac{1}{2}-N} \\
&\quad\times G\left(N+\frac{1}{2},N-\frac{1}{2};N+\frac{1}{2};\frac{1}{2}-\frac{m\left(1-\rho^2\right)}{4\mid z\mid}+\frac{\rho z}{2\mid z\mid}\right).
\end{aligned} \tag{2.5}$$

Using the property that

$$G\left(a,b;a;x\right)=(1-x)^{-b},$$

(2.5) can be reduced to

$$I(z)=B\left(N+\frac{1}{2},N-\frac{1}{2}\right)z^{\frac{1}{2}-N}\left\{\frac{1}{2}+\frac{m\left(1-\rho^2\right)}{4\mid z\mid}-\frac{\rho z}{2\mid z\mid}\right\}^{\frac{1}{2}-N}. \tag{2.6}$$

The result of the theorem follows by combining (2.3), (2.4) and (2.6). ■

Figure 1 below illustrates possible shapes of the pdf (2.1) for a range of values of ρ and N. Note the change in the shape as N increases.

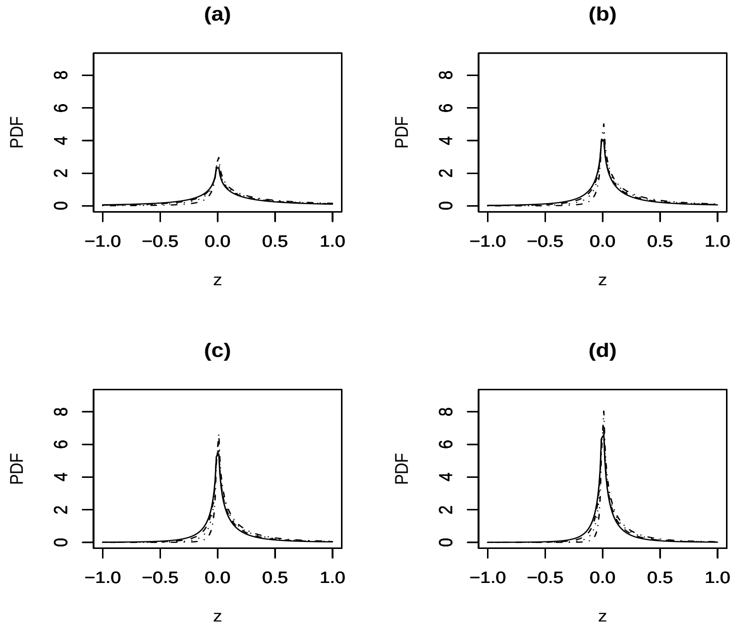

Figure 1. Plots of the pdf (2.1) for $m = 1$, and (a): $N = 2$; (b): $N = 3$; (c): $N = 4$; and (d): $N = 5$. In each plot, there are four curves: the unbroken curve ($\rho = 0.2$), the curve of dashes ($\rho = 0.4$), the curve of dots ($\rho = 0.6$), and the curve of dashes and dots ($\rho = 0.8$).

Theorems 2 and 3 below consider particular forms for the pdf of Z involving only elementary forms. Theorem 2 considers integer values for N while Theorem 3 considers half-integer values for N.

Theorem 2. *Suppose X and Y are jointly distributed according to (1.1) with $N \geq 2$ taking integer values. Then, the pdf of $Z = XY$ can be expressed as*

$$f(z) = \frac{(2N-3)!!m^{N-1}(1+\rho)^{N-\frac{1}{2}}}{2^{N-1}(N-2)!\left\{m(1+\rho)+2z\right\}^{N-\frac{1}{2}}} \tag{2.7}$$

for $-\infty < z < \infty$, provided that $\left\{m\left(1-\rho^2\right)-2\rho z\right\}^2 < 4z^2$ for $N = 2, 3, 4, \ldots$ or that $\left\{m\left(1-\rho^2\right)-2\rho z\right\}^2 > 4z^2$ for $N = 3, 5, 7, \ldots$.

Proof. The proof follows by direct application of Lemma 2 to calculate the integral in (2.3). ■

Theorem 3. *Suppose X and Y are jointly distributed according to (1.1) with $N >$*

1 *taking half integer values. Then, the pdf of* $Z = XY$ *can be expressed as*

$$f(z) = \frac{(N-1)(N-\frac{3}{2})! m^{N-1} 2^{2N-4} \left(1-\rho^2\right)^{N-\frac{1}{2}}}{\pi (2N-2)! z} \times \left\{m\left(1-\rho^2\right) + 2z\left(\rho+1\right)\right\}^{\frac{1}{2}-N} \qquad (2.8)$$

for $-\infty < z < \infty$*, provided that* $m\left(1-\rho^2\right) > 2\rho z$*.*

PROOF. By applying Lemma 3 to calculate the integral in (2.3), one can express

$$f(z) = -\frac{(N-1)(N-\frac{5}{2})! m^{N-1} 2^{2N-4} \left(1-\rho^2\right)^{N-\frac{1}{2}}}{\pi (2N-2)!} \times \frac{\partial}{\partial a}\left[z^{-1}\left\{\frac{m\left(1-\rho^2\right)}{2} + z\left(\rho+\sqrt{a}\right)\right\}^{\frac{3}{2}-N}\right]\Bigg|_{a=1}. \qquad (2.9)$$

The result of the theorem follows by taking the derivative in (2.9). ■

3. Percentiles

In the appendix, we provide tabulations of percentage points z_p associated with $Z = XY$ when (X, Y) has the joint pdf (1.1) with $N = \frac{(m+2)}{2}$ (recall that this case corresponds to the well-known t distribution which has major applications in statistics and applied sciences). The values of z_p are obtained by numerically solving the equation

$$\int_{-\infty}^{z_p} f(z)dz = p, \qquad (3.1)$$

where $f(\cdot)$ is given by (2.1). Evidently, this involves computation of the Gauss hypergeometric function and routines for this are widely available. We used the function hypergeom $(\cdot)$ in the algebraic manipulation package, MAPLE. Tables 1–20 in the appendix provide the numerical values of z_p for $p = 0.01$, 0.05, 0.1, 0.9, 0.95, 0.99, $m = 1, 2, \ldots, 20$, $N = 2$ and $\rho = 0.1, 0.2, \ldots, 0.9$. We hope these tables will be of use to the practitioners of the bivariate t distribution (see Section 1 above). Similar tabulations could easily be derived for other values of p, m, N and ρ by using the hypergeom $(\cdot)$ function in MAPLE. A program in MAPLE for computing z_p is also shown in the appendix.

ACKNOWLEDGEMENTS

The authors would like to thank the referee and the editor for carefully reading the paper and for their great help in improving the paper.

References

[1] M.S. Abu-Salih, Distributions of the product and the quotient of power-function random variables, *Arab Journal of Mathematics* **4** (1983), 77–90.

[2] J.W. Atkinson, *An introduction to motivation*, Van Nostrand, Princeton, 1964.

[3] R.P. Bhargava and C.G. Khatri, The distribution of product of independent beta random variables with application to multivariate analysis, *Annals of the Institute of Statistical Mathematics* **33** (1981), 287–96.

[4] R.L. Decker, A study of three specific problems in the measurement and interpretation of employee attributes, *Psychological Monographs* **69** (Whole No. 401), (1955).

[5] R.B. Ewen, Weighting components of job satisfaction, *Journal of Applied Psychology* **51** (1967), 68–73.

[6] K.-T. Fang, S. Kotz and K.W. Ng, *Symmetric multivariate and related distributions*, Chapman and Hall, London, 1980.

[7] M.S. Feldstein, The error of forecast in econometric models when the forecast-period exogenous variables are stochastic, *Econometrica* **39** (1971), 55–60.

[8] M. Garg, V. Katta and M.K. Gupta, The distribution of the products of powers of generalized Dirichlet components, *Kyungpook Mathematical Journal* **42** (2002), 429–36.

[9] J.H. Goldthorpe *et al.*, *The affluent worker: industrial attitudes and behavior*, Cambridge University Press, Cambridge, 1968.

[10] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products* (6th edn), Academic Press, San Diego, 2000.

[11] H.G. Grubel, Internationally diversified portfolios: welfare gains capital flows, *American Economic Review* **58** (1968), 1299–1314.

[12] H.L. Harter, On the distribution of Wald's classification statistic, *Annals of Mathematical Statistics* **22** (1951), 58–67.

[13] M.E. Johnson, *Multivariate statistical simulation*, John Wiley and Sons, New York, 1987.

[14] A.L. Kalleberg, Work values, job rewards, and job satisfaction: a theory of the equality of work experience, unpublished PhD dissertation, University of Wisconsin, 1975.

[15] A.L. Kalleberg, Work values and job rewards: a theory of job satisfaction, *American Sociological Review* **42** (1977), 124–43.

[16] S. Kotz and S. Nadarajah, *Multivariate t distributions and their applications*, Cambridge University Press, New York, 2004.

[17] H.J. Malik and R. Trudel, Probability density function of the product and quotient of two correlated exponential random variables, *Canadian Mathematical Bulletin* **29** (1986), 413–18.

[18] S. Nadarajah, Sums, products, and ratios for the bivariate Lomax distribution, *Computational Statistics and Data Analysis* **49** (2005), 109-29.

[19] E. Palmore and P.E. Hammond, Interacting factor in juvenile delinquency, *American Sociological Review* **29** (1964), 848–54.

[20] H. Podolski, The distribution of a product of n independent random variables with generalized gamma distribution, *Demonstratio Mathematica* **4** (1972), 119–23.

[21] A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev, *Integrals and series* (3 vols), Gordon and Breach Science Publishers, Amsterdam, 1986.

[22] P.N. Rathie and H.G. Rohrer, The exact distribution of products of independent random variables, *Metron* **45** (1987), 235–45.

[23] M. Rokeach and P. Kliejunas, Behavior as a function of attitude-toward-object and attitude-toward-situation, *Journal of Personality and Social Psychology* **22** (1972), 194–201.

[24] A.M. Rugman, *International diversification and the multinational enterprise*, Lexington Books, Lexington, MA., 1979.

[25] H. Sakamoto, On the distributions of the product and the quotient of the independent and uniformly distributed random variables, *Tohoku Mathematical Journal* **49** (1943), 243–60.

[26] R.H. Schaffer, Job satisfaction as related to need satisfaction at work, *Psychological Monographs* **67** (Whole No. 364), (1953).

[27] M.D. Springer and W.E. Thompson, The distribution of products of beta, gamma and Gaussian random variables, *SIAM Journal of Applied Mathematics* **18** (1970), 721–37.

[28] B.M. Steece, On the exact distribution for the product of two independent beta-distributed random variables, *Metron* **34** (1976), 187–90.

[29] A. Stuart, Gamma-distributed products of independent random variables, *Biometrika* **49** (1962), 564–65.

[30] J. Tang and A.K. Gupta, On the distribution of the product of independent beta random variables, *Statistics & Probability Letters* **2** (1984), 165–68.

[31] C.M. Wallgren, The distribution of the product of two correlated t variates, *Journal of the American Statistical Association* **75** (1980), 996–1000.

[32] L.K. Waters, The utility of importance weights in predicting overall job satisfaction and dissatisfaction, *Educational and Psychological Measurement* **29** (1969), 519–22.

Appendix

Program and Tables

The following program in MAPLE returns the value of $z_p$defined by (3.1) for given values of p, m, N and ρ.

```
cc:=((N-1)/Pi)*(m**(N-1))*Beta(N+1/2,N-1/2)*((1-rho*rho)/z)**(N-1/2):
f1:=(1/2+m*(1-rho*rho)/(4*abs(z))-rho*z/(2*abs(z)))**(1/2-N):
tt:=1-((m*(1-rho*rho)-2*rho*z)/(2*z))**2:
f2:=hypergeom([N/2+1/4,N/2-1/4],[N+1/2],tt):
g1:=int(cc*f2,z=-infinity..0):
g2:=int(cc*f1,z=0..(m/2)*(1-rho)):
g3:=int(cc*f2,z=(m/2)*(1-rho)..infinity):
if p<=g1 then ff:=int(cc*f2,z=-infinity..zz): end if:
if p>g1 and p<=g1+g2 then ff:=g1+int(cc*f1,z=0..zz): end if:
if p>g1+g2 and p<=g1+g2+g3 then ff:=g1+g2+int(cc*f2,z=(m/2)*(1-rho)..zz): end if:
percent:=fsolve(ff=p,z=-100..100):
```

The tables below have been generated using this MAPLE program for $p = 0.01$, 0.05, 0.1, 0.9, 0.95, 0.99, $m = 1, 2, \ldots, 20$, $N = 2$ and $\rho = 0.1, 0.2, \ldots, 0.9$.

Table 1—Percentage points of Z for $m = 1$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-575.7261	-22.58014	-5.39408	8.5352	35.60001	900.9405
0.2	-438.9795	-17.61385	-4.17473	10.43456	43.19773	1101.526
0.3	-329.5630	-13.25534	-3.113158	12.55992	51.95437	1296.743
0.4	-236.7208	-9.52442	-2.223868	15.17299	62.33748	1607.543
0.5	-166.3933	-6.517157	-1.495763	17.74899	71.7088	1826.277
0.6	-102.8059	-4.057788	-0.9081795	20.70867	84.17212	2102.313
0.7	-58.03747	-2.191686	-0.4718468	24.59820	101.2746	2620.975
0.8	-25.53795	-0.962922	-0.1852100	28.29269	115.4463	2918.121
0.9	-6.32683	-0.2191793	-0.02829353	33.39404	135.2549	3397.879

Table 2—Percentage points of Z for $m = 2$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-26.04568	-4.696836	-2.005748	2.842461	6.507423	36.04552
0.2	-21.73859	-3.859848	-1.632392	3.316275	7.54232	41.69504
0.3	-17.59645	-3.085113	-1.287803	3.852243	8.652078	47.59597
0.4	-14.06802	-2.414353	-0.9772882	4.375830	9.79825	53.41304
0.5	-10.65072	-1.789824	-0.7017181	4.921167	10.97575	59.48458
0.6	-7.365179	-1.253693	-0.4731813	5.557683	12.3622	66.06722
0.7	-4.74411	-0.7700781	-0.2710454	6.213188	13.72014	74.10055
0.8	-2.572563	-0.3866314	-0.1210232	6.924048	15.20092	82.14072
0.9	-0.8820174	-0.1128258	-0.02193244	7.780498	16.95016	90.70052

Table 3—Percentage points of Z for $m = 3$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-10.95778	-3.003705	-1.494204	2.081580	4.062016	14.48318
0.2	-9.263014	-2.516161	-1.238000	2.391581	4.62573	16.22175
0.3	-7.65956	-2.05102	-0.9902574	2.711098	5.177446	18.08424
0.4	-6.133325	-1.618351	-0.762322	3.066674	5.815078	20.36022
0.5	-4.741768	-1.232056	-0.5647783	3.421389	6.452203	22.25094
0.6	-3.468876	-0.871491	-0.3819387	3.804473	7.120816	24.41185
0.7	-2.333030	-0.5615504	-0.2285019	4.217935	7.832291	26.85502
0.8	-1.314351	-0.2938824	-0.1042622	4.614855	8.473997	29.1064
0.9	-0.4880983	-0.0928057	-0.02058833	5.087784	9.343676	31.30466

Table 4—Percentage points of Z for $m = 4$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-7.291827	-2.430657	-1.304847	1.786174	3.247371	9.54243
0.2	-6.236027	-2.054898	-1.087808	2.039441	3.665111	10.66123
0.3	-5.181237	-1.684054	-0.8741585	2.313084	4.111843	11.79099
0.4	-4.207570	-1.350352	-0.68135	2.592529	4.556362	12.92863
0.5	-3.294503	-1.030855	-0.5055656	2.896668	5.073729	14.31122
0.6	-2.462901	-0.7467488	-0.3481009	3.19641	5.557596	15.54597
0.7	-1.67461	-0.4869354	-0.2111714	3.512714	6.04826	16.85186
0.8	-0.976122	-0.2598604	-0.09813772	3.842068	6.599692	18.50624
0.9	-0.3817444	-0.08486976	-0.02003236	4.166598	7.108797	19.77882

Table 5—Percentage points of Z for $m = 5$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-5.769646	-2.157232	-1.200227	1.63572	2.857949	7.461198
0.2	-4.931474	-1.829176	-1.003833	1.868770	3.22469	8.36924
0.3	-4.195103	-1.507373	-0.807365	2.101478	3.586183	9.198803
0.4	-3.447246	-1.221259	-0.6389314	2.368142	4.004349	10.24292
0.5	-2.70426	-0.9395269	-0.4764418	2.612497	4.390254	11.21691
0.6	-2.027705	-0.676116	-0.3288718	2.882068	4.809029	12.06016
0.7	-1.397675	-0.4475589	-0.2010153	3.154543	5.2519	13.10798
0.8	-0.8290223	-0.2432436	-0.0957114	3.452966	5.684744	14.04334
0.9	-0.3307593	-0.08091072	-0.01977628	3.751192	6.139264	15.22259

Table 6—Percentage points of Z for $m = 6$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-4.987624	-1.998174	-1.143482	1.537006	2.614484	6.443343
0.2	-4.292285	-1.691918	-0.9540507	1.755970	2.960135	7.171303
0.3	-3.637818	-1.414151	-0.7811087	1.984197	3.311553	7.973013
0.4	-2.970815	-1.128967	-0.6080795	2.224037	3.675636	8.820244
0.5	-2.379094	-0.8758727	-0.4563549	2.448383	4.032254	9.569227
0.6	-1.794484	-0.6363859	-0.3163133	2.70081	4.399409	10.27285
0.7	-1.240988	-0.4205316	-0.1925273	2.961031	4.793095	11.18267
0.8	-0.738407	-0.2312252	-0.09183566	3.22118	5.15167	11.95975
0.9	-0.3021492	-0.07760573	-0.01925864	3.497727	5.567699	12.88463

Table 7—Percentage points of Z for $m = 7$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-4.531517	-1.886664	-1.095870	1.489097	2.485673	5.779526
0.2	-3.896402	-1.605522	-0.9206758	1.691107	2.791582	6.43596
0.3	-3.291525	-1.336301	-0.7506883	1.902122	3.117807	7.161076
0.4	-2.751052	-1.078245	-0.5915429	2.122073	3.43929	7.821474
0.5	-2.169034	-0.8386773	-0.4415677	2.353401	3.781169	8.542757
0.6	-1.648727	-0.6105573	-0.3068902	2.585443	4.12955	9.258249
0.7	-1.142172	-0.4050232	-0.1902167	2.813675	4.467478	9.98083
0.8	-0.6883294	-0.2238262	-0.09104082	3.062001	4.810012	10.63624
0.9	-0.2854556	-0.0747731	-0.01905901	3.330159	5.228898	11.46000

Table 8—Percentage points of Z for $m = 8$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-4.193554	-1.809184	-1.066787	1.442887	2.385339	5.407572
0.2	-3.621327	-1.543755	-0.8950158	1.634821	2.674131	5.976704
0.3	-3.092641	-1.289860	-0.732017	1.835751	2.971723	6.643536
0.4	-2.543256	-1.044226	-0.5777428	2.055649	3.30413	7.214718
0.5	-2.058487	-0.8157284	-0.4348236	2.266405	3.604813	7.83322
0.6	-1.541834	-0.5943818	-0.3023089	2.491144	3.926679	8.515626
0.7	-1.086331	-0.3937142	-0.1859968	2.722387	4.265364	9.171639
0.8	-0.650213	-0.2170623	-0.08861354	2.969777	4.628914	9.854049
0.9	-0.2678789	-0.0739203	-0.01864914	3.206926	4.961614	10.59909

Table 9—Percentage points of Z for $m = 9$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.948970	-1.761236	-1.047467	1.411089	2.29671	5.08357
0.2	-3.456744	-1.506019	-0.879997	1.597272	2.580992	5.625994
0.3	-2.918614	-1.243553	-0.7183954	1.801941	2.880748	6.200269
0.4	-2.423122	-1.016653	-0.5682835	2.007416	3.185581	6.82509
0.5	-1.945779	-0.7924505	-0.4262074	2.211661	3.478983	7.466756
0.6	-1.462151	-0.5754703	-0.2971736	2.425472	3.792651	8.060835
0.7	-1.028987	-0.3866122	-0.1841994	2.655454	4.108783	8.639637
0.8	-0.621938	-0.2137884	-0.08831543	2.888733	4.452213	9.288379
0.9	-0.2594530	-0.07368185	-0.01879642	3.11921	4.775363	9.967772

Table 10—Percentage points of Z for $m = 10$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.815345	-1.712120	-1.029515	1.389831	2.255280	4.889544
0.2	-3.311625	-1.468760	-0.8669173	1.573365	2.518134	5.406895
0.3	-2.775760	-1.218370	-0.7066627	1.767087	2.79559	5.950004
0.4	-2.324181	-0.992404	-0.5594533	1.960335	3.088387	6.502795
0.5	-1.855102	-0.7722169	-0.4202323	2.162273	3.373229	7.04638
0.6	-1.413902	-0.5636959	-0.2930012	2.384651	3.684622	7.666676
0.7	-0.9902364	-0.3788015	-0.1820957	2.590290	3.997573	8.18949
0.8	-0.5986408	-0.2087728	-0.0869599	2.817959	4.309529	8.774683
0.9	-0.249483	-0.07172889	-0.01821677	3.059500	4.648936	9.429459

Table 11—Percentage points of Z for $m = 11$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.677147	-1.68661	-1.016185	1.361358	2.201275	4.709677
0.2	-3.197047	-1.439716	-0.8553949	1.548698	2.463445	5.213978
0.3	-2.715089	-1.200809	-0.695535	1.734892	2.742957	5.734809
0.4	-2.252465	-0.978571	-0.5548879	1.935117	3.017410	6.23724
0.5	-1.792270	-0.7662858	-0.418534	2.14146	3.317238	6.814329
0.6	-1.360856	-0.5608097	-0.2926214	2.34662	3.609411	7.363093
0.7	-0.9679367	-0.3731323	-0.1800271	2.546553	3.88935	7.893548
0.8	-0.5853122	-0.2070345	-0.08633323	2.772639	4.219333	8.485662
0.9	-0.2453981	-0.07165806	-0.0184642	3.002688	4.521286	9.039971

Table 12—Percentage points of Z for $m = 12$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.584158	-1.660769	-1.004330	1.349213	2.158708	4.549818
0.2	-3.126661	-1.418738	-0.8457436	1.531225	2.426463	5.062448
0.3	-2.632941	-1.179377	-0.6923801	1.707543	2.687641	5.567363
0.4	-2.195187	-0.964123	-0.548977	1.913059	2.973350	6.088693
0.5	-1.765217	-0.7481825	-0.4100242	2.102309	3.241855	6.542337
0.6	-1.330863	-0.5496193	-0.2893885	2.303885	3.538131	7.142022
0.7	-0.9339713	-0.3693421	-0.1792674	2.512380	3.830009	7.671507
0.8	-0.5698536	-0.2058558	-0.08677245	2.741406	4.134409	8.24995
0.9	-0.2385981	-0.07056024	-0.01847504	2.941978	4.430885	8.760019

Table 13—Percentage points of Z for $m = 13$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.489558	-1.628901	-0.9916999	1.335454	2.134315	4.440679
0.2	-3.017948	-1.394935	-0.8363866	1.511910	2.384411	4.911279
0.3	-2.589712	-1.172481	-0.6893468	1.697836	2.645685	5.394334
0.4	-2.136967	-0.9502217	-0.5423062	1.883647	2.928797	5.896248
0.5	-1.718908	-0.7409976	-0.4098492	2.078506	3.195825	6.432302
0.6	-1.315682	-0.5445785	-0.2861363	2.286865	3.483502	6.976463
0.7	-0.9218953	-0.3636978	-0.1771569	2.488057	3.768092	7.483346
0.8	-0.5608599	-0.2050295	-0.08646439	2.701852	4.064076	8.015914
0.9	-0.2380039	-0.07068119	-0.01849360	2.907959	4.360012	8.470607

Table 14—Percentage points of Z for $m = 14$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.417063	-1.609026	-0.9851817	1.321788	2.103936	4.318686
0.2	-2.968805	-1.375277	-0.8264777	1.495166	2.356771	4.810646
0.3	-2.540178	-1.157462	-0.680062	1.681958	2.621337	5.319814
0.4	-2.091014	-0.9384699	-0.5376384	1.87063	2.887134	5.822761
0.5	-1.692939	-0.7306554	-0.4057702	2.065482	3.160777	6.320287
0.6	-1.284445	-0.5423824	-0.2857678	2.268464	3.447326	6.793236
0.7	-0.9100303	-0.361769	-0.1772765	2.468462	3.723774	7.29464
0.8	-0.5554973	-0.201852	-0.08556871	2.668099	4.015075	7.827362
0.9	-0.2349496	-0.07077845	-0.01837967	2.884709	4.309932	8.305454

Table 15—Percentage points of Z for $m = 15$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.369042	-1.599169	-0.9815545	1.308732	2.077018	4.256417
0.2	-2.917824	-1.371522	-0.8243538	1.485896	2.324588	4.707422
0.3	-2.485499	-1.147457	-0.6786837	1.671785	2.595851	5.215883
0.4	-2.067534	-0.9293967	-0.5352936	1.852493	2.849667	5.667842
0.5	-1.655886	-0.7249514	-0.4045843	2.044785	3.114850	6.159863
0.6	-1.263834	-0.5353704	-0.2829333	2.24768	3.40505	6.65905
0.7	-0.8891449	-0.3573048	-0.1753387	2.434652	3.674754	7.161794
0.8	-0.5438476	-0.2013127	-0.0854149	2.648935	3.971911	7.659545
0.9	-0.2297673	-0.06991807	-0.01833736	2.861185	4.255515	8.202117

Table 16—Percentage points of Z for $m = 16$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.306330	-1.588345	-0.975545	1.306673	2.062150	4.182349
0.2	-2.869695	-1.355177	-0.8198858	1.474202	2.309835	4.645341
0.3	-2.43324	-1.129997	-0.672014	1.653737	2.556692	5.105617
0.4	-2.037326	-0.9240351	-0.5322273	1.843849	2.825900	5.591043
0.5	-1.610186	-0.7203308	-0.4009714	2.027615	3.078989	6.01478
0.6	-1.243151	-0.5301101	-0.2819897	2.222257	3.359775	6.550245
0.7	-0.8795996	-0.3553984	-0.1748246	2.421017	3.625949	7.010561
0.8	-0.5384102	-0.2002799	-0.08582174	2.621756	3.912884	7.540569
0.9	-0.2285637	-0.06990817	-0.01828425	2.846765	4.221369	8.076916

Table 17—Percentage points of Z for $m = 17$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.263097	-1.577600	-0.972889	1.298196	2.055573	4.122835
0.2	-2.821946	-1.343109	-0.8114497	1.470103	2.283773	4.572698
0.3	-2.420097	-1.129198	-0.6718766	1.645620	2.549250	5.046735
0.4	-2.000458	-0.9124714	-0.5290936	1.826227	2.793880	5.504151
0.5	-1.603695	-0.7158773	-0.3993558	2.016741	3.058174	5.969449
0.6	-1.236857	-0.5288776	-0.2812824	2.214933	3.334614	6.441338
0.7	-0.8719545	-0.3538223	-0.1750794	2.400928	3.602577	6.902576
0.8	-0.5319261	-0.1982294	-0.08468536	2.596120	3.864048	7.377263
0.9	-0.2242597	-0.06899771	-0.01812863	2.812118	4.155429	7.874631

Table 18—Percentage points of Z for $m = 18$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.220876	-1.558697	-0.961797	1.287797	2.027091	4.073656
0.2	-2.798141	-1.337898	-0.8151006	1.459546	2.277054	4.499051
0.3	-2.387818	-1.118237	-0.6646692	1.633456	2.524818	4.975343
0.4	-1.985649	-0.9125769	-0.5259377	1.821317	2.772953	5.424434
0.5	-1.606539	-0.7108555	-0.3981024	2.00927	3.036320	5.894292
0.6	-1.220374	-0.5285972	-0.2814086	2.2014	3.307564	6.363901
0.7	-0.8578483	-0.3516286	-0.1733905	2.388743	3.574221	6.837261
0.8	-0.5291324	-0.1982026	-0.08463706	2.590735	3.846527	7.343967
0.9	-0.2238275	-0.06870137	-0.0181483	2.801962	4.135193	7.799765

Table 19—Percentage points of Z for $m = 19$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.164981	-1.553950	-0.9606198	1.279169	2.013349	4.036668
0.2	-2.777964	-1.329772	-0.8115468	1.461841	2.265301	4.487451
0.3	-2.356365	-1.113636	-0.6650267	1.620899	2.500122	4.923285
0.4	-1.961676	-0.9028863	-0.5247754	1.802951	2.752937	5.351938
0.5	-1.577885	-0.7103667	-0.3978176	1.990984	3.008967	5.811036
0.6	-1.208037	-0.5217122	-0.2786722	2.188746	3.289410	6.30242
0.7	-0.8525232	-0.3542327	-0.1745973	2.373153	3.548271	6.723971
0.8	-0.5231273	-0.1970522	-0.08417345	2.574265	3.817259	7.240467
0.9	-0.2237265	-0.06879045	-0.01805158	2.782465	4.101312	7.709405

Table 20—Percentage points of Z for $m = 20$.

ρ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	-3.167752	-1.540658	-0.955967	1.280138	2.008272	3.993921
0.2	-2.730288	-1.323645	-0.8082057	1.446894	2.242233	4.420654
0.3	-2.348489	-1.113086	-0.6649561	1.629605	2.493034	4.86473
0.4	-1.9524	-0.9041114	-0.526153	1.795266	2.733825	5.312453
0.5	-1.558249	-0.7023797	-0.3940396	1.987784	3.003293	5.735147
0.6	-1.191947	-0.5195472	-0.2791065	2.178237	3.264892	6.235225
0.7	-0.850882	-0.3508011	-0.1752822	2.370529	3.530829	6.67244
0.8	-0.5226046	-0.1962689	-0.08438214	2.564348	3.795812	7.142065
0.9	-0.2194294	-0.06846013	-0.01818054	2.759668	4.05702	7.577329