SOME SUPERSOLVABILITY CONDITIONS FOR FINITE GROUPS

By

F. Barry

and

D. MacHale*
Department of Mathematics, University College, Cork, Ireland

and

Á. Ní Shé
Department of Mathematics, Cork Institute of Technology, Ireland

(Communicated by T.D. Spearman, m.r.i.a.)

[Accepted 18 August 2005. Published 13 October 2006.]

Abstract

To investigate the intuitive notion that the closer a group is to being abelian the more likely it is to be supersolvable, we investigate situations where the following indicators of commutativity are 'large'—the reciprocal of the index $(G : Z(G))$; the reciprocal of $|G'|$; $Pr(G) = \frac{k(G)}{|G|}$; $f(G) = \sum_{i=1}^{k} \frac{d_i}{|G|}$ and $l(G)$, the maximum proportion of elements inverted by an automorphism. Applications to groups that satisfy the converse of Lagrange's Theorem are deduced.

1. Introduction

In this paper, we investigate a variety of numerical conditions which force a finite group to be supersolvable. These conditions are based on the intuitive notion that the closer a group is to being abelian, the more likely it is to be supersolvable, so we examine several 'indicators of commutativity' to establish our results. We deduce several results on conditions under which a finite group satisfies the converse of Lagrange's Theorem.

Much of the material in this paper is based on the doctoral theses of the first and third authors under the supervision of the second author.

2. Notation and Terminology

Throughout, G will denote a finite group with centre $Z(G)$, commutator subgroup G' and Sylow p-subgroup G_p.

*Corresponding author, e-mail: d.machale@ucc.ie

A_4 is the alternating group on four symbols. It has order 12, trivial centre and $A_4' \simeq C_2 \times C_2$; it has exactly four conjugacy classes and the sum of the degrees of its irreducible complex representations is 6. A group is called CLT if it satisfies the converse of Lagrange's Theorem—A_4 is non-CLT (NCLT) since it has no subgroup of order 6; A_4 is solvable but not supersolvable and it is the smallest such finite group.

$G(75)$ will denote the unique non-abelian group of order 75; $Z(G(75))$ is trivial and $(G(75))' \simeq C_5 \times C_5$; it has precisely 11 conjugacy classes and the sum of the degrees of its irreducible complex characters is 27; $G(75)$ is NCLT since it has no element of order 15 and therefore no subgroup of order 15, while 15 divides 75; $G(75)$ is solvable but not supersolvable and is the smallest such odd order group.

$\mathcal{G}_p$ is the set of all finite groups with order divisible by the prime p but by no smaller prime.

G has $k(G)$ conjugacy classes and $Pr(G) = \frac{k(G)}{|G|}$ is the probability that a randomly chosen pair of elements of G will commute with each other.

$f(G) = \frac{\sum_{i=1}^{k(G)} d_i}{|G|}$, where the d_i are the degrees of the irreducible complex representations of G.

We recall that a group G is supersolvable if it has a series of subgroups A_i such that

$$\{1\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_r = G;$$

with $A_i \lhd G$ for each i, $0 \leq i \leq r$, and each factor group $\frac{A_{i+1}}{A_i}$ is cyclic for $0 \leq i \leq r-1$. It is clear that if G is nilpotent, then G is supersolvable; but not conversely.

3. Preliminary results

We need a number of preliminary results, some of which are of interest in their own right.

Lemma 3.1. *If G' is cyclic, then G is supersolvable.*

PROOF. See [14], for example. ■

Lemma 3.2. *If $G \in \mathcal{G}_p$ and G is non-abelian, then*

$$Pr(G) \leq \frac{1}{p^2}\left[1 + \frac{p^2-1}{|G'|}\right].$$

PROOF. See [10]. ■

Lemma 3.3. *If $G \in \mathcal{G}_p$ and G is non-abelian, then*

(i)

$$f(G) \leq \frac{1}{p}\left[1 + \frac{p-1}{|G'|}\right]$$

and

(ii)

$$f(G) \leq \frac{2p-1}{p^2}.$$

PROOF.

(i)

$$\begin{aligned} f(G) &= \frac{1}{|G|}\sum_{i=1}^{k} d_i \\ &= \frac{1}{|G'|} + \frac{1}{|G|}\sum_{i=(G:G')+1}^{k} d_i \\ &\leq \frac{1}{|G'|} + \frac{1}{|G|}\sum_{i=(G:G')+1}^{k} \frac{d_i^2}{p} \text{ (since } d_i \mid |G|) \\ &\leq \frac{1}{|G'|} + \frac{1}{p}\left(1 - \frac{1}{|G'|}\right) \\ &= \frac{1}{p}\left[1 + \frac{p-1}{|G'|}\right]. \end{aligned}$$

(ii) Since $|G'| \geq p$, the above gives

$$f(G) \leq \frac{2p-1}{p^2}.$$

■

Lemma 3.4. *G is supersolvable if and only if* $\frac{G}{Z(G)}$ *is supersolvable.*

PROOF. Since every factor group of a supersolvable group is supersolvable, one part of the lemma follows at once.

Now suppose that $\frac{G}{Z(G)}$ is supersolvable and let

$$\frac{A_0}{Z(G)} \lhd \frac{A_1}{Z(G)} \lhd \ldots \lhd \frac{A_r}{Z(G)} \tag{3.1}$$

be a normal series for $\frac{G}{Z(G)}$, where $A_0 = Z(G)$, $A_r = G$, $\frac{A_i}{Z(G)} \lhd \frac{G}{Z(G)}$ for $0 \leq i \leq r$, and $\left(\frac{A_{i+1}}{Z(G)}\right) / \left(\frac{A_i}{Z(G)}\right)$ is cyclic for $0 \leq i \leq r-1$.

By the third isomorphism theorem, $\frac{A_{i+1}}{A_i}$ is cyclic also.

Now $Z(G)$, being a finite abelian group, is supersolvable, so let

$$\{1\} = B_0 \lhd B_1 \lhd \ldots \lhd B_n = Z(G) \tag{3.2}$$

be a normal series for $Z(G)$, with $B_i \lhd Z(G)$, $0 \leq i \leq n$ and $\frac{B_{j+1}}{B_j}$ cyclic for $0 \leq i \leq n-1$. It is now clear that

$$\{1\} = B_0 \lhd B_1 \lhd \ldots \lhd Z(G) \lhd A_1 \lhd \ldots \lhd G \tag{3.3}$$

is a normal series for G in which each subgroup is normal in G, because central subgroups are normal and $\frac{A_i}{Z(G)} \lhd \frac{G}{Z(G)}$ if and only if $A_i \lhd G$. Moreover, the required factor groups are cyclic, so G is supersolvable. ■

[While this result is all we need for the purposes of this paper, it is clear that it can be strengthened to the following:

Let $N \lhd G$ and suppose that N and $\frac{G}{N}$ are both supersolvable. If every normal subgroup of N is normal in G, then G is supersolvable.

Note that both A_4 and $G(75)$ show that N and $\frac{G}{N}$, which are both supersolvable, do not necessarily imply that G is supersolvable.]

From Lemma 3.4 we deduce:

Lemma 3.5. *If Aut G is supersolvable, then G is supersolvable.*

The quaternion group Q_2 of order 8 shows that the converse of this result is false.

We note that 'supersolvable' in Lemma 3.5 cannot be replaced by 'CLT' because $S_4 \simeq \text{Aut } A_4$ is CLT but A_4 is NCLT.

However, it can be shown that $\frac{G}{Z(G)}$ CLT implies G CLT, but again the converse is false, since $A_4 \times C_2$ is CLT but $(A_4 \times C_2)/Z(A_4 \times C_2) \simeq A_4$ is NCLT.

From Lemma 3.4 we deduce the following improvement on Lemma 3.1:

Lemma 3.6. *If $\frac{G'}{G' \cap Z(G)}$ is cyclic, then G is supersolvable.*

Proof. Since $\frac{G'}{G' \cap Z(G)}$ is cyclic, so is $\left(\frac{G}{Z(G)}\right)'$. Thus by Lemma 3.1 $\frac{G}{Z(G)}$ is supersolvable, so, by Lemma 3.4, G is supersolvable. ■

Lemma 3.7. *If G has a cyclic, characteristic, non-central subgroup, then there is no group K such that $G \simeq K'$. [We say G is incompetent].*

Proof. See [11]. ■

Lemma 3.8. *If $G \in \mathcal{G}_p$ and $G' \simeq C_p \times C_p$, where p is an odd prime, then G is nilpotent.*

PROOF. First of all we show that if $G_q \subseteq C_G(G')$, for any prime q which divides $|G|$, then $G_q \lhd G$.

Now $G_q \subseteq C_G(G')$ implies that $G' \subseteq C_G(G_q)$.

Then

$$G' \subseteq C_G(G_q) \subseteq N_G(G_q)$$

implies that $N_G(G_q) \lhd G$. Now G_q is a Sylow q-subgroup of G, so G_q is a Sylow q-subgroup of $N_G(G_q)$.

The Frattini argument now implies that

$$G = N_G(G_q)N_G(G_q) = N_G(G_q),$$

so $G_q \lhd G$.

Next, $\frac{N_G(G')}{C_G(G')}$ can be embedded in Aut G', which has order $p(p-1)^2(p+1)$. Since p is odd, $(p-1)^2$ and $p+1$ are both even and relatively prime to p; also

$$((p-1)^2(p+1), |G|) = 1,$$

since $G \in \mathcal{G}_p$, so $|\frac{G}{C_G(G')}| = 1$ or p.

If $|\frac{G}{C_G(G')}| = 1$, $C_G(G') = G$, so $G' \subseteq Z(G)$ and G is nilpotent of class 2.

If $|\frac{G}{C_G(G')}| = p$, then $G_q \subseteq C_G(G')$ for all primes $q \neq p$, so $G_q \lhd G$.

Thus every Sylow subgroup of G is normal in G, so G is nilpotent, as claimed.
■

Remark 3.1.
(1) G is in fact isomorphic to $G_p \times A$, where A is an abelian p'-group.
(2) Lemma 3.8 does not necessarily hold for $p = 2$, with A_4 being an obvious counterexample.

Lemma 3.9. *If $|G' \cap Z(G)| = 1$, then $|\frac{G}{Z(G)}| \leq |G'||Aut\ (G')|$.*
In particular, if $|Z(G)| = 1$, then $|G| \leq |G'||Aut\ (G')|$.

PROOF. See [12]. ■

Lemma 3.10. *If G is supersolvable and $d \mid |G|$, then G has a subgroup of order d, i.e., G is CLT.*

PROOF. See [6]. ■

Lemma 3.11. (i) *If $N \lhd G$ then $Pr(\frac{G}{N}) \geq Pr(G)$. If $N \cap G' = \{1\}$, then equality holds.*
(ii) *If $G \in \mathcal{G}_p$ and G is non-abelian, then*

$$Pr(G) \leq \frac{1}{p^3}\left[p^2 + p - 1\right].$$

(iii) *If* $G \in \mathcal{G}_p$ *and* $Pr(G) > \frac{1}{p}$*, then* $G' \simeq C_p$.
(iv) *If* $G \in \mathcal{G}_p$ *and* $G' \simeq C_p$ *then* G *is class 2 nilpotent.*

Proof. See [8] and [13]. ■

Lemma 3.12. *If* G *has an abelian subgroup* A *of index 2, then* G *is supersolvable.*

Proof. We may assume G is non-abelian; let $G = \langle A, x\rangle$, where $x^2 \in A$ and x induces an automorphism α of order 2: $(a)\alpha = x^{-1}ax$, $\forall a \in A$. Let

$$C_A(x) = \{a \in A | x^{-1}ax = a\} = Fix(\alpha) = Z(G).$$

Let G be a minimum counterexample to the the assertion that if G has an abelian subgroup of index two then G is supersolvable. We assume that the assertion is true for all groups of order $< |G|$ which have an abelian subgroup of index two. There are two cases to consider:

Case I: $Z(G)$ is non-trivial. Then $|\frac{G}{Z(G)}| < |G|$ and $\frac{G}{Z(G)}$ has an abelian subgroup $\frac{A}{Z(G)}$ of index 2. By hypothesis, $\frac{G}{Z(G)}$ is supersolvable, so by Lemma 3.4, G is supersolvable, which is a contradiction.

Case II: Let

$$Z(G) = \{a \in A | x^{-1}ax = a\}$$

be trivial. Then α is a fixed-point-free automorphism of order 2 of A, and $(a)\alpha = a^{-1}$ for all $a \in A$, so every subgroup of A is normal in G. Now A, being abelian, is supersolvable, so let

$$\{1\} = A_0 \lhd A_1 \lhd A_2 \lhd \ldots \lhd A_n = A$$

be a normal series for A, with $A_i \lhd G$ for $0 \leq i \leq n$ and with $\frac{A_{i+1}}{A_i}$ cyclic for $1 \leq i \leq n-1$. In addition, $A \lhd G$ and $\frac{G}{A}$ is cyclic, so G is supersolvable, which, again, is a contradiction. ■

Remark 3.2.
(i) It follows that if G has an abelian subgroup of index 2, then G is CLT.
(ii) Lemma 3.12 does not necessarily hold for groups in $\mathcal{G}_p$ (p odd) with an abelian subgroup of index p. $G(75) \in \mathcal{G}_3$ has an abelian subgroup of index 3 and is neither supersolvable nor CLT.

Lemma 3.13. *If* $G' \simeq C_2 \times C_2$*, then either*
(i) G *is nilpotent or*
(ii) $Pr(G) = \frac{1}{3}$*, with* $\frac{G}{Z(G)} \simeq A_4$ *and* $G' \cap Z(G) = \{1\}$.

Proof. $G' \cap Z(G) \simeq C_2 \times C_2$, C_2 or $\{1\}$.
If

$$G' \cap Z(G) \simeq C_2 \times C_2,$$

then $G' \subseteq Z(G)$, so G is nilpotent (of class 2).

If $G' \cap Z(G) \simeq C_2$, then

$$\left(\frac{G}{Z(G)}\right)' \simeq \frac{G'}{G' \cap Z(G)} \simeq C_2.$$

By Lemma 3.11 (iv), $\frac{G}{Z(G)}$ is nilpotent of class 2 so G is nilpotent (of class 3).

Finally, suppose that $G' \cap Z(G) \simeq \{1\}$. By Lemma 3.9, $|\frac{G}{Z(G)}| \leq |G'||\mathrm{Aut}\ G'| = 4.6 = 24$. Now, since $G' \cap Z(G) = \{1\}$, $\frac{G}{Z(G)}$ is non-nilpotent so, by [3], the only possibility is $\frac{G}{Z(G)} \simeq A_4$.

Then $\frac{1}{3} = Pr(A_4) = Pr\left(\frac{G}{Z(G)}\right) = Pr(G)$, by Lemma 3.11 (i). ■

Lemma 3.14. *If $G' \simeq Q_2$, the quaternion group of order 8, then $Pr(G) \leq \frac{1}{3}$.*

PROOF. By a famous result of Burnside [4], G cannot be a 2-group. Now G cannot be nilpotent either, because then $G = G_2 \times A$, where A is abelian of odd order, and then $Q_2 \simeq G' = G_2'$, which is a contradiction.

Now we consider $G' \cap Z(G)$, which is a characteristic abelian subgroup of Q_2. Thus $G' \cap Z(G) \simeq C_2$, with $G' \cap Z(G) = \{1\}$ being ruled out because Q_2 has a unique involution which is central in G.

Thus

$$\left(\frac{G}{Z(G)}\right)' \simeq \frac{G'}{G' \cap Z(G)} \simeq C_2 \times C_2.$$

By Lemma 3.13, either $\frac{G}{Z(G)}$ is nilpotent, which is not possible, or $Pr\left(\frac{G}{Z(G)}\right) = \frac{1}{3}$. Then $\frac{1}{3} = Pr\left(\frac{G}{Z(G)}\right) \geq Pr(G)$, as desired, by Lemma 3.11 (i). ■

Remark 3.3. Using a deeper result of Joseph [7], it can be shown that if $G' \simeq Q_2$, then

$$Pr(G) = \frac{1}{6} + \frac{1}{2^{2s+1}} \ s \geq 1,$$

so that the maximum value that $Pr(G)$ can have is $\frac{7}{24} < \frac{1}{3}$, realised when $G \simeq SL(2,3)$.

4. Main results

We now consider five group theoretic 'indicators of commutativity' and show that if any of these is sufficiently large for a group G, then G is supersolvable. We consider the cases G arbitrary and $|G|$ odd separately.

We first examine the fraction

$$\frac{1}{(G : Z(G))},$$

noting that G is abelian if and only if $\frac{1}{(G:Z(G))} = 1$; however $\frac{1}{(G:Z(G))}$ is a rather crude indicator of commutativity, as $(G : Z(G))$ is always an integer.

Theorem 4.1. *If* $\frac{1}{(G:Z(G))} > \frac{1}{12}$ *(=0.08333...), then G is supersolvable.*

Proof. Since all groups of order less than 12 are supersolvable, $\frac{G}{Z(G)}$ is supersolvable. By Lemma 3.4, G is supersolvable, as claimed.

Since $(A_4 : Z(A_4)) = 12$, the above result is the best possible. ■

We immediately deduce:

Theorem 4.2. *If* $\frac{1}{(G:Z(G))} > \frac{1}{12}$*, then G is a CLT group.*

Again, the NCLT group A_4 shows that this result is the best possible.

Theorem 4.3. *If* $|G|$ *is odd and* $\frac{1}{(G:Z(G))} > \frac{1}{75}$ *(=0.01333...), then G is supersolvable.*

Proof. Now $\frac{G}{Z(G)}$ has odd order less than 75. It is easily checked that $\frac{G}{Z(G)}$ is supersolvable. By Lemma 3.4, G is supersolvable, as claimed.

Since $(G(75) : Z(G(75))) = 75$ and $G(75)$ is not supersolvable, this result is the best possible. ■

We immediately deduce:

Theorem 4.4. *If* $|G|$ *is odd and* $\frac{1}{(G:Z(G))} > \frac{1}{75}$*, then G is a CLT group.*

Again $G(75)$ shows that this result is the best possible, since $G(75)$ is NCLT.

We note that both Theorems 4.3 and 4.4 are true under the weaker assumption that $(G : Z(G))$, rather than $|G|$, is odd. We remark that it is fortuitous that both A_4 and $G(75)$ are NCLT as well as being nonsupersolvable, because there exist nonsupersolvable groups which are CLT—S_4, for example.

Our next indicator of commutativity, $\frac{1}{|G'|}$, is also a crude one, as $|G'|$ is always an integer. We note that $\frac{1}{|G'|} = 1$ if and only if G is abelian.

Theorem 4.5. *If* $\frac{1}{|G'|} > \frac{1}{4}$ *(=0.2500), then G is supersolvable.*

Proof. We have $|G'| \leq 3$, so G' is cyclic and G is supersolvable by Lemma 3.1. Since $|A_4'| = 4$, this result is the best one possible. ■

We at once deduce:

Theorem 4.6. *If* $\frac{1}{|G'|} > \frac{1}{4}$ *(=0.2500), then G is CLT.*

Again, A_4 shows that this result is the best possible.

Theorems 4.5 and 4.6 arise in another context. In [2] , Bianchi *et al.* consider finite groups where the conjugacy class sizes are consecutive integers. They find that there are three such types of groups with the following properties:

(i) Abelian groups;
(ii) groups with $|G'| = 2$; and
(iii) groups with $\frac{G}{Z(G)} \simeq S_3$.

It is easy to see [11] that in case (iii), $G' \simeq C_3$. Thus we have:

Theorem 4.7. *If G is a group where conjugacy class sizes are consecutive integers, then G is both supersolvable and CLT.*

Turning our attention to groups of odd order we have:

Theorem 4.8. *Let $|G|$ be odd and suppose that $\frac{1}{|G'|} > \frac{1}{25}$ (=0.0400). Then G is supersolvable.*

Proof. We have $|G|$ odd and $|G'|$ is an odd number less than 25. If $|G'| = 1$ or a prime number, the G is supersolvable by Lemma 3.1. If $|G'| = 15$, again G' is cyclic and we are done. If $|G'| = 9$, then either G' is cyclic and we are done, or $G' \simeq C_3 \times C_3$. Since $|G|$ is odd, $G \in \mathcal{G}_3$ and, by Lemma 3.9, G is supersolvable. Finally, if $|G'| = 21$, then either G is cyclic and we are done, or G' is the unique non-abelian group of this order. But this group has a cyclic, characteristic, non-central subgroup, contradicting Lemma 3.8. ■

This result is the best one possible because $|G(75)'| = 25$ and $G(75)$ is not supersolvable.

We at once deduce:

Theorem 4.9. *Let $|G|$ be odd and suppose that $\frac{1}{|G'|} > \frac{1}{25}$. Then G is CLT.*

Again, the NCLT group $G(75)$ shows that this result is the best one possible.

We turn now to a more precise indicator of commutativity,

$$Pr(G) = \frac{k(G)}{|G|},$$

which is the probability that a randomly chosen pair of elements of G will commute with each other. We note that $Pr(G) = 1$ if and only if G is abelian.

Theorem 4.10. *Suppose that $Pr(G) > \frac{1}{3}$ =(0.3333...). Then G is supersolvable.*

Proof. We may suppose that G is non-abelian. Now $\frac{p^2+p-1}{p^3} < \frac{1}{3}$ for $p \geq 5$, so we may assume by Lemma 3.11 (ii) that $G \in \mathcal{G}_p$ for $p = 2$ or 3.

If $p = 3$, by Lemma 3.11 (iii), $|G'| = 3$, so G is supersolvable by Lemma 3.1. Thus we may assume that $p = 2$.

By Lemma 3.2, $\frac{1}{3} < Pr(G) \leq \frac{1}{4}(1 + \frac{3}{|G'|})$ which means that $|G'| < 9$.

Since G is supersolvable if G' is cyclic, we are left with the following possibilities for G':

$$C_2 \times C_2, S_3, D_4, C_4 \times C_2, Q_2 \text{ and } C_2 \times C_2 \times C_2.$$

Now $C_2 \times C_2$ is eliminated by Lemma 3.13; S_3 and D_4 are eliminated by Lemma 3.7; and Q_2 is eliminated by Lemma 3.14.

We are left with the cases $G' \simeq C_4 \times C_2$ and $G' \simeq C_2 \times C_2 \times C_2$, which we consider in turn.

(i) If $G' \simeq C_4 \times C_2$, then

$$G' \cap Z(G) \simeq C_4 \times C_2,\ C_4,\ C_2 \times C_2,\ C_2 \textit{ or } \{1\}.$$

If $G' \cap Z(G) \simeq C_4 \times C_2$, then G is nilpotent of class 2 and hence supersolvable.

If $|G' \cap Z(G)| = 4$, then

$$\frac{G'}{G' \cap Z(G)} \simeq C_2 \simeq \left(\frac{G}{Z(G)}\right)',$$

so $\frac{G}{Z(G)}$ is nilpotent of class 2 by Lemma 3.14, so G is nilpotent of class 3 and hence supersolvable.

If $G' \cap Z(G) \simeq C_2$, then

$$\frac{G'}{G' \cap Z(G)} \simeq \left(\frac{G}{Z(G)}\right)'$$

is either cyclic of order 4 or $C_2 \times C_2$. If $\left(\frac{G}{Z(G)}\right)'$ is cyclic, then G is supersolvable by Lemma 3.6.

If

$$\left(\frac{G}{Z(G)}\right)' \simeq C_2 \times C_2,$$

then by Lemma 3.13, $\frac{G}{Z(G)}$ is nilpotent, and we are done, or $Pr\left(\frac{G}{Z(G)}\right) = \frac{1}{3}$. Then, by Lemma 3.11,

$$\frac{1}{3} = Pr\left(\frac{G}{Z(G)}\right) \geq Pr(G) > \frac{1}{3},$$

which is a contradiction.

The case $G' \simeq C_4 \times C_2$ and $G' \cap Z(G) = \{1\}$ does not arise, since $C_4 \times C_2$ has a unique element of order 2, which is a square in G' and hence central in G.

(ii) Let $G' \simeq C_2 \times C_2 \times C_2$. Then

$$G' \cap Z(G) \simeq G' \simeq C_2 \times C_2 \times C_2,\ C_2 \times C_2,\ C_2 \ or \ \{1\}.$$

If $|G' \cap Z(G)| = 8$, 4 or 2 as in (i), the result follows.
We are then left with the case where $G' \cap Z(G) = \{1\}$.
Let G be a minimum counter-example to the theorem with $Pr(G) > \frac{1}{3}$ and G non-supersolvable.

If $Z(G)$ is non-trivial, then $Pr\left(\frac{G}{Z(G)}\right) > \frac{1}{3}$ and $|\frac{G}{Z(G)}| < |G|$, so $\frac{G}{Z(G)}$ is supersolvable. But by Lemma 3.4, G is supersolvable, which is a contradiction. We may thus assume that $Z(G)$ is trivial, so by Lemma 3.9,

$$|G| \leq |G'||Aut(G')| = 8.168 = 1344.$$

But using GAP [15], we find there are no groups G with the properties:
(i) $G' \simeq C_2 \times C_2 \times C_2$
(ii) $Z(G) = \{1\}$
(iii) $Pr(G) > \frac{1}{3}$
(iv) $|G| \leq 1344$.
This completes the proof of the theorem. ■

Since $Pr(A_4) = \frac{1}{3}$, the above result is the best one possible.
It is interesting to contrast Theorem 4.10 with a result of Cartwright [5]:

Let G be supersolvable. Then $k(G) > (0.6)\log_2 |G|$.

From Theorem 4.10, we at once deduce:

Theorem 4.11. *If $Pr(G) > \frac{1}{3}$, then G is CLT.*

Again, A_4 shows that this result is the best possible.

Theorem 4.12. *If $|G|$ is odd and $Pr(G) > \frac{11}{75}$ =(0.14666...), then G is supersolvable.*

PROOF. Suppose $G \in \mathcal{G}_p$ for $p \geq 11$. Then, by Lemma 3.11,

$$Pr(G) \leq \frac{11^2 + 11 - 1}{11^3} = \frac{131}{1331} = 0.09842... < 0.141666... = \frac{11}{75},$$

which is a contradiction.
Thus $G \in \mathcal{G}_p$ for $p = 3$, 5 or 7.
If $p = 7$,

$$\frac{11}{75} < Pr(G) \leq \frac{1}{49}\left[1 + \frac{48}{|G'|}\right]$$

gives $|G'| < 7.7586...$, so $|G'| = 7$ and G is supersolvable by Lemma 3.1.

If $p = 5$,

$$\frac{11}{75} < Pr(G) \leq \frac{1}{25}\left[1 + \frac{24}{|G'|}\right]$$

gives $|G'| < 9$, so $|G'| = 5$ or 7 and Lemma 3.1 again applies.

Thus,

$$p = 3, \frac{11}{75} < Pr(G) \leq \frac{1}{9}\left[1 + \frac{8}{|G'|}\right].$$

This gives $|G'| < 25$, and the result now follows from Theorem 4.8.

Again, this result is the best possible because $G(75)$ has exactly 11 classes and is not supersolvable. ■

We immediately deduce:

Theorem 4.13. *If $|G|$ is odd and $Pr(G) > \frac{11}{75}$, then G is CLT.*

The NCLT group $G(75)$ again shows this result is the best one possible.

Theorems 4.10–4.13 can be expressed in the following striking form.

(i) If the average size of a conjugacy class of G is less than 3, then G is both supersolvable and CLT; A_4 shows that this is the best possible result.

(ii) If $|G|$ is odd and the average size of a conjugacy class of G is less than $6\frac{9}{11}$, then G is both supersolvable and CLT. $G(75)$ shows that this result is the best possible in both cases.

Next we prove similiar results for

$$f(G) = \sum_{i=1}^{k(G)} \frac{d_i}{|G|},$$

which is another precise indicator of commutativity. We note that $f(G) = 1$ if and only if G is abelian. In the case of G arbitrary, we are fortunate to have a complete classification of all non-abelian groups with $f(G) > \frac{1}{2}$, due to Berkovich [1]. Among the consequences of this theorem are the following:

If G is non-abelian and $f(G) > \frac{1}{2}$, then either

(i) G has an abelian subgroup A of index 2, or

(ii) G is nilpotent.

By Lemma 3.12 we have:

Theorem 4.14. *If $f(G) > \frac{1}{2}$, then G is supersolvable.*

Since $f(A_4) = \frac{1}{2}$, we see that this result is the best possible.

We at once deduce:

Theorem 4.15. *If $f(G) > \frac{1}{2}$, then G is CLT.*

Again A_4 shows that this result is the best possible.

Turning our attention to groups of odd order, we have:

Theorem 4.16. *If $|G|$ is odd and $f(G) > \frac{9}{25}$ =(0.3600), then G is supersolvable.*

PROOF. If $G \in \mathcal{G}_p$ for $p \geq 5$, then by Lemma 3.3,

$$f(G) \leq \frac{2.5 - 1}{5^2} = \frac{9}{25},$$

which is a contradiction. Thus $G \in \mathcal{G}_3$, so

$$f(G) \leq \frac{1}{3}\left[1 + \frac{3-1}{|G'|}\right].$$

Thus

$$\frac{9}{25} < \frac{1}{3}\left[1 + \frac{2}{|G'|}\right],$$

which gives $|G'| < 25$, so, by Theorem 4.8, G is supersolvable.

Since $f(G(75)) = \frac{9}{25}$, this result is the best possible. ■

We deduce at once:

Theorem 4.17. *If $|G|$ is odd and $f(G) > \frac{9}{25}$ then G is CLT.*

Again, $G(75)$ shows that this result is the best possible.

Finally, we examine one other indicator of commutativity that has been extensively studied.

Let α be an automorphism of G and let $S_\alpha = \{g \in G \mid g\alpha = g^{-1}\}$. Define $l(\alpha)$ to be $\frac{|S_\alpha|}{|G|}$, the proportion of elements of G inverted by α, and let

$$l(G) = \text{Max } l(\alpha) \text{ for } \alpha \in \text{ Aut}G.$$

We note that $l(G) = 1$ if and only if G is abelian.

Liebeck and MacHale [9] have classified groups with $l(G) > \frac{1}{2}$. From their results it follows that if G is a non-abelian group with $l(G) > \frac{1}{2}$, then either

(i) G has an abelian subgroup of index 2, or

(ii) G is nilpotent.

We at once have:

Theorem 4.18. *If $l(G) > \frac{1}{2}$, then G is supersolvable.*

Since $l(A_4) = \frac{1}{2}$, this result is the best possible.

We immediately deduce:

Theorem 4.19. *If $l(G) > \frac{1}{2}$ then G is CLT.*

Again, the NCLT group A_4 shows this result is the best possible.

The situation for $|G|$ odd is less satisfactory in this case. By Liebeck and MacHale [9], if $l(G) > \frac{1}{3}$, then G is in fact abelian; and if $l(G) = \frac{1}{3}$, then either

(i) G has an abelian subgroup A of index 3, which does not contain all of the elements of order 3 in G (and conversely), or

(ii) G is nilpotent.

For the record, we state the result in this case as follows:

Theorem 4.20. *If $|G|$ is odd and $l(G) > \frac{1}{3}$, then G is supersolvable.*

Now $|G(75)|$ is odd and $G(75)$ has an abelian subgroup of index 3 which does not contain all the elements of order 3 in $G(75)$. Thus $l(G(75)) = \frac{1}{3}$, so our result is the best possible.

We again deduce:

Theorem 4.21. *If $|G|$ is odd and $l(G) > \frac{1}{3}$, then G is CLT.*

$G(75)$ again shows that this result is the best possible.

Acknowledgements

We are very grateful for the advice and help of the following people at various stages in the genesis of this paper:

Graham Ellis, Trevor Hawkes, Peter Hegarty, Eamonn O'Brien, Peter Palfy and Gary Sherman.

References

[1] Y.G. Berkovich and K.R. Nekrasov, Finite groups with large sums of degrees of irreducible characters, (Russian) *Publicationes Mathematicae Debrecen* **33** (1986), 333–54.

[2] M. Bianchi, D. Chillag, A. Mauri, M. Herzog and C. Scoppola, Applications of a graph related to conjugacy classes in finite groups, *Archiv der Mathematik* **58** (1992), 126–32.

[3] J. Burns, G. Ellis, D. MacHale, P. Ó Murchú, R. Sheehy and J. Wiegold, Lower central series of groups with small upper central factors, *Proceedings of the Royal Irish Academy,* **97**A (1997), 113–22.

[4] W. Burnside, *Theory of groups of finite order*, Dover Publications, New York, 1955.

[5] M. Cartwright, The number of conjugacy classes of certain finite groups, *Quarterly Journal of Mathematics Oxford* (2) **36** (1985), 393–404.

[6] C.D.H. Cooper, Subgroups of a supersolvable group, *Classroom Notes, American Mathematical Monthly* **78** (1971), 1007.

[7] K.S. Joseph, *Commutativity in non-Abelian groups*, unpublished PhD thesis, University of California at Los Angeles, 1969.

[8] P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *Journal of Algebra*, **177** (1995), 847–69.

[9] H. Liebeck and D. MacHale, Groups with automorphisms inverting most elements, *Mathematische Zeitschrift*, **124** (1972), 51–63.

[10] D. MacHale, Universal power-automorphisms in finite groups, *Journal of the London Mathematical Society* (2) **11** (1975), 366–8.

[11] D. MacHale and P. Ó Murchú, Commutator subgroups of groups with small central factor groups, *Proceedings of the Royal Irish Academy* **93**A (1993), 123–9.

[12] G.M. Ratay, 1974 *On the relationship between the commutator and central factor groups*, unpublished PhD thesis, Adelphi University, New York.

[13] D.J. Rusin, What is the probability that two elements of a finite group commute? *Pacific Journal of Mathematics*, **82** (1979), 237–47.

[14] W.R. Scott, *Group theory*, Prentice-Hall, New Jersey, 1964.

[15] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.2, Aachen–St Andrews, 1999, available at `http://www-gap.dcs.st-and.ac.uk/~gap` (accessed 10 October 2006).