ON GROUPS ALL OF WHOSE ELEMENTS HAVE PRIME POWER ORDER

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Dedicated to Martin Newell on his turn from duty to pleasure

Abstract

Groups described in the title have been characterized according to whether they are soluble or locally soluble. Using the progress achieved in finite simple groups, the remaining locally finite groups are described in this paper: infinite groups are perfect, they are extensions of 2-groups of bounded nilpotency class and exponent by a simple group; this simple group belongs to one of five isomorphism classes. Eight isomorphism classes are possible for simple groups, and only one isomorphism class of non-perfect groups exists.

1. Introduction

For brevity, we call groups satisfying the condition of the title *CP-groups*. G. Higman [4] considered finite soluble CP-groups. Now, almost fifty years later, it may be appropriate to look at these groups again, using the results that have emerged about finite simple groups (see [1; 2; 6]), and to extend the scope to locally finite groups. The structure of the locally soluble CP-groups was given by Yang and Zhang [8]; it follows from the structure of the finite soluble CP-groups (see Higman [4]). We state their result for completeness:

Theorem 1. *If the group G is a locally soluble CP-group, then there is a pair of normal subgroups $N,\ K$ of G, such that $1 \subseteq N \subseteq K \subseteq G$ and*

(i) *N is a p-group for some prime p,*
(ii) *$\frac{K}{N}$ is a q-group for some prime $q \neq p$; it is cylic or locally cyclic for $q \neq 2$ and is a subgroup of the non-split extension of C_{2^∞} by C_2 for $q = 2$,*
(iii) *$\frac{G}{K}$ is a cyclic p-group, and $|\frac{G}{K}|$ divides $q - 1$.*

In the case of finite, non-soluble CP-groups, we will have a much more restricted situation: there is a severe restriction on possible non-abelian chief factors, and abelian chief factors are 2-groups (see, for instance, Propositions 2 and 3 below). These results lead to corresponding restrictions for the locally finite case. The use of [A] and [B] reduces the work to a tolerable amount in sections 2 and 6 below.

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2. First structure statements for finite CP-groups

For the study of the locally finite case, we begin with a statement reminiscent of Theorem 1.

Proposition 2. *If G is a finite, non-soluble CP-group, then there are normal subgroups $B,\ C$ of G, such that $1 \subseteq B \subseteq C \subseteq G$ and*

(i) *B is a 2-group,*
(ii) *$\frac{C}{B}$ is nonabelian and simple,*
(iii) *$\frac{G}{C}$ is a p-group for some prime p and cyclic or generalized quaternion,*
(iv) *if $B \neq 1$, then $p = 2$.*

Proof. Using Theorem 1, we obtain for G that it is not soluble, but all soluble subgroups have orders that are divisible by two primes only. By a theorem of P. Hall (see Higman [4, theorem 4]) we obtain that $\frac{C}{B}$ is nonabelian and simple and that B and $\frac{G}{C}$ are p-groups for the same prime p. Let $r \neq p$ be a prime divisor of $|\frac{C}{B}|$ and let $\frac{R}{B}$ be a Sylow r-subgroup of $\frac{C}{B}$. Then $N(R)C = G$ and $\frac{N(R)}{(N(R)\cap B)} \cong \frac{G}{C}$. Denote a Sylow p-subgroup of $N(R)$ by P. Now PR is a CP-group, so P must be cyclic or (generalized) quaternion, and the same applies for $\frac{G}{C} \cong \frac{PC}{C}$. This shows (iii). It remains to show that $p = 2$ provided that $B \neq 1$.

Assume that $B \neq 1$ is not a 2-subgroup but a q-subgroup for some odd prime q. The nonabelian simple group $\frac{C}{B}$ possesses an elementary abelian subgroup $\frac{K}{B}$ of order 4, and K is not a CP-group. This contradiction shows that we must have $p = 2$ in this case; thus Proposition 2 is proved. ∎

3. Nonabelian simple CP-groups

We want to find all groups that may appear in Proposition 2(ii), and our work relies heavily on the *Atlas of finite groups* [1] and the book by Gorenstein [2]. We have to check all the possible groups using the classification in these two publications.

3.1. The linear groups

The groups $L_2(q)$ are simple for prime powers $q > 3$. If q is odd, they have cyclic subgroups of order $\frac{1}{2}(q+1)$ and $\frac{1}{2}(q-1)$. If q is a power of 2, then there are cyclic subgroups of order $q+1$ and $q-1$. In both cases, a necessary condition for $L_2(q)$ to be a CP-group is that q^2-1 is divisible by only two different prime divisors, and this is true for $q = 4, 5, 7, 8, 9, 17$. The groups $L_3(q)$ have a subgroup that is an extension of an abelian p-group (where q is a power of the prime p) by a group having $L_2(q)$ as quotient group. This shows that only $L_3(q)$ for $q = 2, 3, 4, 8$ have to be checked. We find $L_3(2) \cong L_2(7)$, so this is a CP-group, while the Sylow 3-subgroup of $L_3(3)$ is not a CP-group, so this group is excluded. Close inspection shows that $L_3(4)$ is a CP-group. The group $L_3(8)$ possesses a subgroup isomorphic to $C_7 \times L_2(8)$, which clearly is not a CP-group. As the next higher level, we have to consider $L_4(q)$ for $q = 2, 4$. There is an element of order 15 in $L_4(2)$ and also in $L_4(4)$, so both of them are not CP-groups. We have found:

(A) The linear CP-groups are $L_2(4) \cong L_2(5)$, $L_2(7) \cong L_3(2)$, $L_2(8)$, $L_2(9)$, $L_2(17)$ and $L_3(4)$.

(B) Alternating groups.
A_7 has an element of order 6, so A_n is not a CP-group for $n > 6$; and $A_5 \cong L_2(5)$; $A_6 \cong L_2(9)$. We see that we do not find new CP-groups.

(C) Symplectic groups.
Here $S_2(q) \cong L_2(q)$ and $S_{2n}(q)$ contains a subgroup isomorphic to $S_2(q) \times S_{2n-2}(q)$. This shows that no groups will occur for $q \neq 2$; and $S_4(2) \cong L_2(4)$ appears also in subsection A.

(D) Orthogonal groups.
Consider first the case that q is odd.We know that $O_3(q) \cong L_2(q)$, and these groups we have already reviewed. Also, we know that $O_5(q) \cong S_4(q)$, which is never a CP-group. Both $O_4^+(q)$ and $O_4^-(q)$ possess cyclic subgroups of order $q-1$ and $q+1$, so we are left with $q = 3$. We know that $O_4^-(3) \cong L_2(9)$ is a CP-group, while $O_4^+(3)$ is soluble. For $q = 2^n$ we obtain that $O_6^+(2) \cong A_8$ and $O_6^-(2) \cong O_5(3)$ are not CP-groups. The same is true for $O_6^+(4)$, $O_6^+(8)$, $O_6^-(4)$ and $O_6^-(8)$. Further, $O_5(2) \cong A_6$ is a CP-group, while for $q > 2$ we have that $O_5(q)$ contains a subgroup isomorphic to $L_2(q) \times L_2(q)$ and is not a CP-group. Since $O_4^-(q) \cong L_2(q^2)$, we have that $O_4^-(4)$ and $O_4^-(8)$ are not CP-groups and $O_4^-(2) \cong L_2(4)$ is a CP-group. All CP-groups mentioned here have been mentioned in subsection A.

(E) Ree groups.
Here, the Sylow 3-subgroups are nonabelian and the order of their normalizer is even, so there are elements of order 6. No Ree group is a CP-group.

(F) Suzuki groups.
The Suzuki group $Sz(2^m)$ possesses two cyclic groups A and B, such that $|A||B| = 2^{2m}+1$, where m is an odd integer. So if $Sz(2^m)$ is a CP-group, we obtain necessarily that $2^{2m}+1$ is divisible by two different prime divisors only. This happens only for $m = 3$ and $m = 5$. Close inspection shows that the groups $Sz(8)$ and $Sz(32)$ are in fact CP-groups.

(G) Other groups of exceptional Lie algebras.
Here we make use of the Dynkin diagrams: if a group H corresponds to a certain Dynkin diagram and another one, K, corresponds to a Dynkin digram that is contained in the first, and if both groups are defined over the same field, then K is isomorphic to a quotient group of a subgroup of H; so if K is not a CP-group, the same applies to H. This shows that we have to check only $G_2(q)$ for $q = 2, 3, 4, 7, 8, 17$; $E_4(q)$ for $q = 2, 3, 4$; and $E_6(2)$.
Before we proceed with our argument, we prove a fact that we will use later: let G be a finite group such that p but not p^2 divides $|G|$. Write $|G| = ap+bp^2$ where $1 \leq a \leq p-1$. If a is not a power of a prime or not a divisor of $p-1$,

then G is not a CP-group. For a proof, remember that the normalizer of the Sylow p-subgroup is a CP-group only if it is isomorphic to a subgroup of $Hol(C_p)$, and that its index in G must be congruent to 1 modulo p.
We return to the groups of exceptional Lie algebras that are left. We have $|G_2(q)| = q^6(q^6-1)(q^2-1)$. If we put $m = q^2 - q + 1$, we have $|G_2(q)| = m(m+2q)(m+q-2)^2 + xm^2 = 2mq(q+2)^2 + ym^2$, where x and y are integers. Now we choose a prime r dividing m in $G_2(q)$ and obtain a contradiction for $(q,r) = (17,13), (8,19), (7,43), (5,7), (4,13)$ and $(3,7)$. For $G_2(2)$ we consider the prime 7. For the groups of the form $F_4(q)$ we have $|F_4(q)| = q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$. Here, the argument on the normalizers of the Sylow r-subgroups works for $(q,r) = (4,13), (3,73), (2,257)$. For $E_6(2)$ we consider the prime 13.

(H) The sporadic groups.
These groups always possess an element of order 6, so they are not CP-groups.

3.2. Summary

Proposition 3. *If G is a finite nonabelian simple CP-group, then G is isomorphic to one of the following:*

(i) $L_2(q)$ *for* $q = 5, 7, 8, 9, 17$,
(ii) $L_3(4)$,
(iii) $Sz(8)$, $Sz(32)$.

4. Automorphism groups

In view of Proposition 2, we want to know which extensions of CP-groups mentioned in Section 3 are also CP-groups. We know that these extensions have to be subgroups of the automorphism group. For $Sz(8)$ and $Sz(32)$, it is known that the automorphism group is an extension of the group of inner automorphisms by a cyclic group of order 3 resp. 5 (the field automorphisms). The normalizer of the Sylow 2-subgroup of this extension is soluble and of order divisible by three different primes and is not a CP-group. So in this case no proper extension is possible.

We obtain a contradiction in the same way, considering the Sylow 2-subgroup, for $L_3(4)$.

For $L_2(q)$ we obtain $PGL(2,q)$ as extension for $q = 5, 7, 17$. Here, the normalizer of the (cyclic) Sylow 3-subgroup is divisible by 4 and is not a CP-group. The only possible extension of $L_2(8)$ is effected by a field automorhism of order 3, and here the normalizer of the Sylow 2-subgroup leads to a contradiction.

It remains to consider $L_2(9) \cong A_6$. It is known that the quotient group of the automorphism group modulo the group of inner automorphisms is isomorphic to the elementary abelian group of order 4; in other words, there are 3 subgroups of index 2 in $Aut(L_2(q))$, and we denote them by U, V, W. One of them, say U, is isomorphic to

S_6 and is not a CP-group; another one, V, is isomorphic to $PGL(2,9)$ and possesses a cyclic subgroup of order 10. Close inspection shows that the remaining subgroup W is in fact a CP-group.
We have found:

Proposition 4. *If G is a finite, non-soluble CP-group with trivial Fitting subgroup, then either G is simple or G is isomorphic to W, as defined above in section 4.*

Slightly closer inspection yields, in addition:

Proposition 5. *If G is a finite, non-soluble CP-group with nontrivial Fitting subgroup, then G is perfect.*

Proof. We know that G is not perfect if $\frac{G}{Fit(G)} \cong W$, where W is as in section 4. But W possesses non-cyclic Sylow 3-subgroups, so an extension of a 2-group by W is not a CP-group, and $Fit(G)$ has to be a 2-group by Proposition 2. So Proposition 5 is true. ■

5. Locally finite simple CP-groups

We will prove:

Lemma. *Every locally finite simple CP-group is finite.*

Proof. We may restrict ourselves to non-locally-soluble groups since locally soluble simple CP-groups are of order a prime p. Assume now that G is an infinite simple CP-group that is locally finite. Consequently, we have finitely generated subgroups of our group G that are not soluble, and they contain a perfect finite subgroup T_1. Every subgroup of G that contains T_1 is also not soluble and possesses a perfect subgroup containing T_1. We find an ascending sequence $\{T_i\}$ of perfect subgoups. By Proposition 2 we have that the groups T_i possess a normal 2-subgroup V_i such that $\frac{T_i}{V_i}$ is simple. By construction we have for $j > i$ that $\frac{T_i}{V_i}$ is isomorphic to a quotient group of a subgroup of $\frac{T_j}{V_j}$. By Proposition 3, there are only finitely many possible quotient groups $\frac{T_i}{V_i}$. Therefore, there is always a simple group L such that $\frac{T_i}{V_i} \cong L$ for almost all i. If different sequences lead to different quotients L, we take the subgroups generated by the i-th terms of both sequences for a new sequence. This shows that there is a finite simple group S such that for every finite perfect subgroup V of G we find that $\frac{V}{Fit(V)}$ is isomorphic to the quotient group of some subgroup of S; also, we may choose S such that there is R with $\frac{R}{Fit(R)} \cong S$. By finiteness of S, we obtain for all finite perfect subgroups X containing R that they satisfy the relation $\frac{X}{Fit(X)} \cong \frac{R}{Fit(R)}$ and that $X = RFit(X)$. By Proposition 5, $S \ncong W$, since G is infinite. So every non-soluble finite subgroup of G is perfect by Proposition 4.

Let ν be the variety generated by S, and let $w(H)$ be the word subgroup of H that corresponds to the variety ν. For every finite subgroup $F \supseteq R$ of G, we obtain that $w(F)$ is a 2-group, and so $w(G)$ is also a 2-group and $\frac{R}{Fit(R)} = \frac{R}{w(R)}$, with $R \cap w(G) = w(R)$. Now $\frac{G}{w(G)}$ possesses a direct factor isomorphic to S and is a CP-group. So $\frac{G}{w(G)} \cong S$, and, by simplicity, $w(G) = 1$ and G must be finite (isomorphic to S), contrary to assumption. This demonstrates the Lemma. ■

6. Proposition 2 revisited

We are now able to be more explicit about the structure of locally finite CP-groups.

Theorem 6. *Let G be a locally finite CP-group and let N be its maximal normal 2-subgroup. Then G satisfies one of the following statements:*

(i) *G is locally soluble;*
(ii) *G is isomorphic to $L_2(7), L_2(9), L_3(4)$ or W as defined in section 3;*
(iii) *N is elementary abelian and $\frac{G}{N}$ is isomorphic to $L_2(5)$ or $L_2(17)$;*
(iv) *N is abelian and $\frac{G}{N}$ is isomorphic to $L_2(8)$;*
(v) *N is nilpotent of class 6 and $\frac{G}{N}$ is isomorphic to $Sz(8)$ or $Sz(32)$.*

Proof. Proposition 2 shows that G is either locally soluble or it is an extension of a 2-group by a group mentioned as G or $\frac{G}{N}$ in (ii)–(v). We have to show the statements on N. The group $L_2(7)$ possesses a (soluble) subgroup of order 21, thus an extension of a nontrivial 2-group by this group can not be a CP-group; $L_2(9)$ has a noncyclic Sylow 3-subgroup and an extension of a nontrivial 2-group by this group is not a CP-group either. $L_3(4)$ and W have a subgroup isomorphic to $L_2(9)$. So if $\frac{G}{N}$ is isomorphic to one of the groups mentioned in (ii), then $N = 1$, thereby showing (ii). In (v) we have that N is a locally finite 2-group with fixed-point free automorphism of order 5. By Higman [3, p. 331], we deduce that N is of class 6, since all finite subgroups that are invariant under this automorphism have this property. This shows (v). In (iii) and (iv) we deduce that N possesses a fixed-point free automorphism of order 3, and a similar argument as before yields that N is of nilpotency class 2. For (iii), we choose a subgroup $\frac{K}{N} \cong A_4$ of $\frac{G}{N}$. Such a subgroup is known to exist (see [5, Satz II, 8.18(b)]). Now K' is a 2-group with fixed-point free automorphism of order 3, so K' is nilpotent of class 2. In particular, N' is centralized by K' and by G, since N' is a normal subgroup of G. This shows that $N' = 1$. Let x be an element of K' and let y be an element of N. Then $[x^2, y] = [x, [x, y]] = 1$ and therefore also $[x, y^2] = 1$. This shows that K' centralizes N^2 and again $N^2 \subseteq Z(G) = 1$. We have therefore shown (iii).

For (iv) we choose two elements sN, tN of $\frac{G}{N}$ that are of order 3, and put $u = tst^{-1}$. By Straus and Szekeres [7, lemma 2], we have for all $z \in N$ and all $w \in \{s, t, u\}$ the identity $[z, w^{-1}zw] = 1$. Let $x = zy^{-1} \in N$, then $[xy, w^{-1}xyw] = 1$ leads to $[x, w^{-1}yw] = [w^{-1}xw, y]$ and, in particular,

$$[x, y] = [sxs^{-1}, s^{-1}ys] = [sxs^{-1}, t^{-1}u^{-1}tyt^{-1}ut] = [t^{-1}sxs^{-1}t, u^{-1}tyt^{-1}u]$$

$$= [u^{-1}t^{-1}sxs^{-1}tu, tyt^{-1}] = [tu^{-1}t^{-1}sxs^{-1}tut^{-1}, y].$$

Since $tut^{-1}N = t^{-1}stN$, we have $[[x,[s,t]],y] = 1$. In $\frac{G}{N} \cong L_2(8)$ there are sN, tN, such that $[s,t]N$ is of odd order and $[s,t]N \neq N$. So, in this case $[N,[s,t]] = N$ and $N' = [[N,[s,t]],N] = 1$, and Theorem 6 is proved completely. ■

The statements of Theorem 6 can be completed by bounds for the exponent of N for the remaining cases. Before we begin with the statement, we introduce the following notation: $[x,y] = [x,_1 y]$ and, by induction, $[[x,_n y],y] = [x,_{n+1} y]$ for all n.

Proposition 7. *Let G and N be as in Theorem 6. Then $Z(N)$ is of bounded exponent.*

PROOF. We review the possibilities for $\frac{G}{N}$; notice that the statement is true for $\frac{G}{N} \cong L_2(5),\ L_2(17)$ by Theorem 6(iii). If $\frac{G}{N} \cong L_2(8)$, consider the normalizer $N(S)$ of a Sylow 2-subgroup S of G. This is a 2-group extended by a cyclic group of order 7, which operates fixed-point-freely. By Higman [3], there is a bound $k(7)$ for the nilpotency class of S. We choose an element $x \in S$ and know $x^2 \in N$. Let $u \in Z(N)$. Then $[u,x^2] = 1 = [u,_{k(7)} x]$. Since u belongs to an abelian normal subgroup, we have also $[u,x]^{2^{k(7)-1}} = 1$, and we find that the characteristic subgroup $Z(N)^{2^{k(7)-1}}$ of N centralizes S and also all conjugates of S and G. This shows that $Z(N)$ has exponent $2^{k(7)-1}$ at most.

If $\frac{G}{N} \cong Sz(8)$, we choose the normalizer $N(S)$ of a Sylow 2-subgroup S of G. Again, $N(S)$ is the extension of a 2-group by a cyclic group of order 7 and S is nilpotent of class $k(7)$. Let $x \in S$, then $x^4 \in N$ and $x^2 \in S'N$. Define m by $2m-1 \leq k(7) \leq 2m$. For every $u \in Z(N)$ we have $[u,x^4] = 1 = [u,_m x^2)$ and $[u,x^2]^{2^{m-1}} = 1$. Arguing as in the previous paragraph, we have that $Z(N)$ is of exponent 2^{m-1} at most, where $m = \frac{1}{2}(k(7)+1)$.

If $\frac{G}{N} \cong Sz(32)$, the argument is analogous to the previous one, with $N(S)$ being an extension of a 2-group by a cyclic group of order 31 operating without fixed points. The 2-group is nilpotent of class $k(31)$ at most, and we define n by $2n-1 \leq k(31) \leq 2n$. We find that $Z(N)$ is of exponent 2^{n-1} at most, where $n = \frac{1}{2}(k(31)+1)$. ■

Corollary 1. *Let G and N be as in Theorem 6. Then N is of bounded exponent.*

PROOF. If T is a group and $Z(T)$ is of exponent e, then the exponent of $\frac{Z_{i+1}(T)}{Z_i(T)}$ is also bounded by e for all i. Now the corollary follows from Theorem 6 and Proposition 7. ■

Remark 1. *The exact values of $k(p)$ for $p > 5$ seem to be unknown; combining the results of Higman [3] and Kreknin (see Huppert [5, p. 500]), we have* $12 \leq k(7) < 10^{49}$ *and* $240 \leq k(31) < 10^{10^{10}}$.

7. Closer inspection of 2-modules

If G is a non-solvable CP-group and N is its maximal normal 2-subgroup, if, further, $K \subset N$ is a normal subgroup of G that is maximal with this property, then $\frac{N}{K}$ can be considered as an irreducible 2-module of $\frac{G}{N}$; and since G is a CP-group, we have, further, that all elements of odd order of $\frac{G}{N}$ will operate on $\frac{N}{K}$ without fixed points. This property can be read off from the Brauer character tables, which are assembled conveniently in [6].

We first consider $L_2(17)$. Here, the character table (see [6, p. 11]) shows that there is no irreducible 2-module on which elements of order 3 operate without fixed elements. This shows: if G is a CP-group with $\frac{G}{N} \cong L_2(17)$, then $N = 1$.

In the remaining four possibilities for $\frac{G}{N}$ we have groups that have a classical description as linear groups over some field $GF(2^n)$, and the order of this field is attached to the group notation. Considering the Brauer character tables (see [6, pp 2, 6, 63, 197]), we see that these canonical descriptions as linear groups lead to the only 2-modules on which $\frac{G}{N}$ operates such that elements of odd order have no fixed elements. For this, it suffices to look at the character of the element of smallest odd order.

We summarize:

Theorem 8. *Let G and N be as in Theorem 6. Then, in addition,*

(i) $N = 1$ *if* $\frac{G}{N} \cong L_2(17)$*;*
(ii) *if* $\frac{G_1}{N_1}, \frac{G_2}{N_2}$ *are isomorphic to one of* $L_2(4), L_2(8), Sz(8), Sz(32)$ *and* N_1, N_2 *are minimal normal subgroups, then* $G_1 \cong G_2$ *if and only if* $\frac{G_1}{N_1} \cong \frac{G_2}{N_2}$.

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Added in proof: The author was not aware of the following paper—
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