THE ANALYTIC ALGEBRAS OF HIGHER RANK GRAPHS

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Abstract

We begin the study of a new class of operator algebras that arise from higher rank graphs. Every higher rank graph generates a Fock space Hilbert space and creation operators that are partial isometries acting on the space. We call the weak operator topology closed algebra generated by these operators a 'higher rank semigroupoid algebra'. A number of examples are discussed in detail, including the single vertex case and higher rank cycle graphs. In particular, the cycle graph algebras are identified as matricial multivariable function algebras. We obtain reflexivity for a wide class of graphs and characterize semisimplicity in terms of the underlying graph.

1. Introduction

In [20], Kumjian and Pask introduced k-graphs as an abstraction of the combinatorial structure underlying the higher rank graph C*-algebras of Robertson and Steger [29; 30]. A k-graph generalizes the set of finite paths of a countable directed graph when viewed as a partly defined, multiplicative semigroup with vertices considered as degenerate paths. The C*-algebras associated with k-graphs include k-fold tensor products of graph C*-algebras, and much more [2; 19; 24; 25; 28]. On the other hand, as a generalization of the non-self-adjoint free semigroup algebras $\mathfrak{L}_n$ [3; 6; 7; 8; 16; 26; 27], the authors [17; 18] have recently studied free semigroupoid algebras $\mathfrak{L}_G$ associated with directed countable graphs G. In particular, it was shown that these algebras are reflexive. (See also [11; 12; 13; 14; 15; 21; 22; 23; 32] for related recent work.) As it turns out, these algebras arise from the left regular representation of the 1-graph of the directed graph G. In the present paper, we consider the higher rank versions of these algebras, the k-graph algebras $\mathfrak{L}_{(\Lambda,d)}$ associated with the k-graph (Λ, d), as well as their norm closed subalgebras. To our knowledge, such non-self-adjoint higher rank graph algebras have not been considered previously. However, from the perspective of contemporary operator algebra theory, they

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evidently form a natural class, and one that may play an important role in more general higher rank operator algebra considerations.

The algebras $\mathfrak{L}_n$ are also referred to as the noncommutative analytic Toeplitz algebras, and in the case $n = 1$, one obtains the usual algebra H^∞ acting on the Hardy space of the circle [9; 10; 31]. In the present paper, we examine eigenvalues, reflexivity, hyper-reflexivity and semisimplicity for the algebras $\mathfrak{L}_{(\Lambda,d)}$, which can be viewed as the H^∞ algebras of higher rank graphs.

In §2 we outline the nomenclature associated with higher rank graphs (Λ, d). In §3 we introduce higher rank semigroupoid algebras $\mathfrak{L}_{(\Lambda,d)}$ and derive some basic properties. We follow this in §4 by presenting a diverse collection of examples, and in §5 we consider the single vertex algebras. In particular, we determine the eigenvalues for the adjoint algebras and the Gelfand space of the norm closed subalgebras $\mathcal{A}_{(\Lambda,d)}$. In the next section (§6), we prove that the algebras $\mathfrak{L}_{(\Lambda,d)}$ are reflexive or hyper-reflexive for various diverse graphs. In the final section (§7), we find a graph condition that characterizes when $\mathfrak{L}_{(\Lambda,d)}$ is semisimple and we give an explicit description of the Jacobson radical in the finite vertex case.

2. Higher Rank Graphs

Let $G = (V, E)$ be a countable directed graph, and let Λ_G denote the set of all directed paths $\lambda = e_r e_{r-1} \cdots e_1$, where e_1 is an edge (v_2, v_1) directed from v_1 to v_2 and where e_k is an edge (v_{k+1}, v_k), for $1 \leq k \leq r$, where $v_1, v_2, \ldots, v_r$ are the vertices of the path, in their directed order, possibly with repetitions. Let $d : \Lambda_G \to \mathbb{N}$ be the length function. Then the pair (Λ_G, d) is an example of a 1-graph in the sense of the definition below.

The set Λ_G has a natural, partially defined multiplication that, together with vertices as degenerate paths, makes $\Lambda_G \cup V$ into a (discrete) *semigroupoid* with vertices as units. (This is the terminology of [17; 18].) However, Λ_G can be viewed as a set of morphisms between elements of V, and as such Λ_G forms a small category with V as the set of objects. ('Small' since the objects form a set.) It is this viewpoint that is extended in the definition below.

Definition 2.1. [20] A k-graph (Λ, d) consists of a countable small category Λ, with range and source maps r and s, respectively, together with a functor $d : \Lambda \to \mathbb{Z}_+^k$ satisfying the factorization property: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{Z}_+^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$, such that $\lambda = \mu\nu$ and $d(\mu) = m$ and $d(\nu) = n$.

By the factorization property, we may identify the objects $\mathrm{Obj}(\Lambda)$ of Λ with the subset $\Lambda^0 \equiv d^{-1}(0, \ldots, 0)$. We also write Λ^n for $d^{-1}(n)$, $n \in \mathbb{Z}_+^k$. Observe that the factorization property implies that left and right cancellation hold in Λ. We let $\delta : \Lambda \to \mathbb{Z}_+$ denote the grading function defined by $\delta(\lambda) = |d(\lambda)| = n_1 + \ldots + n_k$, where $d(\lambda) = (n_1, \ldots, n_k)$. As indicated above, the conventional definition of a directed graph, with its semigroupoid of paths, is captured in the special case of 1-graphs. The set Λ^n corresponds to the directed paths of length n in the graph.

We give a variety of examples of k-graphs in § 4. However, let us consider here the simple example given by the 2-graph $\Lambda_{(G_1\times G_2,d)}$ arising from the set $\Lambda_{G_1\times G_2}$ of paths in the direct product directed graph $G_1 \times G_2$, where G_1, G_2 are directed graphs, together with the natural map $d : \Lambda_{G_1\times G_2} \to \mathbb{Z}^2_+$. In this case, if $e = (v_2, v_1)$ is an edge of G_1 and $f = (w_2, w_1)$ is an edge of G_2, then $e \times f = ((v_2, w_2), (v_1, w_1))$ is an edge of $G_1 \times G_2$ and $d(e \times f) = (1, 1)$. Also, by definition, $G_1 \times G_2$ includes all edges $((v, w_2), (v, w_1))$ and $((v_2, w), (v_1, w))$, with d-degrees $(0, 1)$ and $(1, 0)$, respectively, for each vertex v of G_1 and w of G_2. More generally, a direct product $G_1 \times \ldots \times G_k$ of k directed graphs generates a k-graph in this way.

It is of immediate interest to see k-graphs that do not arise as direct products. For an elementary example, let $\Lambda^0 = \{v\}$, $\Lambda^{(1,0)} = \{a\}$, $\Lambda^{(0,1)} = \{b\}$ be singleton sets. Suppose, moreover, that composition of the morphisms a, b generate all morphisms of the category Λ. In view of the uniqueness required by the factorization property, it soon becomes clear that all the morphisms $w = a^{n_1}b^{m_1}a^{n_2}\cdots b^{m_r}$ with degree defined by $d(w) = (n, m)$, $n = n_1+\ldots+n_r$, $m = m_1+\ldots+m_r$ must coincide. Thus, set $\Lambda^{(n,m)} = \{a^nb^m\}$, and in this way we obtain a 2-graph $\Lambda = \cup_{n\in\mathbb{Z}^2_+}\Lambda^n$.

Note that the 2-graph above is generated by the units and elements of total degree 1, subject to a simple commutation relation. We now give a similar such description of more general 2-graphs, in which a commutation rule $\alpha\times\beta = \theta(\alpha\times\beta)$ is built in to ensure the factorization property.

Let $A = \Lambda_{(G_1,d_1)}$ and $B = \Lambda_{(G_2,d_2)}$ be 1-graphs such that $A^0 = B^0$, so that the underlying graphs G_1 and G_2 have the same number of vertices and the vertex sets are identified. Let v_1, v_2 be two vertices and consider the following sets of pairs of edges,

$$E(v_2, v_1) = \Big\{\alpha \times \beta \in A^1 \times B^1 \;\Big|\; s(\alpha) = r(\beta),\, s(\beta) = v_1,\, r(\alpha) = v_2\Big\}$$
$$F(v_2, v_1) = \Big\{\beta \times \alpha \in B^1 \times A^1 \;\Big|\; s(\beta) = r(\alpha),\, s(\alpha) = v_1,\, r(\beta) = v_2\Big\}.$$

Suppose also that these sets have the same cardinality for all vertex pairs, and that θ is a bijection mapping each $\alpha\times\beta$ in $E(v_2, v_1)$ to an element $\theta(\alpha\times\beta)$ in $F(v_2, v_1)$, for all vertex pairs. To construct the 2-graph $(\Lambda, d) = A *_\theta B$, define

$$\Lambda^0 = V = V(G_1) = V(G_2),$$

$$\Lambda^{(1,0)} = \big\{\alpha \times v \in A^1 \times V \;\big|\; s(\alpha) = v\big\},$$

$$\Lambda^{(0,1)} = \big\{v \times \beta \in V \times B^1 \;\big|\; r(\beta) = v\big\},$$

$$\Lambda^{(1,1)} = \bigcup_{v,w\in V} E(v, w) = \bigcup_{v,w\in V} F(v, w),$$

where the last equality arises from the identifications $\alpha\times\beta = \theta(\alpha\times\beta)$. Plainly, the factorization property holds for morphisms in $\Lambda^{(1,1)}$. Finally, define $\Lambda^{(n,m)}$ as the set of morphisms obtained from arbitrary finite compositions of morphisms in $\Lambda^{(1,0)}$ and $\Lambda^{(0,1)}$ subject to the relations generated by the identifications $(\alpha\times v)(v\times\beta) = (\beta_1 \times v)(v \times \alpha_1)$ if $\alpha_1 \times \beta_1 = \theta(\alpha \times \beta)$. It is routine to check that $\Lambda^{(n,m)}$ satisfies the factorization property with the natural map $d : \Lambda \to \mathbb{Z}_+^2$ (where Λ is the union of all the sets $\Lambda^{(n,m)}$), and thus the pair (Λ, d) is a 2-graph.

Remark 2.2. It convenient to view a 2-graph as being specified through a directed graph in which edges are of two types, red or blue, according to degree, $(1, 0)$ or $(0, 1)$, together with a set of relations that ensure (or are implied by) the factorization property. Thus, in the construction above of $(\Lambda, d) = A *_\theta B$, we can view the red edges of A and the blue edges of B as determining a bichromatic graph. The set $E(v_2, v_1)$ (resp. $F(v_2, v_1)$) is the set of directed paths in this graph that start at v_1, end at v_2, have length two and are coloured blue-then-red (resp. red-then-blue). The bijection θ (for the pair (v_1, v_2)) determines the basic commutation relations, namely, which red-then-blue pairs are equal to blue-then-red pairs. It is thus understood that a given path in the chromatic graph represents an equivalence class of paths under the commutation relations.

The same remarks apply to a k-graph, which yields a k-coloured graph $\Lambda^{(e_1)} \cup \ldots \cup \Lambda^{(e_k)}$. For $k \geq 3$, however, the factorisation property may also depend on (or imply) further commutation relations. (See [30].)

Remark 2.3. Let us clarify our use of the terminology 'semigroupoid' and 'freeness'. To each directed graph G one can associate the (universal) graph C*-algebra, and under mild hypotheses this is isomorphic to a (topological) groupoid C*-algebra $C^*(\mathcal{G})$ associated with the topological path groupoid $\mathcal{G}$ of G. We have no cause in this paper to consider this groupoid, but we do find it convenient to use terminology that derives from the (discrete) groupoid of an undirected graph G. This consists of paths in the edges e of G and their formal inverses e^{-1}, together with the vertices viewed as degenerate edges forming units. With the understanding that the only identification of paths is through the relations $ee^{-1} = r(e)$ and $e^{-1}e = s(e)$, it is natural to refer to this groupoid as the *free* groupoid $\mathbb{F}(G)$ of G. Indeed, in the case of a single vertex graph, this groupoid is the free group on n-generators where n is the number of edges of G. Moreover, if $\mathcal{E}$ is any discrete groupoid generated by elements $e_1, e_2, \ldots$ together with units $x_1, x_2, \ldots$, then this set of generators determines a graph, G say. If $\alpha : \mathcal{E} \to \mathcal{F}$ is a discrete groupoid homomorphism, then there is a lifting $\beta : \mathbb{F}(G) \to \mathcal{F}$ such that $\beta = \pi \circ \alpha$ where $\pi : \mathbb{F}(G) \to \mathcal{E}$ is the natural map. Thus, $\mathbb{F}(G)$ is the free object in the category of discrete groupoids with generators labelled by the graph G.

Similarly, if we omit the formal inverses e^{-1} of the edges of a graph G, then we identify a unital semigroupoid in $\mathbb{F}(G)$, which we denote as $\mathbb{F}^+(G)$ and refer to as the (discrete) free semigroupoid of G. Thus, a 1-graph coincides with the pair $(\mathbb{F}^+(G), d)$, where d is the length function and G is the graph arising from elements of total degree 1.

3. Higher Rank Semigroupoid Algebras

Let (Λ, d) be a k-graph. Let $\mathcal{H}_\Lambda$ be the Fock space of Λ, which we define to be the Hilbert space with orthonormal basis $\{\xi_\lambda : \lambda \in \Lambda\}$. For $\lambda \in \Lambda$ define the operator

L_λ on $\mathcal{H}_\lambda$, such that

$$L_\lambda \xi_\mu = \begin{cases} \xi_{\lambda\mu} & \text{if } s(\lambda) = r(\mu) \\ 0 & \text{if } s(\lambda) \neq r(\mu) \end{cases}.$$

It follows from the factorization property that each L_λ is a partial isometry. Moreover, L_v, $v \in \Lambda^0$, is the projection onto the closed subspace $\text{span}\{\xi_\lambda : r(\lambda) = v\}$.

Definition 3.1. The *semigroupoid algebra* $\mathfrak{L}_{(\Lambda,d)}$ of the k-graph (Λ, d) is the weak operator topology closed linear span of $\{L_\lambda : \lambda \in \Lambda\}$.

Recalling the definition of the grading function δ, it is evident that $\mathcal{H}_\lambda$ is naturally graded as $\mathcal{H}_\lambda = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$, where $\mathcal{H}_n$ is the closed span of the ξ_λ with $\delta(\lambda) = n$.

By arguing exactly as in the case of free semigroupoid algebras [17], one can obtain the following proposition. For brevity we write $\mathfrak{L}_\Lambda$ for $\mathfrak{L}_{(\Lambda,d)}$ and $\mathfrak{R}_\Lambda$ for the analogue of $\mathfrak{L}_\Lambda$ for right actions.

Proposition 3.2. *If $A \in \mathfrak{L}_\Lambda$, then A is the* SOT*-limit of the Cesaro sums*

$$\sum_{\delta(\lambda)\leq n} \Big(1 - \frac{\delta(\lambda)}{n}\Big) a_\lambda L_\lambda,$$

where $a_\lambda \in \mathbb{C}$ is the coefficient of ξ_λ in $A\xi_v = \sum_{s(\lambda)=v} a_\lambda \xi_\lambda$, for $v \in \Lambda^0$.

Thus, elements of $\mathfrak{L}_\Lambda$ have Fourier expansions $A \sim \sum_{\lambda\in\Lambda} a_\lambda L_\lambda$. In particular, this leads to the following description of the commutant:

Proposition 3.3. *The commutant of $\mathfrak{R}_\Lambda$ is $\mathfrak{L}_\Lambda$.*

PROOF. As in the directed graph case, we can consider the Cesaro operators associated with the partition $I = E_0 + E_1 + \ldots$, where E_n is the projection onto $\mathcal{H}_n$. These operators are given by:

$$\Sigma_n(A) = \sum_{\delta(\lambda)=m<n} \Big(1 - \frac{\delta(\lambda)}{n}\Big) \Phi_m(A),$$

where the operators $\Phi_m(A) = \sum_{n\geq \max\{0,-m\}} E_n A E_{n+m}$ are the diagonals of A with respect to the block matrix decomposition associated with the partition. The operators $\Sigma_n(A)$ converge to A in the strong operator topology for all $A \in \mathcal{B}(\mathcal{H}_\Lambda)$.

It is clear that $\mathfrak{L}_\Lambda$ is contained in $\mathfrak{R}'_\Lambda$, thus for the converse we fix $A \in \mathfrak{R}'_\Lambda$. We will show that $A_v \equiv AL_v$ belongs to $\mathfrak{L}_\Lambda$ for all $v \in \Lambda^0$. This will finish the proof, since $A = \sum_{v\in\Lambda^0} AL_v$, the sum converging SOT when Λ^0 is infinite. Let

$A\xi_v = R_v A_v \xi_v = \sum_{s(\lambda)=v} a_\lambda \xi_\lambda$. Define operators in $\mathfrak{L}_\Lambda$ by

$$p_n(A_v) = \sum_{\delta(\lambda)<n;\, s(\lambda)=v} \Big(1 - \frac{\delta(\lambda)}{n}\Big) a_\lambda L_\lambda.$$

We will prove that $A_v = \text{SOT}-\lim_{n\to\infty} p_n(A_v)$ by showing that $p_n(A_v) = \Sigma_n(A_v)$. First, note that $\Phi_m(A_v)$ belongs to $\mathfrak{R}'_\Lambda$ for all m, since A_v belongs to $\mathfrak{R}'_\Lambda$ and $E_{n+1}R_\lambda = R_\lambda E_n$ for all n and $\lambda \in \Lambda^1$, while $\Phi_m(A_v)$ commutes with each projection R_w, since $R_w E_n = E_n R_w$ is the projection onto span$\{\xi_\lambda : \delta(\lambda) = n,\ s(\lambda) = w\}$ for all n. It follows that $\Sigma_n(A_v)$ belongs to $\mathfrak{R}'_\Lambda$ for $n \geq 1$.

Now it is enough to show that $\Sigma_n(A_v)\xi_v = p_n(A_v)\xi_v$. If this is the case, then for $\lambda \in \Lambda$ with $r(\lambda) = v$ we have:

$$\Sigma_n(A_v)\xi_\lambda = R_\lambda \Sigma_n(A_v)\xi_v = R_\lambda p_n(A_v)\xi_v = p_n(A_v)\xi_\lambda,$$

whereas, if $r(\lambda) = w$ with $w \neq v$, then

$$\Sigma_n(A_v)\xi_\lambda = \Sigma_n(A_v)L_v L_w \xi_\lambda = 0 = p_n(A_v)L_w\xi_\lambda = p_n(A_v)\xi_\lambda,$$

since L_v commutes with each E_n.

Observe that $\Phi_0(A_v)\xi_v = E_0(A_v)E_0\xi_v = a_v\xi_v$, and $\Phi_m(A_v)\xi_v = 0$ for $m > 0$. Further, for $m < 0$ we have:

$$\Phi_m(A_v)\xi_v = (E_{-m}A_v)\xi_v = E_{-m}\sum_{s(\lambda)=v} a_\lambda\xi_\lambda = \sum_{s(\lambda)=v;\, \delta(\lambda)=-m} a_\lambda\xi_\lambda.$$

Hence, it follows that

$$\Sigma_n(A_v)\xi_v = \sum_{\delta(\lambda)<n;\, s(\lambda)=v} \Big(1 - \frac{\delta(\lambda)}{n}\Big) a_\lambda \xi_\lambda = p_n(A_v)\xi_v,$$

as required. Thus, each $A_v = AL_v$ belongs to $\mathfrak{L}_\Lambda$, and this completes the proof. ∎

Given Λ, let Λ^t be a category with the same functor d, with $\mathrm{Obj}(\Lambda^t) = \mathrm{Obj}(\Lambda)$ and morphisms for each $\lambda \in \Lambda$ denoted by λ^t with $s(\lambda^t) = r(\lambda)$ and $r(\lambda^t) = s(\lambda)$. Then a simple argument shows that $\mathfrak{L}_\Lambda$ and $\mathfrak{R}_{\Lambda^t}$ are unitarily equivalent via the unitary $U : \mathcal{H}_{\Lambda^t} \to \mathcal{H}_\Lambda$ defined by $U\xi_{\lambda^t} = \xi_\lambda$.

Corollary 3.4. *The commutant of* $\mathfrak{L}_\Lambda$ *is* $\mathfrak{R}_\Lambda$.

Proof. If U is the unitary above, then $\mathfrak{R}'_{\Lambda^t} = (U^*\mathfrak{L}_\Lambda U)' = U^*\mathfrak{L}'_\Lambda U$. Hence, by Proposition 3.3 we have $\mathfrak{R}_\Lambda = U\mathfrak{L}_{\Lambda^t}U^* = U\mathfrak{R}'_{\Lambda^t}U^* = \mathfrak{L}'_\Lambda$. ∎

Corollary 3.5. $\mathfrak{L}_\Lambda$ *is its own second commutant,* $\mathfrak{L}_\Lambda = \mathfrak{L}''_\Lambda$.

4. Examples

We now describe a number of examples of higher rank semigroupoid algebras, starting with some elementary direct product k-graphs.

Example 4.1. Let C_1 be the directed graph with a single vertex v and loop edge e. Then the Fock space $\mathcal{H}_{\Lambda_{C_1}}$ may be identified with the Hardy space H^2, and under this identification $\mathfrak{L}_{\Lambda_{C_1}}$ is unitarily equivalent to the analytic Toeplitz algebra H^∞ [9; 10; 31]. Consider, as in § 2, the natural direct product $\Lambda = \Lambda_{C_1} \times \Lambda_{C_1}$ and let $\overline{v} = v \times v$, $a = e \times v$ and $b = v \times e$. Then $\Lambda^0 = \{\overline{v}\}$, $\Lambda^{(1,0)} = \{a\}$, $\Lambda^{(0,1)} = \{b\}$, and it becomes clear that Λ is the simple 2-graph discussed in § 2. The standard basis for the Fock space $\mathcal{H}_\Lambda \cong H^2 \otimes H^2$ may be identified with the vertices in the 2-lattice of positive integers $\mathbb{Z}_+^2$ and $\mathfrak{L}_\Lambda$ is unitarily equivalent to $H^\infty \otimes H^\infty$.

More generally, given directed graphs $G_1, \ldots, G_k$, the standard basis for the Fock space $\mathcal{H}_{\Lambda_{G_1} \times \ldots \times \Lambda_{G_k}}$ may be identified with the standard basis for $\mathcal{H}_{G_1} \otimes \ldots \otimes \mathcal{H}_{G_k}$, and this identification yields the unitary equivalence $\mathfrak{L}_{\Lambda_{G_1} \times \ldots \times \Lambda_{G_k}} \cong \mathfrak{L}_{G_1} \otimes \ldots \otimes \mathfrak{L}_{G_k}$. For example, if F_n is the directed graph with a single vertex and $n \geq 2$ distinct loop edges, then $\mathfrak{L}_{F_n} = \mathfrak{L}_n$ is the free semigroup algebra (the noncommutative analytic Toeplitz algebra) that acts on unrestricted n-variable Fock space $\mathcal{H}_{F_n} \equiv \mathcal{H}_n$. The standard basis for $\mathcal{H}_n$ is identified with the set of all words from an alphabet with n noncommuting letters. Thus, given positive integers $n_1, \ldots, n_k$, we have $\mathcal{H}_{\Lambda_{F_{n_1}} \times \ldots \times \Lambda_{F_{n_k}}} \cong \mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_k}$ and $\mathfrak{L}_{\Lambda_{F_{n_1}} \times \ldots \times \Lambda_{F_{n_k}}} \cong \mathfrak{L}_{n_1} \otimes \ldots \otimes \mathfrak{L}_{n_k}$.

Example 4.2. Let G be the connected directed graph with two edges $a_1 = (x_2, x_1)$, $a_2 = (x_3, x_2)$. Then the free semigroupoid algebra $\mathfrak{L}_G$ is unitarily equivalent to the operator algebra of matrices

$$\begin{bmatrix} \alpha & 0 & 0 \\ \delta & \beta & 0 \\ \kappa & \epsilon & \gamma \end{bmatrix} \quad \oplus \quad \begin{bmatrix} \beta & 0 \\ \epsilon & \gamma \end{bmatrix} \quad \oplus \quad \begin{bmatrix} \gamma \end{bmatrix},$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \kappa \in \mathbb{C}$, acting on the Fock space

$$\mathcal{H}_G = \big(\mathbb{C}\xi_{x_1} + \mathbb{C}\xi_{a_1} + \mathbb{C}\xi_{a_2 a_1}\big) \ \oplus\ \big(\mathbb{C}\xi_{x_2} + \mathbb{C}\xi_{a_2}\big) \ \oplus\ \mathbb{C}\xi_{x_3}.$$

We can construct the finite 2-graph $\Lambda = \Lambda_G *_\theta \Lambda_G$ described in § 2 as follows:

$$\Lambda^0 = \{x_1, x_2, x_3\},$$

$$\Lambda^{(1,0)} = \{a_1, a_2\}, \quad \Lambda^{(0,1)} = \{b_1, b_2\},$$

$$\Lambda^{(1,1)} = \{b_2 a_1\} = \{a_2 b_1\},$$

with range and source maps such that $x_1 = s(a_1) = s(b_1)$, $r(a_1) = x_2 = s(a_2)$, $r(b_1) = x_2 = s(b_2)$, $x_3 = r(a_2) = r(b_2)$. The rest of Λ consists of $\Lambda^{(2,0)} = \{a_2 a_1\}$ and $\Lambda^{(0,2)} = \{b_2 b_1\}$. Thus, $\Lambda^1 = \Lambda^{(1,0)} \cup \Lambda^{(0,1)}$ includes two 'red' and two 'blue' edges and the relation $b_2 a_1 = a_2 b_1$ specifies all possible commutation relations within Λ.

The Fock space $\mathcal{H}_\Lambda$ is naturally identified with the vertices of three disjoint downward directed graphs, with vertices for the basis vectors $\{\xi_{x_1}, \xi_{x_2}, \xi_{x_3}\}$ at level one, for $\{\xi_{a_1}, \xi_{b_1}, \xi_{a_2}, \xi_{b_2}\}$ at level two, and for $\{\xi_{a_2a_1}, \xi_{b_2a_1} = \xi_{a_2b_1}, \xi_{b_2b_1}\}$ at level three. The action of L_λ, $\lambda \in \Lambda$, is given as the appropriate downward (partial) shift. In particular, $\mathfrak{L}_\Lambda$ can be identified as a matrix algebra on a ten-dimensional Hilbert space.

Example 4.3. (*Higher rank cycle algebras*) Let C_n be the directed cycle graph with n edges $e_i = (x_{i+1}, x_i)$, $1 \leq i \leq n$ $(i+1 \mod n)$. We define k-graphs $C_n^{(k)}$ that are the higher rank variants of these graphs and identify their operator algebras as matrix function algebras. Assume first that $k = 2$. Define $C_n^{(2)}$ to be the 2-graph Λ such that:

$$\Lambda^{(0)} = \{x_1, \dots, x_n\},$$

$$\Lambda^{(1,0)} = \{e_1, \dots, e_n\} \quad \text{and} \quad \Lambda^{(0,1)} = \{f_1, \dots, f_n\},$$

where f_i and e_i have the same sources and same ranges, and where

$$f_{i+1}e_i = e_{i+1}f_i \quad \text{for} \quad 1 \leq i \leq n.$$

In fact, Λ is the unique 2-graph arising from the $*_\theta$ construction with $A = B = C_2$. The Fock space $\mathcal{H}_\Lambda$ has a basis $\{\xi_\lambda\}$, which is in natural correspondence with the vertices of n disjoint graphs, each of which is a downward directed rectangular lattice. The generators L_{e_i}, L_{f_i} can be realized as downward partial shifts, with leftward and rightward actions, respectively. Each vertex carries a label of the form ξ_λ where:

$$\lambda = f_{p+q} \cdots f_{p+1} e_p \cdots e_{i+1} e_i.$$

Thus, we may identify $\mathcal{H}_\Lambda$ with n copies of $H^2 \otimes H^2 = H^2(z,w)$, the Hardy space for the torus $\mathbb{T}^2 = \{(z,w) : |z| = |w| = 1\}$, with its basis $\{z^p w^q : p, q \in \mathbb{Z}_+\}$.

However, there is a more useful related n-fold decomposition of $\mathcal{H}_\Lambda$. We first illustrate this in the case $n = 3$. In this case, the identification above is $\mathcal{H}_\Lambda \cong \mathbb{C}^3 \otimes H^2(z,w)$ with orthonormal basis

$$\{g_i \otimes z^p w^q : 1 \leq i \leq 3,\ p, q \in \mathbb{Z}_+\},$$

where $\{g_1, g_2, g_3\}$ is an orthonormal basis for $\mathbb{C}^3$. Consider now the decomposition $\mathcal{H}_\Lambda = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ where:

$$\begin{aligned}
\mathcal{H}_1 &= \operatorname{span}\big\{g_i \otimes z^{p_i} w^{q_i} : p_1 + q_1 \equiv 0,\ p_2 + q_2 \equiv 2,\ p_3 + q_3 \equiv 1\big\} \\
\mathcal{H}_2 &= \operatorname{span}\big\{g_i \otimes z^{p_i} w^{q_i} : p_2 + q_2 \equiv 0,\ p_3 + q_3 \equiv 2,\ p_1 + q_1 \equiv 1\big\} \\
\mathcal{H}_3 &= \operatorname{span}\big\{g_i \otimes z^{p_i} w^{q_i} : p_3 + q_3 \equiv 0,\ p_1 + q_1 \equiv 2,\ p_2 + q_2 \equiv 1\big\},
\end{aligned}$$

where each of these subspaces is closed and the prescribed addition is modulo 3. Let us dispense with the $\mathbb{C}^3 \otimes H^2(z,w)$ identification above and identify $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ afresh, with $H^2(\mathbb{T}^2)$ in the natural way. Now $\mathcal{H}_\Lambda = H^2(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2)$

and we see that the operators L_{e_1}, L_{e_2}, L_{e_3}, L_{f_1}, L_{f_2}, L_{f_3} are represented by the operator matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ T_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & T_z & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & T_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ T_w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & T_w & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & T_w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

while $\alpha L_{x_1} + \beta L_{x_2} + \gamma L_{x_3}$ is represented by

$$\begin{bmatrix} \alpha I & 0 & 0 \\ 0 & \beta I & 0 \\ 0 & 0 & \gamma I \end{bmatrix}.$$

It follows readily now that $\mathfrak{L}_\Lambda$ is unitarily equivalent to the matrix function algebra

$$\begin{bmatrix} H^\infty_{3,0}(z,w) & H^\infty_{3,2}(z,w) & H^\infty_{3,1}(z,w) \\ H^\infty_{3,1}(z,w) & H^\infty_{3,0}(z,w) & H^\infty_{3,2}(z,w) \\ H^\infty_{3,2}(z,w) & H^\infty_{3,1}(z,w) & H^\infty_{3,0}(z,w) \end{bmatrix},$$

where $H^\infty_{3,i}(z,w)$ is the closed span of the basis elements $\{z^p w^q : p+q \equiv i \mod 3\}$, for $i = 1, 2$.

For the general case, $\Lambda = C_n^{(k)}$ (with $n \neq 3$, $k \neq 2$), we have k sets of morphisms/edges of total δ-degree 1, say

$$\Lambda^{(1,0,\dots,0)} = \{e_1^1, \dots, e_n^1\}, \ \dots, \ \Lambda^{(0,\dots,0,1)} = \{e_1^k, \dots, e_n^k\},$$

and all other morphisms arise from compositions, subject only to identifications through the relations

$$e^s_{i+1} e^r_i = e^r_{i+1} e^s_i$$

for all i ($i+1 \mod n$), and all $1 \leq r \neq s \leq k$. We identify the subspace of $\mathcal{H}_\Lambda$, which is spanned by $\{\lambda : r(\lambda) = x_i\}$ with $H^2(\mathbb{T}^k)$. Then $\mathcal{H}_\Lambda$ is isomorphic to $H^2(\mathbb{T}^k) \oplus \ldots \oplus H^2(\mathbb{T}^k)$ and we identify, as before, $\mathfrak{L}_\Lambda$ with the matrix function algebra

$$\begin{bmatrix} H^\infty_{n,0}(\mathbb{T}^k) & H^\infty_{n,n-1}(\mathbb{T}^k) & \cdots & H^\infty_{n,1}(\mathbb{T}^k) \\ H^\infty_{n,1}(\mathbb{T}^k) & H^\infty_{n,0}(\mathbb{T}^k) & & \vdots \\ \vdots & & \ddots & \\ H^\infty_{n,n-1}(\mathbb{T}^k) & \cdots & & H^\infty_{n,0}(\mathbb{T}^k) \end{bmatrix},$$

where $H^\infty_{n,i}(\mathbb{T}^k)$ is the weak-$*$ closed subspace of $H^\infty(\mathbb{T}^k)$ spanned by the monomials $z_1^{i_1} \cdots z_k^{i_k}$ with $i_1 + \ldots + i_k \equiv i \mod n$.

In view of the matrix function identifications in Alaimia and Peters [1], it is now possible to see that for the higher rank cycle graph $C_n^{(k)}$, its higher rank semigroupoid algebra is equal to the higher rank σ-weakly closed semicrossed product $\mathbb{C}^n \times_\alpha^\sigma \mathbb{Z}_+^k$, where the action $\alpha : \mathbb{Z}_+^k \to \mathrm{Aut}(\mathbb{C}^n)$ is given by $\alpha(m_1, \ldots, m_k) = \sigma^{m_1+\ldots+m_k}$ where σ is the cyclic shift.

5. The Algebras $\mathcal{A}_{\underline{n},\theta}$ and $\mathfrak{L}_{\underline{n},\theta}$

We now consider single vertex k-graphs and their non-self-adjoint operator algebras. In fact, it is convenient to consider somewhat more general algebras, namely $\mathcal{A}_{\underline{n},\theta}$ (norm closed) and $\mathfrak{L}_{\underline{n},\theta}$ (WOT-closed). First, we determine the co-dimension one invariant subspaces of $\mathfrak{L}_{\underline{n},\theta}$ and identify the natural connection with the Gelfand space of the quotient of the algebra $\mathcal{A}_{\underline{n},\theta}$ by its norm-closed commutator ideal. As we shall see, this Gelfand space is homeomorphic to a subspace of the closed polyball $\overline{\mathbb{B}}_{\underline{n}} = \overline{\mathbb{B}}_{n_1} \times \ldots \times \overline{\mathbb{B}}_{n_k}$ determined by a complex algebraic variety associated with the relations θ latent in the k-graph.

Let us now consider the semigroup of a single vertex k-graph explicitly in terms of edge generators and commutation relations. Let $\underline{n} = (n_1, \ldots, n_k)$, where n_i is the number of edges e of degree $d(e) = \delta_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ labelled $e_1^{(i)}, e_2^{(i)}, \ldots, e_{n_i}^{(i)}$. Let θ denote a set $\{\theta_{i,j} : 1 \le i < j \le k\}$ of permutations associated with the relations

$$e_p^{(i)} e_q^{(j)} = \big(\theta_{i,j}(e_p^{(i)} e^{(j)})_q\big)^{\mathrm{op}},$$

where $d(e_p^{(i)}) = \delta_i$, and where $(ef)^{\mathrm{op}}$ denotes the opposite product fe. Thus, $\theta_{i,j}$ is a permutation of the $n_i n_j$ products $e_p^{(i)} e_q^{(j)}$, which we enumerate in the natural order

$$e_1^{(i)} e_1^{(j)}, e_1^{(i)} e_2^{(j)}, \ldots e_1^{(i)} e_{n_j}^{(j)}, e_2^{(i)} e_1^{(j)}, \ldots\ldots, e_{n_i}^{(i)} e_{n_j}^{(j)}.$$

When $k = 2$ there is a single permutation and there is no restriction on this permutation. For $k \ge 3$ the permutations satisfy additional relations to ensure the factorisation property. We remark that one can define the operator algebras on Fock space for permutation sets whose semigroups have left and right cancellation and Theorem 5.1 below holds in this context also.

Consider now a path λ in $\Lambda = \Lambda_{\underline{n},\theta}$, with unique factorization $\lambda = \lambda_1 \lambda_2 \cdots \lambda_k$, where each λ_j is a free word in the loop edges of degree δ_j. For a point $\alpha^{(j)} \in \mathbb{C}^{n_j}$ define the corresponding word $\lambda_j(\alpha^{(j)})$, corresponding to evaluation in $\mathbb{C}$ of the word λ_j, by letterwise substitution, at $\alpha^{(j)} = (\alpha_1^{(j)}, \ldots, \alpha_{n_j}^{(j)})$ in $\mathbb{C}^{n_j}$. Finally, for $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$ in $\mathbb{C}^{n_1} \times \ldots \times \mathbb{C}^{n_k}$ define

$$\lambda(\alpha) = \lambda_1(\alpha^{(1)}) \lambda_2(\alpha^{(2)}) \ldots \lambda_k(\alpha^{(k)}).$$

Thus, we only evaluate the general path λ if it is expressed in its uniquely factored form. If α lies in the open ball product $\mathbb{B}_{n_1} \times \ldots \times \mathbb{B}_{n_k}$, then we may define the unit vector $\nu_\alpha = \frac{\omega_\alpha}{||\omega_\alpha||_2}$ in the Fock space $\mathcal{H}_\Lambda$, where $\omega_\alpha = \sum_{\lambda\in\Lambda} \lambda(\alpha)\xi_\lambda$. Indeed,

$$\begin{aligned} ||\omega_\alpha||_2^2 &= \sum_{\lambda\in\Lambda} |\lambda(\alpha)|^2 \\ &= \sum_{\lambda_1\in\mathbb{F}_{n_1}^+} \cdots \sum_{\lambda_k\in\mathbb{F}_{n_k}^+} |\lambda_1(\alpha^{(1)})|^2 \ldots |\lambda_k(\alpha^{(k)})|^2 \\ &= \prod_{i=1}^{k} \big(1 - ||\alpha^{(i)}||_2^2\big)^{-1}. \end{aligned}$$

Suppose first that the commutation relations given by $\theta = \{\theta_{i,j} : 1 \le i < j \le k\}$ are the commuting relations arising when each $\theta_{i,j}$ is the identity permutation of $d^{-1}(\delta_i)d^{-1}(\delta_j)$. In particular, for each generating edge $e = e_j^{(i)}$ with degree δ_i, we have

$$e\lambda = e\lambda_1\lambda_2\cdots\lambda_k = \lambda_1\cdots\lambda_{i-1}(e\lambda_i)\lambda_{i+1}\cdots\lambda_k,$$

and so $(e\lambda)(\alpha) = \alpha_j^{(i)}\lambda(\alpha)$. We deduce that $L_e^*\omega_\alpha = \alpha_j^{(i)}\omega_\alpha$. Indeed, for all $\lambda \in \Lambda$,

$$\begin{aligned} \big\langle L_e^*\omega_\alpha, \xi_\lambda\big\rangle &= \big\langle \omega_\alpha, \xi_{e\lambda}\big\rangle = (e\lambda)(\alpha) \\ &= \alpha_j^{(i)}\lambda(\alpha) = \alpha_j^{(i)}\big\langle \omega_\alpha, \xi_\lambda\big\rangle = \big\langle \alpha_j^{(i)}\omega_\alpha, \xi_\lambda\big\rangle. \end{aligned}$$

Thus, with $N = n_1+\ldots+n_k$, we have shown that each N-tuple in the product $\mathbb{B}_{n_1} \times \ldots \times \mathbb{B}_{n_k}$ is a joint eigenvalue for the N-tuple $\big\{L^*_{e_1^{(1)}}, \ldots, L^*_{e_{n_k}^{(k)}}\big\}$ with eigenvector ω_α, and that $\{\omega_\alpha\}^\perp$ is therefore a codimension one subspace in $\operatorname{Lat}\mathfrak{L}_\Lambda$. Here, as we have already noted in Example 4.1 above, $\mathfrak{L}_{\underline{n},\theta}$ is naturally identifiable with the spatial tensor product $\mathfrak{L}_{n_1} \otimes \ldots \otimes \mathfrak{L}_{n_k}$.

Suppose now that $\theta = \{\theta_{i,j} : 1 \le i < j \le k\}$ is a family of permutations as before. For $1 \le i \le k$, let $z_{i,1}, \ldots, z_{i,n_i}$ be the coordinate variables for $\mathbb{C}^{n_i}$, so that there is a natural bijective correspondence $e_k^{(i)} \to z_{i,k}$ between edges and coordinate variables. Define $V_\theta \subseteq \mathbb{C}^{n_1} \times \ldots \times \mathbb{C}^{n_k}$ to be the complex algebraic variety determined by the equation set

$$\big\{z_{i,p}z_{j,q} - \hat{\theta}_{i,j}(z_{i,p}z_{j,q}) : 1 \le p \le n_i,\, 1 \le q \le n_j,\, 1 \le i < j \le k\big\},$$

where $\hat{\theta}_{i,j}$ is the permutation induced by $\theta_{i,j}$ and the bijective correspondence.

Theorem 5.1. *Let $\mathfrak{L}_{\underline{n},\theta}$ and $\mathcal{A}_{\underline{n},\theta}$ be the operator algebras associated with a single vertex k-graph. Then*

(i) *Each invariant subspace of $\mathfrak{L}_{\underline{n},\theta}$ of codimension one has the form $\{\omega_\alpha\}^\perp$ for some α in $\mathbb{B}_{\underline{n}} \cap V_\theta$.*

(ii) *The character space $\mathcal{M}(\mathcal{A}_{\underline{n},\theta})$ with the weak star topology is homeomorphic to the set $\Omega_\theta = \mathbb{B}_{\underline{n}} \cap V_\theta$ under the map φ given by:*

$$\varphi(\rho) = \big(\rho(L_{e_1^{(1)}}), \ldots, \rho(L_{e_{n_k}^{(k)}})\big), \quad \textit{for} \quad \rho \in \mathcal{M}(\mathcal{A}_{\underline{n},\theta}).$$

PROOF. To see (ii), first let $\rho \in \mathcal{M}(\mathcal{A}_{\underline{n},\theta})$. Since ρ is a multiplicative linear functional it is completely contractive. Thus, since the row operator $R_i = [L_{e_1^{(i)}}\ L_{e_2^{(i)}}\ \cdots\ L_{e_{n_i}^{(i)}}]$ satisfies $R_iR_i^* \le I$, it follows that the scalar row matrix $[\rho(L_{e_1^{(i)}})\ \rho(L_{e_2^{(i)}})\ \cdots\ \rho(L_{e_{n_i}^{(i)}})]$ is a contraction, and hence $\alpha^{(i)} = (\rho(L_{e_k^{(i)}}))_{k=1}^{n_i}$ lies in $\overline{\mathbb{B}}_{n_i}$. Let $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(k)})$ be the point in $\overline{\mathbb{B}}_{\underline{n}}$ that derives in this way from ρ. We have thus shown that the map φ maps $\mathcal{M}(\mathcal{A}_{\underline{n},\theta})$ into the product ball. In view of the relations $e_p^{(i)}e_q^{(j)} = (\theta_{i,j}(e_p^{(i)}e_q^{(j)}))^{\text{op}}$, we have $e_p^{(i)}e_q^{(j)} = e_s^{(j)}e_r^{(i)}$ for some r, s depending on p, q, thus

$$\begin{aligned}\alpha_p^{(i)}\alpha_q^{(j)} &= \rho(L_{e_p^{(i)}}L_{e_q^{(j)}}) = \rho(L_{e_p^{(i)}}L_{e_q^{(j)}})\\ &= \rho(L_{e_p^{(i)}e_q^{(j)}}) = \rho(L_{e_s^{(j)}e_r^{(i)}})\\ &= \rho(L_{e_s^{(j)}})\rho(L_{e_r^{(i)}}) = \alpha_s^{(j)}\alpha_r^{(i)},\end{aligned}$$

and so, in particular, the polynomial

$$z_{i,p}z_{j,q} - \hat{\theta}_{i,j}(z_{i,p}z_{j,q}) = z_{i,p}z_{j,q} - z_{j,s}z_{i,r}$$

vanishes on α. This is true for all appropriate p, q, i, j and so $\alpha \in \Omega_\theta$.

On the other hand, suppose that $\alpha \in V_\theta$. Then it follows that for any $e = e_j^{(i)}$, we have $(e\lambda)(\alpha) = \alpha_j^{(i)}\lambda(\alpha)$. Thus, for any (unfactored) path λ in the edges of the k-graph, the substitutional evaluation of λ at α coincides with the evaluation of the factored form of λ at α, which we denote $\lambda(\alpha)$. If, in addition, $\alpha \in \mathbb{B}_{\underline{n}} \cap V_\theta$, then our earlier calculation shows that the vector ω_α is an eigenvector for the joint eigenvalue α for the N-tuple $(L^*_{e_1^{(1)}}, \ldots, L^*_{e_{n_k}^{(k)}})$. It follows that the unit vector $\nu_{\overline{\alpha}}$ defines a vector functional

$$\rho(A) = \langle A\nu_{\overline{\alpha}}, \nu_{\overline{\alpha}}\rangle\,,$$

which defines a character ρ in $\mathcal{M}(\mathcal{A}_{\underline{n},\theta})$ with $\varphi(\rho) = \alpha$. Since $\mathcal{M}(\mathcal{A}_{\underline{n},\theta})$ is a compact Hausdorff space, and since we have shown that the range of φ contains $\mathbb{B}_{\underline{n}} \cap V_\theta$ and is contained in Ω_θ, it follows that the range of φ is precisely Ω_θ. From this, part (ii) of the theorem now follows.

To see (i), suppose now that ν is a unit vector such that $\{\nu\}^\perp$ is invariant for $\mathfrak{L}_{\underline{n},\theta}$, and hence that ν is a joint eigenvector for the $|\underline{n}|$-tuple $(L^*_{e_1^{(1)}}, \ldots, L^*_{e_{n_k}^{(k)}})$ with corresponding eigenvalue $\beta = (\beta^{(1)}, \ldots, \beta^{(k)})$. Since the column operators R_i^*, $1 \le i \le k$, are contractions, it follows that $\beta^{(i)} \in \overline{\mathbb{B}}_{n_i}$. Since ν is a joint eigenvector, it follows that the map $L_e \mapsto \langle L_e\nu, \nu\rangle$ extends to a multiplicative linear functional on $\mathcal{A}_{\underline{n},\theta}$ and so from the calculation above $\overline{\beta}$, and hence β, lies in Ω_θ.

Suppose now that $\nu = \sum_{\lambda\in\Lambda} b_\lambda \xi_\lambda$. Then

$$b_\lambda = \langle \nu, \xi_\lambda\rangle = \langle L_\lambda^*\nu, \xi_x\rangle = \lambda(\beta)\langle \nu, \xi_x\rangle = \lambda(\beta)b_x,$$

where, as before, $\lambda(\beta)$ represents the factored form substitution of the word λ in Λ. Our earlier calculation of the norm of ν_α applies here and the finiteness of $\sum |\lambda(\beta)|^2$ implies that $\beta \in \mathbb{B}_{n_1} \cap V_\theta$. ■

Remark 5.2. As an illustration of the theorem, let $\underline{n} = (n_1, n_2)$ and let $z_1, \ldots, z_{n_1}$ and $w_1, \ldots, w_{n_2}$ be coordinate variables for $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$. If θ is a simple cyclic permutation of all the $n_1 n_2$ pairs $\{z_i w_j\}$, then one can check that V_θ is the variety

$$V_\theta = E_{n_1} \times E_{n_2} \cup (\mathbb{C}^{n_1} \times \{0\}) \cup (\{0\} \times \mathbb{C}^{n_2}),$$

where

$$E_{n_1} \times E_{n_2} = \{(z, w) : z_1 = \ldots = z_{n_1},\, w_1 = \ldots = w_{n_2}\}.$$

Thus, the Gelfand space is the direct product of two discs with radii $n_1^{-1/2}$, $n_2^{-1/2}$ and two unit balls. This in fact is the minimal such subset of $\overline{\mathbb{B}}_{\underline{n}}$ associated with a permutation. On the other hand, it can be shown that, in general, these algebras are not isometrically isomorphic, and so the Gelfand space with its geometric and holomorphic structure is only a partial classifying invariant for isometric isomorphism.

6. Reflexivity

Recall that a (WOT-closed) operator algebra $\mathfrak{A}$ is *reflexive* if $\mathfrak{A}$ coincides with the algebra of operators that leave every subspace in the invariant subspace lattice for $\mathfrak{A}$ invariant; $\mathfrak{A} = \operatorname{Alg}\operatorname{Lat}\mathfrak{A}$. On the other hand, a measure of the distance to an operator algebra $\mathfrak{A}$ is given by $\beta_{\mathfrak{A}}(X) = \sup_{L \in \operatorname{Lat}\mathfrak{A}} ||P_L^\perp X P_L||$, where P_L is the projection onto the subspace L. Clearly, $\beta_{\mathfrak{A}}(X) \leq \operatorname{dist}(X, \mathfrak{A})$, and $\mathfrak{A}$ is said to be *hyper-reflexive* if there is a constant C such that $\operatorname{dist}(X, \mathfrak{A}) \leq C\beta_{\mathfrak{A}}(X)$ for all X. We begin by identifying a new class of hyper-reflexive algebras.

As a generalization of terminology from [17; 18], we define the 'double pure cycle property' for a higher-rank graph. Firstly, a *pure cycle* (a monochromatic cycle) is one composed of edges of the same minimal degree and, secondly, Λ has the *double pure cycle property* if, for every $v \in \Lambda^0$, there is a path $\lambda \in \Lambda$ with $s(\lambda) = v$ and $r(\lambda) = w$, such that w lies on a *double pure cycle* in the sense that there is a pair of distinct pure cycles $\lambda_i = w\lambda_i w$, $i = 1, 2$, of the same colour, neither of which may be written as a product (concatenation) of cycles.

Lemma 6.1. *If Λ satisfies the double pure cycle property, then $\mathfrak{L}_\Lambda$ contains a pair of isometries with mutually orthogonal ranges.*

PROOF. We may construct isometries $U, V \in \mathfrak{L}_\Lambda$ with $U^*V = 0$ in a direct manner as follows. Let $\lambda_1 \neq \lambda_2$ be a double pure cycle with $s(\lambda_i) = r(\lambda_i) = v$ for some $v \in \Lambda^0$. We may assume that for all $w \in \Lambda^0$ there is a $\lambda_w \in \Lambda$, such that $s(\lambda_w) = w$ and $r(\lambda_w) = v$. The general case follows easily from this special case. By hypothesis and from the factorization property, for $k \geq 1$ the paths $\lambda_1^k \lambda_2$ are cycles over v; and the partial isometries $L_{\lambda_1^k \lambda_2}$, $k \geq 1$, have mutually orthogonal ranges and initial projection L_v. Let $w \mapsto \{k_a^w, k_b^w\}$ be a one-to-two map from Λ^0 to the positive integers $\mathbb{N}$. As the desired isometries, we may define

$$U = \sum_{w \in \Lambda^0} L_{\lambda_1}^{k_a^w} L_{\lambda_2} L_{\lambda_w} \qquad \text{and} \qquad V = \sum_{w \in \Lambda^0} L_{\lambda_1}^{k_b^w} L_{\lambda_2} L_{\lambda_w},$$

the sums converging SOT when Λ^0 is infinite. ■

Theorem 6.2. *If Λ^t satisfies the double pure cycle property, then $\mathfrak{L}_\Lambda$ is hyper-reflexive with distance constant at most 3.*

PROOF. As $\mathfrak{L}_{\Lambda^t}$ is unitarily equivalent to $\mathfrak{R}_\Lambda = \mathfrak{L}'_\Lambda$, the previous lemma shows that $\mathfrak{L}'_\Lambda$ contains a pair of isometries with mutually orthogonal ranges. Thus, the result follows as a direct application of Bercovici's hyper-reflexivity theorem [4]. ■

As an immediate consequence we obtain the following:

Corollary 6.3. *Let $\underline{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, and suppose that $n_j \geq 2$ for some j. Then $\mathfrak{L}_{\underline{n},\theta}$ is hyper-reflexive for all choices of θ.*

Note that the single vertex algebras $\mathfrak{L}_\Lambda$ that do not satisfy the hypothesis of Corollary 6.3 are each unitarily equivalent to $\mathbb{C}$ or $H^\infty(\mathbb{T}^k) \cong (H^\infty)^{\otimes k}$ for some $k \geq 1$, and these algebras are known to be reflexive [31]. (We note that the problem of hyper-reflexivity for $H^\infty(\mathbb{T}^k)$, $k \geq 2$, appears to remain unresolved at present. The case $k = 1$ is due to Davidson [5].) Reflexivity of these algebras is well-known, but for the interested reader we mention that this fact may be deduced from the first part of the proof of Theorem 6.5 below. Thus, as hyper-reflexivity subsumes reflexivity, it follows now that every single vertex algebra $\mathfrak{L}_\Lambda$ is reflexive. We use this in the proof below.

We shall prove reflexivity for $\mathfrak{L}_\Lambda$ up to a mild graph constraint. We shall say $v \in \Lambda^0$ is a *radiating vertex* if, for each $\lambda \in \Lambda$, with total degree 1, $r(\lambda) = v$ implies that $s(\lambda) = v$. Such a vertex is *multiplicity one* if there is at most one loop edge at v of each colour. Further we say that a radiating vertex $v \in \Lambda^0$ is *relational* if there are loop edges $\mu \neq \mu'$ at v and paths $\lambda, \lambda' \in \Lambda$ with $s(\lambda) = v = s(\lambda')$ that immediately leave v such that $\lambda\mu = \lambda'\mu'$.

Theorem 6.4. *Let Λ be a higher-rank graph with no multiplicity one relational radiating vertices. Then $\mathfrak{L}_\Lambda$ is reflexive.*

PROOF. Let $A \in \operatorname{Alg}\operatorname{Lat}\mathfrak{L}_\Lambda$. We shall show that if $x \in d^{-1}(0)$ and $A = AL_x$, then $A \in \mathfrak{L}_\Lambda$. Since every operator B on $\mathcal{H}_\Lambda$ is the weak operator topology limit of the sums $\sum_{i\geq 1} BL_{x_i}$, where $x_1, x_2, \dots$ is an enumeration of $d^{-1}(0)$, the proof will be complete.

Given $\mu \in \Lambda$ with $r(\mu) = x$, we have

$$A\xi_\mu = \sum_{s(\lambda)=x} \alpha_\lambda^\mu \xi_{\lambda\mu}$$

for some choice of scalars α_λ^μ. This follows since the subspace $\mathcal{M}$ spanned by $\{\xi_{\lambda\mu} : \lambda \in \Lambda\}$ belongs to $\operatorname{Lat}\mathfrak{L}_\Lambda$. Note that $A\xi_\mu = 0$ if $r(\mu) \neq x$. We shall show that for all paths μ, ν with $r(\mu) = x = r(\nu)$, we have $\alpha_\lambda^\mu = \alpha_\lambda^\nu$ for all paths λ with $s(\lambda) = x$.

If this is the case, then for all $\lambda' \in \Lambda$ with $r(\lambda') = s(\mu)$,

$$R_{\lambda'} A\xi_\mu = \sum_\lambda \alpha_\lambda^\mu \xi_{\lambda\mu\lambda'} = \sum_\lambda \alpha_\lambda^{\mu\lambda'} \xi_{\lambda\mu\lambda'} = A\xi_{\mu\lambda'} = AR_{\lambda'}\xi_\mu,$$

and $R_{\lambda'}A\xi_\mu = 0 = AR_{\lambda'}\xi_\mu$ when $r(\lambda') \neq s(\mu)$. Hence, $A \in \mathfrak{R}'_\Lambda$ and so $A \in \mathfrak{L}_\Lambda$, as desired.

We consider two cases. First, suppose that there is a path ν with $\nu = x\nu y$ and $y \neq x$ ($x, y \in d^{-1}(0)$). Then the range $\mathcal{N}$ of $R_x + R_\nu$ is spanned by the set of vectors $\{\xi_{\lambda x} + \xi_{\lambda\nu} : s(\lambda) = x\}$, and the vectors in this set are pairwise orthogonal. Since $\mathcal{N} \in \operatorname{Lat}\mathfrak{L}_\Lambda$, it follows that

$$A(\xi_x + \xi_\nu) = A(R_x + R_\nu)\xi_x = \sum_{s(\lambda)=x} \gamma_\lambda(\xi_{\lambda x} + \xi_{\lambda\nu})$$

for some choice of scalars γ_λ. But $A\xi_x$ and $A\xi_\nu$ are given in terms of the coefficients α_λ^x, α_λ^ν, respectively, and so $\alpha_\lambda^x = \gamma_\lambda = \alpha_\lambda^\nu$ since $s(\nu) \neq x$. Precisely the same argument holds if we replace x by a path μ with $\mu = x\mu x$, and so we obtain $\alpha_\lambda^\mu = \alpha_\lambda^x = \alpha_\lambda^\nu$ for all λ for such a path at x. It follows that $\alpha_\lambda^{\mu'} = \alpha_\lambda^x$ for all λ and for all paths μ' that terminate at x, as desired.

If there is no such path $\nu = x\nu y$ with $y \neq x$, then we are in the second case, in which $\mu = x\mu x$ whenever $r(\mu) = x$. Plainly, this entails that there is a single vertex induced sub-k-graph Γ of Λ such that $L_x\mathfrak{L}_\Lambda|_{L_x\mathcal{H}_\Lambda}$ is unitarily equivalent to $\mathfrak{L}_\Gamma$. By the discussion preceding the theorem, $\mathfrak{L}_\Gamma$ is reflexive, and so we do at least have $\alpha_\lambda^\mu = \alpha_\lambda^x$, for all $\lambda = x\lambda x$, when $\lambda = x\lambda$ ($= x\lambda x$). We shall now show that this equality also holds for paths λ' with $\lambda' = y\lambda' x$, $y \neq x$, and this will complete the proof.

If the loop edges at x include a double pure loop, then we may argue as above and use the Bercovici Theorem to deduce this equality for all λ'. Further, the equality trivially holds when there are no loops at x. Thus, we may reduce to the case that x is a multiplicity one vertex.

Suppose first that λ' is not of the form $\lambda_1 h$ with $h \in \Gamma$ and with $d(h) \neq 0$. Consider the restriction operator $A_{\lambda'} = L_{\lambda'}^* A|_{\mathcal{H}_\Gamma}$. We show that $A_{\lambda'}$ lies in $\mathfrak{L}_\Gamma$. To this end, let $\mathcal{M} \in \operatorname{Lat}\mathfrak{L}_\Gamma$ and define $\widetilde{\mathcal{M}} = \bigvee_{s(\lambda)=x} L_\lambda \mathcal{M}$ in $\operatorname{Lat}\mathfrak{L}_\Gamma$. Then

$$A_{\lambda'}\mathcal{M} = L_{\lambda'}^* A\mathcal{M} \subseteq L_{\lambda'}^* \widetilde{\mathcal{M}}.$$

In view of the assumption on λ', and the constraint on Λ in the hypothesis, we have that $L_{\lambda'}^* L_\lambda$ is non-zero for $s(\lambda) = x$ only if $\lambda = \lambda' h$ with $h \in \Gamma$. Thus,

$$L_{\lambda'}^* \widetilde{\mathcal{M}} \subseteq \bigvee_{h=xhx} L_h \mathcal{M} \subseteq \mathcal{M}.$$

Since we have shown that $A_{\lambda'}$ belongs to $\operatorname{Alg}\operatorname{Lat}\mathfrak{L}_\Gamma$, it follows that $A_{\lambda'}$ is in $\mathfrak{L}_\Gamma$ and hence that there exist scalars α_h such that $A_{\lambda'} \sim \sum_{h=xhx} \alpha_h L_h$.

Thus, if μ is a path in Γ then

$$\begin{aligned}\sum_{h\in\Gamma}\alpha_h\xi_{\lambda' h\mu} &= L_{\lambda'}\big(A_{\lambda'}\xi_\mu\big)\\ &= L_{\lambda'}L_{\lambda'}^*A\xi_\mu\\ &= L_{\lambda'}L_{\lambda'}^*\Big(\sum_{s(\lambda)=x}\alpha_\lambda^\mu\xi_{\lambda\mu}\Big)\\ &= \sum_{h\in\Gamma}\alpha_{\lambda' h}^\mu\xi_{\lambda' h\mu}.\end{aligned}$$

Therefore, $\alpha^\mu_{\lambda' h} = \alpha_h$ for all $h, \mu \in \Gamma$, and $\alpha^\mu_{\lambda' h} = \alpha^\nu_{\lambda' h}$ for all $\mu, \nu \in \Gamma$. As we are in the second case, it follows that $\alpha^\mu_\lambda = \alpha^\nu_\lambda$ for all μ, ν with $r(\mu) = x = r(\nu)$ and λ with $s(\lambda) = x$, as desired. ■

The following proof of reflexivity for the free semigroup algebras $\mathfrak{L}_n$ is considerably more elementary than other proofs in the literature [3; 8] and gives a more direct generalization of Sarason's approach for $\mathfrak{L}_1 = H^\infty$ [31]. We include it for interest's sake.

Theorem 6.5. $\mathfrak{L}_n$ *is reflexive.*

Proof. Let $A \in \operatorname{Alg}\operatorname{Lat}\mathfrak{L}_n$. For $\alpha = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{B}_n$, we may define eigenvectors for $\mathfrak{L}_n^*$ by $\nu_\alpha = \sum_{w\in\mathbb{F}_n^+}\overline{w(\alpha)}\xi_w$. Then $\{\nu_\alpha\}^\perp$ is invariant for $\mathfrak{L}_n$, and so $A^*v_\alpha = \overline{\lambda}_\alpha v_\alpha$ for some scalar λ_α in $\mathbb{C}$.

Let $A\xi_x = \sum_w a_w\xi_w$. Then

$$\begin{aligned}\lambda_\alpha\big\langle\xi_x, v_\alpha\big\rangle = \big\langle A\xi_x, v_\alpha\big\rangle &= \sum_w a_w\big\langle\xi_w, v_\alpha\big\rangle\\ &= \sum_w a_w\big\langle\xi_x, L_w^*v_\alpha\big\rangle = \sum_w a_w w(\alpha)\big\langle\xi_x, v_\alpha\big\rangle,\end{aligned}$$

and so $\lambda_\alpha = \sum_w a_w w(\alpha)$.

Note that for each $v \in \mathbb{F}_n^+$, the subspace $\mathcal{M}_v$ spanned by $\{\xi_{wv} : w \in \mathbb{F}_n^+\}$ belongs to $\operatorname{Lat}\mathfrak{L}_n$, and so for some scalars b_w^v, $w \in \mathbb{F}_n^+$,

$$A\xi_v = \sum_w b_w^v\xi_{wv}.$$

We show that $b_w = a_w$ for all w. This will complete the proof, since we obtain for any choice of v

$$R_vA\xi_x = \sum_w a_w\xi_{wv} = \sum_w b_w^v\xi_{wv} = A\xi_v = AR_v\xi_x.$$

As $A \in \operatorname{Alg}\operatorname{Lat}\mathfrak{L}_n$ was arbitrary, it follows that $R_vAL_w = AL_wR_v = AR_vL_w$ and hence $R_vA\xi_w = R_vAL_w\xi_x = AR_v\xi_w$ for all w. Thus, $R_vA = AR_v$ for all v, and so A belongs to $\mathfrak{R}_n' = \mathfrak{L}_n$.

Observe that

$$\langle A\xi_v, v_\alpha\rangle = \sum_w b_w^v \langle L_{wv}\xi_x, v_\alpha\rangle = \sum_w b_w^v w(\alpha)v(\alpha)\langle \xi_x, v_\alpha\rangle.$$

Further,

$$\langle A\xi_v, v_\alpha\rangle = \langle \xi_v, A^*v_\alpha\rangle = \lambda_\alpha v(\alpha)\langle \xi_x, v_\alpha\rangle,$$

and hence

$$\lambda_\alpha = \sum_w b_w^v w(\alpha).$$

Now,

$$\sum_w b_w^v w(\alpha) = \sum_w a_w w(\alpha)$$

for all $||\alpha||_2 < 1$. (Observe that in the case of H^∞ or $H^\infty(\mathbb{T}^k)$ this fact finishes the proof.)

In particular, we have $a_x = b_x^v$ for all choices of v. Thus, by replacing A with $A - a_xI$ we may assume that $a_x = b_x^v = 0$ for all v. Suppose now that $k \geq 1$ is minimal, such that $a_w = 0 = b_w^v$ for all $|w| < k$ (where $|w| = \delta(w)$ is word length) and all v. We claim that $L_{w'}^*A$ belongs to $\operatorname{Alg}\operatorname{Lat}\mathfrak{L}_n$ for all $|w'| = k$. If this holds, then observe

$$L_{w'}^*A\xi_x = L_{w'}^*\sum_w a_w\xi_w = \sum_u a_{w'u}\xi_u$$
$$L_{w'}^*A\xi_v = L_{w'}^*\sum_w b_w^v\xi_{wv} = \sum_u b_{w'u}^v\xi_{uv},$$

and hence, by the above argument, $a_{w'} = b_{w'}^v$ for all v. Thus we finish the proof by verifying the claim.

Let $\mathcal{M}$ belong to $\operatorname{Lat}\mathfrak{L}_n$. Without loss of generality assume $\mathcal{M}$ is cyclic. Then an orthonormal basis for $\mathcal{M}$ is given by $\{L_w\eta : w \in \mathbb{F}_n^+\}$, where η is some unit vector. (This is follows from part of the Beurling Theorem for $\mathfrak{L}_n$ [8; 27].) As $A\mathcal{M} \subseteq \mathcal{M}$, we have $A\eta = \sum_w c_wL_w\eta$ for some scalars c_w. By our assumption on the scalars a_w, b_w^v, a 'graded Fock space' type argument can be used to show that $c_w = 0$ for all $|w| < k$. Thus,

$$L_{w'}^*A\eta = \sum_{|w|\geq k} c_wL_{w'}^*L_w\eta = \sum_u c_{w'u}L_u\eta \in \mathcal{M}.$$

More generally, for arbitrary u we have $AL_u\eta = \sum_{|w|\geq k+|u|} c_wL_w\eta$, and similarly $L_{w'}^*AL_u\eta \in \mathcal{M}$. Thus $L_{w'}^*A\mathcal{M} \subseteq \mathcal{M}$ for all $|w'| = k$ and all cyclic subspaces $\mathcal{M} \in \operatorname{Lat}\mathfrak{L}_n$, and it follows that $L_{w'}^*A$ belongs to $\operatorname{Alg}\operatorname{Lat}\mathfrak{L}_n$, as claimed. ■

7. Semisimplicity

We say that an edge λ 'lies on a cycle' when there is a cycle $\mu \in \Lambda$, $s(\mu) = r(\mu)$, that includes λ in at least one of its factorizations as a product of edges. Let $\mathrm{NC}(\Lambda)$ be the edges in Λ^1 that do not lie on a cycle. We show that the Jacobson radical of $\mathfrak{L}_\Lambda$, $\mathrm{rad}(\mathfrak{L}_\Lambda)$, is determined by the operators L_λ, $\lambda \in \mathrm{NC}(\Lambda)$, and in the finite vertex case we obtain a complete description of $\mathrm{rad}(\mathfrak{L}_\Lambda)$. Recall that $\mathrm{rad}(\mathfrak{L}_\Lambda)$ is the largest quasinilpotent ideal of $\mathfrak{L}_\Lambda$, and that $\mathfrak{L}_\Lambda$ is semisimple if and only if $\mathrm{rad}(\mathfrak{L}_\Lambda)$ is the zero ideal.

We begin with a combinatorial lemma that shows extremal paths, in the sense of the distance measure, have special factorization properties. Given $\lambda, \mu \in \Lambda$ with $d(\lambda) = (m_1, \ldots, m_k)$ and $d(\mu) = (n_1, \ldots, n_k)$, we write $\lambda \geq \mu$ when the corresponding lexicographic ordering on these k-tuples is satisfied in $\mathbb{N}^k$.

Lemma 7.1. *Let Γ be a nonempty subset of Λ such that $\delta(\lambda_1) = \delta(\lambda_2)$ for all $\lambda_1, \lambda_2 \in \Gamma$, and let $\gamma \in \Gamma$ satisfy $\gamma \geq \lambda$ for all $\lambda \in \Gamma$. If $\gamma^r = \lambda_1 \cdots \lambda_r$ for some $r \geq 1$ and $\lambda_i \in \Gamma$, then $\gamma = \lambda_i$ for $1 \leq i \leq r$.*

Proof. Suppose $\gamma^r = \lambda_1 \cdots \lambda_r$ with $\gamma, \lambda_i \in \Gamma$ for $1 \leq i \leq r$, and put $d(\gamma) = (n_1, \ldots, n_k)$ and $d(\lambda_i) = (n_1^{(i)}, \ldots, n_k^{(i)})$. Then $rn_j = \sum_{i=1}^r n_j^{(i)}$ for $1 \leq j \leq k$, and since $\gamma \geq \lambda_i$ we have $n_1 \geq n_1^{(i)}$ for all i. This gives (with $j = 1$) $n_1 = n_1^{(i)}$ for all i. As $\gamma \geq \lambda_i$, this forces $n_2 \geq n_2^{(i)}$ for all i and, again (with $j = 2$), $n_2 = n_2^{(i)}$ for all i. Hence, we may proceed inductively to obtain $n_j = n_j^{(i)}$ for all $1 \leq i \leq r$ and $1 \leq j \leq k$. Thus, we have $\gamma^r = \lambda_1 \cdots \lambda_r$ with $d(\gamma) = d(\lambda_i)$ for all i, and the result follows from the factorization property. ∎

Theorem 7.2. *$\mathfrak{L}_\Lambda$ is semisimple if and only if every edge in Λ lies on a cycle. If Λ has finitely many vertices, $|\Lambda^0| = n < \infty$, then* $\mathrm{rad}(\mathfrak{L}_\Lambda)$ *is nilpotent of degree at most n and is equal to the* WOT*-closed, two-sided ideal generated by $\{L_\lambda : \lambda \in \mathrm{NC}(\Lambda)\}$.*

Proof. First, suppose that every edge in Λ lies on a cycle. Let $A \in \mathfrak{L}_\Lambda$ be nonzero. Then $A\xi_\lambda = R_\lambda A \xi_{r(\lambda)}$ for all λ, and so there is some $v \in \Lambda^0$ such that $A\xi_v = \sum_{s(\lambda)=v} a_\lambda \xi_\lambda \neq 0$. Let Γ be the set of $\lambda \in \Lambda$ such that $a_\lambda \neq 0$ and $\delta(\lambda)$ is minimal with this property. Let γ be a maximal element in Γ with respect to the lexicographic ordering discussed above. By assumption, there is a path μ such that $\mu\gamma$ is a cycle, and so the paths $(\mu\gamma)^k$, $k \geq 1$, are also cycles. Further, $\mu\gamma$ is maximal in the set $\mu\Gamma$. Thus, by the minimality of $\delta(\gamma)$ and an application of the previous lemma to $\mu\gamma \in \mu\Gamma$ for each $k \geq 1$, from a consideration of Fourier expansions we have

$$(L_\mu A)^k \xi_v = a_\gamma^k \; \xi_{(\mu\gamma)^k} + \sum_{\lambda' \neq (\mu\gamma)^k} b_{\lambda'} \xi_{\lambda'}.$$

Hence, for $k \geq 1$ this yields

$$||(L_\mu A)^k||^{1/k} \geq \left| \langle (L_\mu A)^k \xi_v, \xi_{(\mu\gamma)^k} \rangle \right|^{1/k} = |a_\gamma| > 0.$$

Thus $L_\mu A$ has positive spectral radius and is not quasinilpotent. Since $0 \neq A \in \mathfrak{L}_\Lambda$ was arbitrary, it follows that $\mathrm{rad}(\mathfrak{L}_\Lambda) = \{0\}$ and $\mathfrak{L}_\Lambda$ is semisimple.

Conversely, suppose that $\lambda \in \mathrm{NC}(\Lambda) \neq \emptyset$. Since λ does not lie on a cycle there are no paths $\mu_1, \mu_2 \in \Lambda$ such that both μ_1 and μ_2 contain λ as an edge and $\mu_1\mu_2$ belongs to Λ. Hence, a consideration of Fourier expansions shows that $(AL_\lambda)^2 = 0$ for all $A \in \mathfrak{L}_\Lambda$. Thus L_λ belongs to $\mathrm{rad}(\mathfrak{L}_\Lambda)$ and $\mathfrak{L}_\Lambda$ has non-zero radical.

It remains to verify the structure of $\mathrm{rad}(\mathfrak{L}_\Lambda)$ in the finite vertex case. Let $\mathcal{J}$ be the WOT-closed two-sided ideal in $\mathfrak{L}_\Lambda$ generated by $\{L_\lambda : \lambda \in \mathrm{NC}(\Lambda)\}$. First suppose that A belongs to $\mathrm{rad}(\mathfrak{L}_\Lambda)$ with expansion $A \sim \sum_\lambda a_\lambda L_\lambda$. We claim that a coefficient a_λ is non-zero only if λ includes an edge $\lambda' \in \mathrm{NC}(\Lambda)$. Since the Cesaro sums for A would then belong to $\mathcal{J}$, and they converge in the strong operator topology to A, this would show that A belongs to $\mathcal{J}$. Suppose, by way of contradiction, that there is a path λ with $a_\lambda \neq 0$ that includes no edges from $\mathrm{NC}(\Lambda)$ and, as above, assume λ is maximal in the lexicographic ordering amongst the paths of minimal length with this property. Then λ belongs to a transitive component of Λ. So we may choose $\mu \in \Lambda$ such that $\mu\lambda$ is a cycle in Λ, and hence $(\mu\lambda)^k$ belongs to Λ for $k \geq 1$. Then, by assumption, $(\mu\lambda)^k$ is a path of minimal length and satisfies the maximality condition amongst the paths in the expansion of $(L_\mu A)^k$ that have non-zero coefficients. Hence, an application of the lemma shows that the coefficient of $L_{(\mu\lambda)^k}$ in this expansion is $(a_\lambda)^k$. Thus, we may argue as above to obtain that $L_\mu A$ has positive spectral radius and hence is not quasinilpotent. This contradiction verifies the claim and shows that $\mathcal{J}$ contains $\mathrm{rad}(\mathfrak{L}_\Lambda)$. Notice that this inclusion does not rely on finitely many vertices.

For the converse inclusion note that, in the case that $|\Lambda^0| = n < \infty$, if $\mu_1, \ldots, \mu_n$ are paths in Λ, each of which has a factorization that includes at least one edge from $\mathrm{NC}(\Lambda)$, then the product $\mu_n \cdots \mu_1$ cannot belong to Λ. It follows that $\mathcal{J}^n = \{0\}$ in this case, because any operator X of the form $X = A_1 \cdots A_n$ with each $A_i \in \mathcal{J}$ can have no non-zero Fourier coefficients. Thus, $\mathcal{J}$ is contained in $\mathrm{rad}(\mathfrak{L}_\Lambda)$ and the result follows. ∎

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