LEFT–RIGHT CONSISTENCY IN RINGS II

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Abstract

A bounded linear operator T is said to be 'left or right consistent' if the spectra of all the products ST and TS satisfy an inclusion relation.

1. Introduction

Suppose A is a semigroup, with identity 1 and invertible group $A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}$, or more generally an abstract category. Elements $a \in A$ induce left and right multiplications on A,

$$L_a : x \mapsto ax \ ; \ R_a : x \mapsto xa \ .$$

It is the relationship between these operators that gives rise to 'left or right consistency'. We recall [1; 2] that an element $a \in A$ is said to be 'left-right consistent' if, for arbitrary $b \in A$, the spectra $\sigma(ba)$ and $\sigma(ab)$ coincide: equivalently ba and ab are invertible or not together. In this note, we break this condition into two parts—according as one spectrum is included in the other, equivalently one invertibility implies the other:

Definition 1. *If $K \subseteq A$ is arbitrary, write*

$$\varpi_{left}(K) = \{a \in A : R_a^{-1}(K) \subseteq L_a^{-1}(K)\} \tag{1}$$

for the set of 'left K consistent' elements of A,

$$\varpi_{right}(K) = \{a \in A : L_a^{-1}(K) \subseteq R_a^{-1}(K)\} \tag{2}$$

for the set of 'right K consistent' elements of A, with

$$\varpi(K) = \varpi_{left}(K) \cap \varpi_{right}(K) \tag{3}$$

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for the set of 'left-right K consistent' $a \in A$.

Like $\varpi(K)$ [2], $\varpi_{left}(K)$ and $\varpi_{right}(K)$ are always sub-semigroups of A:

$$\varpi_{left}(K)\varpi_{left}(K) \subseteq \varpi_{left}(K) \ ; \ \varpi_{right}(K)\varpi_{right}(K) \subseteq \varpi_{right}(K) \ . \tag{4}$$

In this note, we determine $\varpi_{left}(K)$ and $\varpi_{right}(K)$ for the invertible group $K = A^{-1}$ and for the semigroups A^{-1}_{left} and A^{-1}_{right} of left and of right invertibles:

Theorem 1. *For arbitrary A there is equality*

$$\varpi_{left}(A^{-1}) = A^{-1} \cup (A \setminus A^{-1}_{left}) = \varpi_{left}(\{1\}) = \varpi_{left}(A^{-1}_{left}) \tag{5}$$

and

$$\varpi_{right}(A^{-1}) = A^{-1} \cup (A \setminus A^{-1}_{right}) = \varpi_{right}(\{1\}) = \varpi_{right}(A^{-1}_{right}). \tag{6}$$

Proof. If $a \in A^{-1}$ then, as in [2],

$$L_a^{-1}(A^{-1}) = R_a^{-1}(A^{-1}) \text{ and } L_a^{-1}(A^{-1}_{left}) = R_a^{-1}(A^{-1}_{left}),$$

giving $A^{-1} \subseteq \varpi_{left}(A^{-1})$ and $A^{-1} \subseteq \varpi_{left}(A^{-1}_{left})$. If $a \in A \setminus A^{-1}_{left}$, then

$$R_a^{-1}(A^{-1}) \subseteq R_a^{-1}(A^{-1}_{left}) = \emptyset,$$

giving $A \setminus A^{-1}_{left} \subseteq \varpi_{left}(A^{-1})$ and $A \setminus A^{-1}_{left} \subseteq \varpi_{left}(A^{-1}_{left})$. If $a \in A^{-1}_{left} \setminus A^{-1}$, with $a'a = 1 \neq aa'$, then

$$a' \in R_a^{-1}(A^{-1}) \setminus L_a^{-1}(A^{-1}_{left}) \ .$$

This proves (5), and similarly (or by 'reversal of products') (6). ■

For $\varpi_{left}(A^{-1}_{right})$ and $\varpi_{right}(A^{-1}_{left})$, we have to go again to the 'mixed invertible' elements of A,

$$A^{-1}_{mixed} = \{a \in A : 1 \in AaA\}. \tag{7}$$

Theorem 2. *If A is arbitrary, then*

$$\varpi_{left}(A^{-1}_{right}) = A^{-1}_{right} \cup (A \setminus A^{-1}_{mixed}) \tag{8}$$

and

$$\varpi_{right}(A^{-1}_{left}) = A^{-1}_{left} \cup (A \setminus A^{-1}_{mixed}) \ . \tag{9}$$

Proof. It is clear that A^{-1}_{right} is a subset of $\varpi_{left}(A^{-1}_{right})$: if $a \in A^{-1}_{right}$, then

$$xa \in A^{-1}_{right} \Longrightarrow x \in A^{-1}_{right} \Longrightarrow ax \in A^{-1}_{right} \ .$$

It is also clear that $A \setminus A^{-1}_{mixed}$ is a subset of $\varpi_{left}(A^{-1}_{right})$: by [2, theorem 7], it is included in $\varpi(A^{-1}_{right})$. Finally, if $a \in A^{-1}_{mixed} \setminus A^{-1}_{right}$, then it cannot be in $\varpi_{left}(A^{-1}_{right})$:

$$a \in A^{-1}_{mixed} \setminus A^{-1}_{right} \Longrightarrow L_a^{-1}(A^{-1}_{right}) = \emptyset \neq R_a^{-1}(A^{-1}_{right}) \ .$$

This proves (8), and hence also (9). ■

Young Oh Kim and Woo Young Lee [6] have obtained part of Theorem 2 when $A = B(X)$ is the bounded operators on a Banach space X, and also when $A = \frac{B(X)}{K(X)}$ is the Calkin algebra. In a C^*-algebra we get:

Theorem 3. *If A is a C^*-algebra and $a \in A$, then*

$$a \in \varpi_{left}(A^{-1}) \Longleftrightarrow \big(a^*a \in A^{-1} \text{ or } aa^* \notin A^{-1}\big) \Longleftrightarrow a \in \varpi_{left}(A^{-1}_{left}) \tag{10}$$

and

$$a \in \varpi_{right}(A^{-1}) \Longleftrightarrow \big(aa^* \in A^{-1} \text{ or } a^*a \notin A^{-1}\big) \Longleftrightarrow a \in \varpi_{right}(A^{-1}_{right}) \ . \tag{11}$$

In particular, hyponormal elements of $A = B(X)$ are in $\varpi_{left}(A^{-1})$ for a Hilbert space X.

PROOF. $a^*a \in A^{-1} \Longleftrightarrow a \in A^{-1}_{left}$. ■

We recall [3; 4] the 'regular' elements of a semigroup, or more generally a category:

$$\overset{\sqcap}{A} = \{a \in A : a \in aAa\} \ , \tag{12}$$

elements $a = aba$ with *generalized inverses* $b \in A$, together with the 'decomposably regular' elements

$$\overset{\sqcup}{A} = \{a \in A : a \in aA^{-1}a\} \ , \tag{13}$$

elements with invertible generalized inverses $b \in A^{-1}$. It is clear from [2, theorem 4] that the decomposably regular elements (13) satisfy

$$\overset{\sqcup}{A} \subseteq \omega(K) \tag{14}$$

for each $\omega(K)$ of $\varpi_{left}(A^{-1})$, $\varpi_{right}(A^{-1})$, $\varpi_{left}(A^{-1}_{left})$, $\varpi_{right}(A^{-1}_{left})$, $\varpi_{left}(A^{-1}_{right})$ and $\varpi_{right}(A^{-1}_{right})$, and also that if $J \subseteq A$ is [5; 2] a 'completely regular weakly Riesz' two-sided ideal, then ([2, theorem 6])

$$\overset{\sqcup}{A} \cup (A \setminus \overset{\sqcap}{A}) \subseteq \bigcap_{d \in J} (\omega(K) + d), \tag{15}$$

for the same $\omega(K)$.

Associated with the 'consistency' or otherwise of $a \in A$ and its scalar perturbations $a - \lambda$ are related modifications of the spectrum:

Definition 2. If ω is a spectral mapping on an algebra A, then the associated semi consistent spectral mappings are given, for $a \in A$, by

$$\boldsymbol{C} \setminus \omega^{\sim}_{left}(a) = \{\lambda \in \boldsymbol{C} : R^{-1}_{a-\lambda} H_\omega \subseteq L^{-1}_{a-\lambda} H_\omega\} \ , \tag{16}$$

$$\boldsymbol{C} \setminus \omega^{\sim}_{right}(a) = \{\lambda \in \boldsymbol{C} : L^{-1}_{a-\lambda} H_\omega \subseteq R^{-1}_{a-\lambda} H_\omega\} \tag{17}$$

and

$$\boldsymbol{C} \setminus \omega^{\sim}(a) = \{\lambda \in \boldsymbol{C} : R^{-1}_{a-\lambda} H_\omega = L^{-1}_{a-\lambda} H_\omega\} \ , \tag{18}$$

where

$$H_\omega = \{a \in A : 0 \notin \omega(a)\} \ . \tag{19}$$

Thus, if ω is the usual spectrum σ, then H_ω is the group A^{-1} of invertibles. The characterization [2] of left-right consistent elements can be expressed spectrally:

Theorem 4. *If $a \in A$, then*

$$\sigma^{\sim}(a) = \sigma(a) \setminus (\sigma^{left}(a)_\cap \sigma^{right}(a)) \ . \tag{20}$$

Also

$$\sigma^{left\sim}(a) = \sigma^{right\sim}(a) = \sigma(a) \setminus \sigma^{mixed}(a) \ , \tag{21}$$

where

$$\sigma^{mixed}(a) = \{\lambda \in \boldsymbol{C} : a - \lambda \notin A^{-1}_{mixed}\} \ . \tag{22}$$

It follows

$$\sigma^{\sim}(a) \subseteq \ int \ \sigma(a) \ . \tag{23}$$

Proof. For (20) recall ([2, theorem 2]):

$$\omega(A^{-1}) = A^{-1}{}_\cap\big(A \setminus (A^{-1}_{left}{}_\cap A^{-1}_{right})\big) \ , \tag{24}$$

and take complements. For (21) do the same with ([2, theorem 7]):

$$\omega(A^{-1}_{left}) = \omega(A^{-1}_{right}) = A^{-1}{}_\cup\big(A \setminus A^{-1}_{mixed}\big) \ . \tag{25}$$

Now for (23), recall the inclusion

$$\partial\sigma(a) \subseteq \sigma^{left}(a)_\cap \sigma^{right}(a).$$

■

Theorem 4 has 'semi' analogues:

Theorem 5. *If $a \in A$, then*

$$\sigma^{\sim}_{left}(a) = \sigma^{left\sim}_{left}(a) = \sigma(a) \setminus \sigma^{left}(a) \tag{26}$$

and

$$\sigma^{\sim}_{right}(a) = \sigma^{right\sim}_{right}(a) = \sigma(a) \setminus \sigma^{right}(a) \ . \tag{27}$$

Also,

$$\sigma^{left\sim}_{right}(a) = \sigma^{right}(a) \setminus \sigma^{mixed}(a) \tag{28}$$

and

$$\sigma^{right\sim}_{left}(a) = \sigma^{left}(a) \setminus \sigma^{mixed}(a) \ . \tag{29}$$

PROOF. Take complements in (5), (6), (8) and (9). ■

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