# INVARIANCE OF GENERALISED REYNOLDS IDEALS UNDER DERIVED EQUIVALENCES

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Abstract

For any algebraically closed field $k$ of positive characteristic $p$ and any non negative integer $n$, Külshammer defined ideals $T_nA^\perp$ of the centre of a symmetric $k$-algebra $A$. We show that for derived equivalent algebras $A$ and $B$, there is an isomorphism of the centres of $A$ and $B$ mapping $T_nA^\perp$ to $T_nB^\perp$ for all $n$. Recently, Héthelyi, Horváth, Külshammer and Murray showed that this holds for Morita equivalent algebras.

## 1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $A$ be a finite dimensional symmetric $k$-algebra with non degenerate symmetrising bilinear form $( \ , \ )$ on $A$. Külshammer defined in [9] ideals $T_nA^\perp$ of the centre of $A$, by means of the following construction: let $KA$ be the $k$-subspace of $A$ generated by $ab - ba$ for all $a, b \in A$ and set $T_nA := \{x \in A \mid x^{p^n} \in KA\}$. Let $T_nA^\perp$ be the subspace orthogonal to $T_nA$ with respect to the form $( \ , \ )$ on $A$. Note that $T_nA^\perp$ is then an ideal of $ZA$ as $ZA = KA^\perp$, and that $T_nA$ is a $ZA$ submodule of $A$.

In [10] Külshammer shows that the equation $(\zeta_n(z), x)^{p^n} = (z, x^{p^n})$ for any $x, z$ in the centre of $A$ defines a mapping $\zeta_n$ from the centre of $A$ to the centre of $A$. Moreover, $\zeta_n(A) = T_nA^\perp$. Many properties of group algebras can be shown using the ideals $T_nA^\perp$. Concerning the ideals $T_nA^\perp$, Héthelyi *et al.* show in [5] that $Z_0A \subseteq (T_1A^\perp)^2 \subseteq HA$, where $HA$ is the Higman ideal of $A$, and where $Z_0A$ is the sum of the centres of those blocks of $A$ that are simple algebras. They show that for odd $p$ the left inclusion is an equality, whereas for $p = 2$ one gets $Z_0A = (T_1A^\perp)^3 = (T_1A^\perp) \cdot (T_2A^\perp)$. Finally, Héthelyi *et al.* show that $e \cdot (T_nA^\perp) \cdot e = T_n(eAe)^\perp$ for any idempotent $e$ of $A$. They then use the fact that within all algebras Morita equivalent to $A$, there is—up to isomorphism unique—a smallest algebra $B = eAe$ Morita equivalent to $A$, the basic algebra. If $A$ is symmetric, $B$ is symmetric as well, as follows in a more general context by Zimmermann [15]. Multiplication by this idempotent induces an isomorphism between the centers of an algebra $A$ and its basic algebra $B$. Hence, the corresponding ideals $T_nA^\perp$ and $T_nB^\perp$ are sent to each

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other by this isomorphism. Composing two of them gives a corresponding statement for Morita equivalent algebras.

In [5, question 5.4], Héthelyi *et al.* ask whether, for two symmetric algebras $A$ and $B$, the condition that the derived categories of $A$ and of $B$ are equivalent implies the existence of an isomorphism $\varphi$ of their centres, so that $\varphi$ induces an isomorphism between the ideals $T_nA^\perp$ and $T_nB^\perp$ for all $n \in \mathbb{N}$. The main objective of this paper is to give a positive answer to this question.

In doing so, we provide new invariants for equivalences between the triangulated categories $D^b(A)$ and $D^b(B)$ for algebras $A$ and $B$. A number of invariants are known. Suppose $D^b(A) \simeq D^b(B)$ as triangulated categories, then we get an isomorphism of the Hochschild homology $HH_*(A) \simeq HH_*(B)$ and the Hochschild cohomology $HH^*(A) \simeq HH^*(B)$ (cf. Rickard [12]); the cyclic homology $HC_*(A) \simeq HC_*(B)$; the cyclic cohomology $HC^*(A) \simeq HC^*(B)$ of the algebra $A$ (Keller [7]); or, by a result of Thomason and Trobaugh [13], the $K$-theory $K_*(A) \simeq K_*(B)$. Some of these are quite useful in specializing the degree. Thus, $HH^0(A) \simeq Z(A)$ is the centre of the algebra $A$, or $rank_{\mathbb{Z}}(K_0(A))$ equals the number of isomorphism classes of simple $A$-modules. Nevertheless, if these few computable invariants coincide, it is in general very difficult to decide whether two algebras have equivalent derived categories or not. So, invariants that are easier to determine in examples will be very welcome. Our result provides some such invariants.

Our main result, Theorem 1, will be proven in Section 4. Since there is no analogue of a basic algebra for derived equivalences, we need to proceed differently from Héthelyi *et al.*'s proof for Morita equivalence. We remark that the arguments hold equally well for Morita equivalences instead of derived equivalences, and so we give an alternative, more direct proof also for the weaker result of Héthelyi *et al.* Section 2 recalls some of the relevant notation and results from homological algebra, for the convenience of the reader. We use the characterisation [10, (46)] or [5, lemma 2.1] of $T_nA^\perp$ as the image of the mapping $\zeta_n^A$; and we define in Section 3 the mapping $\zeta_n^A$ in a functorial manner by means of a composition of mappings between $A \otimes_k A^{op}$-modules. We apply the derived equivalence to each of the factors and, using results in [15] and some delicate commutativity considerations, we are able to show that the mapping induced by a standard derived equivalence on the morphism sets are indeed as asked. For notations concerning derived categories and equivalences we follow König and Zimmermann [8]. Other references covering the needed background are Gelfand and Manin [2] or Weibel [14]. The notation may differ slightly there.

## 2. A crash course on the relevant homological algebra

For the reader's convenience and to fix notation, we shall recall some basic facts in homological algebra, as they are needed in the sequel. Our basic source is the book by König and Zimmermann [8], and for some more general aspects see Gelfand and Manin [2], Weibel [14] or Rickard [12], as well as [15].

For a commutative noetherian ring $k$ and a finitely generated $k$-algebra $A$, we denote the category of finitely generated left $A$-modules by $A-mod$ and the category of all $A$-modules by $A-Mod$. Let $K(A-mod)$ be the category of complexes in $A-mod$

modulo homotopy. Recall that the derived category $D^b(A)$ of bounded complexes of finitely generated $A$-modules is formed by bounded complexes in $K(A - mod)$ and by formally inverting morphisms that induce isomorphisms on homology. Recall furthermore that $A - mod$ is a full subcategory of $D^b(A)$ by mapping a module $M$ to a complex with homogeneous components 0 in all degrees, except in degree 0 where the homogeneous component is $M$ (cf., for example, Gelfand and Manin [2, III, §5, proposition 2]). Hence, any two objects $M$ and $N$ of $A$-mod may be considered as objects in $D^b(A)$, and then $Hom_{D^b(A)}(M, N) = Hom_A(M, N)$. This fact will be used in various places throughout this paper.

Recall that if $X$ is a complex in $D^b(A)$ whose homogeneous components are all projective, then for any complex $Y$ in $D^b(A^{op})$ one has $Y \otimes_A X = Y \otimes^{\mathbb{L}}_A X$. As usual, we denote by $- \otimes^{\mathbb{L}} -$ the left derived tensor product functor (cf., for example, Weibel [14, 10.6.1]). In the case where $A$ and $B$ are two algebras over a field $k$, then for any $X$ in $D^b(A \otimes_k B^{op})$ there is an $\tilde{X}$ in $D^b(A \otimes_k B^{op})$, so that $X \simeq \tilde{X}$ and so that all homogeneous components of $\tilde{X}$ are projective as $A$-modules and as $B^{op}$-modules (cf. [8, lemma 6.3.12]).

Let $B$ be a $k$-algebra that is projective as $k$-module. By a result due to Keller (cf. [6]) or see [8, chapter 8]), $D^b(A)$ is equivalent to $D^b(B)$ as triangulated categories if and only if there is a complex $X$ in $D^b(B \otimes_k A)$, so that $X \otimes^{\mathbb{L}}_A - : D^b(A) \longrightarrow D^b(B)$ is an equivalence. Such equivalences are called standard and $X$ is called a (two-sided) tilting complex. If $B$ is symmetric, then $A$ is symmetric as well (cf. [15]), and then the inverse equivalence to $X \otimes^{\mathbb{L}}_A -$ is given by $Hom_k(X, k) \otimes^{\mathbb{L}}_B -$. Moreover, Rickard has shown that this $X \otimes^{\mathbb{L}}_A - \otimes^{\mathbb{L}}_A Hom_k(X, k)$ actually defines an equivalence $D^b(A \otimes_k A^{op}) \longrightarrow D^b(B \otimes_k B^{op})$, where the $(A \otimes_k A^{op})$-module $A$ is mapped to $B$ (cf. Rickard [12] or see [8, proposition 6.2.6]). Hence, $X$ induces an isomorphism

$$Z(A) = End_{A \otimes_k A^{op}}(A) = End_{D^b(A \otimes_k A^{op})}(A) \simeq$$
$$End_{D^b(B \otimes_k B^{op})}(B) = End_{B \otimes_k B^{op}}(B) = Z(B).$$

This isomorphism is explicitly exhibited in [8, proposition 6.2.6]. In [15] it is shown that under the equivalence induced by tensoring with $X$, the $(A \otimes_k A^{op})$-module $Hom_k(A, k)$ is mapped to $Hom_k(B, k)$.

We finish with some notation. Let $\mathcal{C}$ be a category and let $X$, $Y$ and $Z$ be any three objects in $\mathcal{C}$. We denote for any morphism $\varphi \in Hom_{\mathcal{C}}(X, Y)$, the induced mapping $Hom_{\mathcal{C}}(Z, \varphi) : Hom_{\mathcal{C}}(Z, X) \longrightarrow Hom_{\mathcal{C}}(Z, Y)$, which is defined by $(Hom_{\mathcal{C}}(Z, \varphi))(\psi) := \varphi \circ \psi$ for any $\psi \in Hom_{\mathcal{C}}(Z, X)$. If $Z$ is clear from the context, we write $Hom_{\mathcal{C}}(Z, \varphi) =: \varphi_*$ for short. Dually, to facilitate the reading, for any $k$-vector space $X$, we denote occasionally by $X^*$ the $k$-linear dual $Hom_k(X, k)$, and for every $k$-linear mapping $\varphi : X \longrightarrow Y$ we note by $\varphi^*$ the induced mapping $Y^* \longrightarrow X^*$ on the dual spaces.

## 3. Interpreting $\zeta$

Recall from Section 2 that $Hom_{D^b(A \otimes_k A^{op})}(A, A) = End_{A \otimes_k A^{op}}(A) \simeq Z(A)$ .

Furthermore, by the adjointness formulas (cf. Mac Lane [11, VI, (8.7)]), we get:

$$Hom_k(A \otimes_{A \otimes_k A^{op}} A, k) \simeq Hom_{A \otimes_k A^{op}}(A, Hom_k(A, k))$$

$$f \mapsto (a \mapsto (b \mapsto f(a \otimes b)))\,,$$

and since, canonically, by the very definition of a tensor product, $A \otimes_{A\otimes_k A^{op}} A \simeq A/KA$, where $KA = \sum_{a,b\in A} k \cdot (ab - ba)$ is the $k$-vector space generated by commutators, we have a functorial isomorphism:

$$\begin{array}{c} Hom_k(A/KA, k) \simeq Hom_{A\otimes_k A^{op}}(A, Hom_k(A, k)) \\ f \mapsto (a \mapsto (b \mapsto f(ab))). \end{array}$$

The mapping $A/KA \ni a \mapsto a^p \in A/KA$ was first defined by Richard Brauer, who called it the Frobenius mapping and proved that it is well defined (cf. Külshammer [9, II]) and semi-linear. Denote by $k^{(n)}$ the $n$ times Frobenius twisted copy of $k$.

The Frobenius mapping induces a well defined mapping:

$$\begin{array}{ccc} Hom_k(A/KA, k) & \stackrel{(Fr^A)^*}{\longrightarrow} & Hom_k(A/KA, k^{(1)}) \\ f & \mapsto & (a \mapsto f(a^p))\,. \end{array}$$

The mapping

$$\begin{array}{rl} {Fr^k}_* : \; Hom_k(A/KA, k) & \longrightarrow Hom_k(A/KA, k^{(1)}) \\ f & \mapsto \; (a \mapsto f(a)^p) \end{array}$$

induces a mapping

$$Hom_{A\otimes_k A^{op}}(A, Hom_k(A, k)) \longrightarrow Hom_{A\otimes_k A^{op}}(A, Hom_k(A, k^{(1)})),$$

and since for any algebra $B$ one has a fully faithful embedding of $B - mod$ into $D^b(B)$, by considering a $B$-module as a complex with differential 0 and modules concentrated in degree 0 only, this in turn gives a mapping:

$$Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A, k)) \longrightarrow Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A, k^{(1)})).$$

Put $A^* := Hom_k(A, k)$. Recall that a $k$-algebra $A$ is symmetric if and only if there is an isomorphism of $A \otimes_k A^{op}$-bimodules $A \simeq A^*$, or equivalently if there is a non degenerate symmetric bilinear form $(\ ,\ ) : A \times A \longrightarrow k$ satisfying $(a, cb) = (ac, b)$ for any $a, b, c \in A$ (cf., for example, [8, chapter 9]). Then the mapping $\zeta_n^A$ is defined by the equation $(\zeta_n(z), x)^{p^n} = (z, x^{p^n})$, which can be written as the composition of the mappings in the following diagram (‡):

$$\begin{array}{ccc} Hom_{D^b(A\otimes_k A^{op})}(A, A) & \longrightarrow & Hom_{D^b(A\otimes_k A^{op})}(A, A^*) \\ & & \downarrow ((Fr^k)_*)^n \\ \uparrow \zeta_n^A & & Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A, k^{(n)}) \qquad (‡) \\ & & \uparrow ((Fr^A)^*)^n \\ Hom_{D^b(A\otimes_k A^{op})}(A, A) & \longrightarrow & Hom_{D^b(A\otimes_k A^{op})}(A, A^*), \end{array}$$

where the horizontal arrows are induced by the isomorphism

$$\begin{array}{rcl} A & \longrightarrow & A^* \\ a & \mapsto & (b \mapsto (a,b)), \end{array}$$

which is coming from the symmetrising bilinear form $(\ ,\ ) : A \otimes_k A \longrightarrow k$ of $A$.

## 4. Behaviour under derived equivalences

In this section we prove our main result.

**Theorem 1.** *Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $A$ and $B$ be finite dimensional $k$-algebras. If $D^b(A) \simeq D^b(B)$ as triangulated categories, then there is an isomorphism $\varphi : ZA \longrightarrow ZB$ between the centres $ZA$ of $A$ and $ZB$ of $B$, so that $\varphi(T_nA^\perp) = T_nB^\perp$ for all positive integers $n \in \mathbb{Z}$.*

*Remark 1.* This answers the positive question 5.4 posed by László Héthelyi, Erszébet Horváth, Burkhard Külshammer and John Murray in [5].

PROOF. Let $F : D^b(A) \longrightarrow D^b(B)$ be a standard derived equivalence with two-sided tilting complex $X$. Let $X'$ be the inverse tilting complex. Then, in [15] it is shown that $X \otimes_A - \otimes_A X'$ induces an equivalence $G : D^b(A \otimes_k A^{op}) \longrightarrow D^b(B \otimes_k B^{op})$, mapping the $A$-$A$-bimodule ${}_AA_A$ to the $B$-$B$-bimodule ${}_BB_B$,

$$G({}_AA_A) = {}_BB_B.$$

From [15, lemma 1], we know that

$$G\left(Hom_k(A,k)\right) = Hom_k(B,k).$$

We shall show that

$$G\left(Hom_k(A,k^{(n)})\right) = Hom_k(B,k^{(n)})$$

for all $n \in \mathbb{Z}$. Indeed, $X \otimes_A - \simeq Hom_A(X', -)$ and $- \otimes_A X' \simeq Hom_A(X, -)$ by the adjointness properties of $Hom$ and $\otimes$-functors. Hence, (cf. [8, proof of corollary 6.3.6]),

$$\begin{array}{rcl} X \otimes_A Hom_k(A,k^{(n)}) \otimes_A X' & \simeq & Hom_A(X', Hom_k(A,k^{(n)})) \otimes_A X' \\ & \simeq & Hom_k(A \otimes_A X', k^{(n)}) \otimes_A X' \\ & \simeq & Hom_k(X', k^{(n)}) \otimes_A X' \\ & \simeq & Hom_A(X, Hom_k(X', k^{(n)})) \\ & \simeq & Hom_k(X \otimes_A X', k^{(n)}) \\ & \simeq & Hom_k(B, k^{(n)}). \end{array} \qquad (\dagger)$$

We now apply $G$ to diagram (‡) of Section 3 and get a commutative diagram:

$$\begin{array}{ccc}
Hom_{D^b(B\otimes_k B^{op})}(B,B) & \longrightarrow & Hom_{D^b(B\otimes_k B^{op})}(B,B^*) \\
{\scriptstyle G}\uparrow \simeq & & {\scriptstyle G}\uparrow \simeq \\
Hom_{D^b(A\otimes_k A^{op})}(A,A) & \longrightarrow & Hom_{D^b(A\otimes_k A^{op})}(A,A^*) \\
 & & \downarrow {\scriptstyle ((Fr^k)_*)^n} \\
\uparrow \zeta_n^A & & Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A,k^{(n)})) \\
 & & \uparrow {\scriptstyle ((Fr^A)^*)^n} \\
Hom_{D^b(A\otimes_k A^{op})}(A,A) & \longrightarrow & Hom_{D^b(A\otimes_k A^{op})}(A,A^*) \\
{\scriptstyle G}\downarrow \simeq & & {\scriptstyle G}\downarrow \simeq \\
Hom_{D^b(B\otimes_k B^{op})}(B,B) & \longrightarrow & Hom_{D^b(B\otimes_k B^{op})}(B,B^*)\ .
\end{array}$$

It is clear that the upper and the lower square are commutative, since they arise as squares induced from applying an equivalence of categories.

Recall the notation we use as explained at the end of Section 2.

We obtain a commutative diagram:

$$\begin{array}{ccc}
Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A,k)) & \xrightarrow{G} & Hom_{D^b(B\otimes_k B^{op})}(B, Hom_k(B,k)) \\
\downarrow {\scriptstyle Hom_k(A,(Fr^k)^n)} & & \downarrow \varphi \\
Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A,k^{(n)})) & \xrightarrow{G} & Hom_{D^b(B\otimes_k B^{op})}(B, Hom_k(B,k^{(n)})),
\end{array}$$

where $\varphi = G \circ Hom_k(A,(Fr^k)^n) \circ G^{-1}$.

We shall need to see that $\varphi = Hom_k(B,(Fr^k)^n)$.

**Claim 1.** *$G \circ Hom(A, Fr^k) = Hom(B, Fr^k) \circ G$.*

Proof. Observe that $G = X \otimes_A - \otimes_A X'$ acts only on the contravariant variables. Going through the isomorphisms (†), since $Fr^k$ acts on the covariant variable only, this proves the claim. ■

Therefore, the diagram:

$$\begin{array}{ccc}
Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A,k)) & \xrightarrow{G} & Hom_{D^b(B\otimes_k B^{op})}(B, Hom_k(B,k)) \\
\downarrow {\scriptstyle Hom_k(A,(Fr^k)^n)} & & \downarrow {\scriptstyle Hom_k(B,(Fr^k)^n)} \\
Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A,k^{(n)})) & \xrightarrow{G} & Hom_{D^b(B\otimes_k B^{op})}(B, Hom_k(B,k^{(n)}))
\end{array}$$

is commutative, and the horizontal morphisms are isomorphisms since $G$ is an equivalence and since the images of the various objects under $G$ in their version $A$ and $B$ correspond to each other.

**Claim 2.** *$G \circ Hom(Fr^A, k) = Hom(Fr^B, k) \circ G$.*

Before we begin the proof, observe the following consequences. Once the claim is established, the diagram

$$\begin{array}{ccc} Hom_{D^b(B\otimes_k B^{op})}(B,B^*) & \stackrel{Hom((Fr^B)^n,k)}{\longrightarrow} & Hom_{D^b(B\otimes_k B^{op})}(B,Hom_k(B,k^{(n)})) \\ \simeq\uparrow G & & \simeq\uparrow G \\ Hom_{D^b(A\otimes_k A^{op})}(A,A^*) & \stackrel{Hom((Fr^A)^n,k)}{\longrightarrow} & Hom_{D^b(A\otimes_k A^{op})}(A,Hom_k(A,k^{(n)})) \end{array}$$

is commutative.

Observe that since $Hom_k(Fr^A,k)$ is not $A\otimes_k A^{op}$-linear, the functor $G$ is not defined on $Hom_k(Fr^A,k)$. Hence, the only way to prove the commutativity of the above diagram is by inspection of the values.

PROOF. We need to make explicit the mappings:

$$G: Hom_{D^b(A\otimes_k A^{op})}(A,A^*) \longrightarrow Hom_{D^b(B\otimes_k B^{op})}(B,B^*)$$

and

$$G: Hom_{D^b(A\otimes_k A^{op})}(A,Hom_k(A,k^{(1)})) \longrightarrow Hom_{D^b(B\otimes_k B^{op})}(B,Hom_k(B,k^{(1)})).$$

For this, it is useful, and possible, to replace $B$ by $X\otimes_A X'$ and $A$ by $X'\otimes_B X$.

We first deal with the first identification. Then, again by the usual adjointness formula between $Hom$ and $\otimes$, one has to make explicit an isomorphism:

$$G: Hom_k(A\otimes_{A\otimes_k A^{op}} A,k) \longrightarrow Hom_k(B\otimes_{B\otimes_k B^{op}} B,k),$$

or, replacing $B$ by $X\otimes_A X'$ and $A$ by $X'\otimes_B X$,

$$G: Hom_k((X'\otimes_B X)\otimes_{A\otimes_k A^{op}}(X'\otimes_B X),k) \to Hom_k((X\otimes_A X')\otimes_{B\otimes_k B^{op}}(X\otimes_A X'),k).$$

Now, we observe that in the tensor product $(X'\otimes_B X)\otimes_{A\otimes_k A^{op}}(X'\otimes_B X)$, the left term $A$ in $A\otimes_k A^{op}$ acts on the right of the left hand $X$ and on the left of the right hand factor $X'$. Similarly, the right term $A^{op}$ in $A\otimes_k A^{op}$ acts on the left of the left hand $X'$ and on the right of the right hand factor $X$. Analogous statements hold, making the left $B\otimes B^{op}$ module structure of the right copy $X\otimes_A X'$ and the right $B\otimes B^{op}$-module structure of the left copy of $X\otimes_A X'$ precise. So, we get the natural isomorphism $\nu$:

$$\begin{array}{rcl} (X'\otimes_B X)\otimes_{A\otimes_k A^{op}}(X'\otimes_B X) & \longrightarrow & (X\otimes_A X')\otimes_{B\otimes_k B^{op}}(X\otimes_A X') \\ (x_1'\otimes x_1)\otimes(x_2'\otimes x_2) & \mapsto & (x_1\otimes x_2')\otimes(x_2\otimes x_1') \end{array}$$

of $k$-vector spaces, given by cyclic permutation of factors. Now, the action of $Fr^A$ consists in tensoring $(X'\otimes_B X)\otimes_A (X'\otimes_B X)$ $p$ times over $A$ and replacing the first tensor $(X'\otimes_B X)\otimes_A -$ by $(X'\otimes_B X)\otimes_{A\otimes A^{op}} -$.

We need to explain the second isomorphism:

$$G: Hom_{D^b(A\otimes_k A^{op})}(A,Hom_k(A,k^{(1)})) \longrightarrow Hom_{D^b(B\otimes_k B^{op})}(B,Hom_k(B,k^{(1)})).$$

Here, we observe that

$$Hom_{D^b(A\otimes_k A^{op})}(A, Hom_k(A,k^{(1)})) \simeq Hom_k(A\otimes_{A\otimes_k A^{op}} A, k^{(1)}),$$

and the very same arguments and constructions as above hold. The only difference is that one needs to consider semi-linear mappings only at the end. The reorganization procedure is just the same. In particular, the action of $Fr^B$ consists in tensoring $(X\otimes_A X')\otimes_B (X\otimes_A X')$ $p$ times over $B$ and replacing the first tensor product over $B$ by a tensor product over $B\otimes B^{op}$. It is now immediately possible to see that this operation commutes with this cyclic permutation of factors as described by explaining $\nu$. So,

$$G\circ Hom(Fr^A,k) = Hom(Fr^B,k)\circ G.$$

This proves our claim. ■

**Claim 3.** *The images of $G\circ\zeta_n^A\circ G^{-1}$ and of $\zeta_n^B$ coincide.*

Proof. Since $\varphi : A \longrightarrow Hom_k(A,k)$ is an isomorphism of $A\otimes_k A^{op}$-modules, and since $G$ is a functor, $G\varphi$ is an isomorphism as well. As we know that choosing an isomorphism $B\longrightarrow Hom_k(B,k)$ is equivalent to choosing a symmetrising form making $B$ into a symmetric algebra, we may well work with this form instead of the original one. Actually, given two different isomorphisms $\phi : B\longrightarrow Hom_k(B,k)$ and $\psi : B \longrightarrow Hom_k(B,k)$, then for all $x\in B$ one has $\phi^{-1}\psi(x) = \lambda x$ for an invertible central $\lambda\in Z(B)^\times$. So, the resulting $\zeta_n^B$ differ by invertible central elements. As a consequence, the images are identical. ■

We shall now finish the proof of the theorem. By the previous claims, the composition

$$\begin{aligned} ZB \stackrel{G\phi}{\longrightarrow} Hom_k(B/KB,k) &\stackrel{(G(Fr^A)^*G^{-1})^n}{\longrightarrow} Hom_k(B/KB,k^{(n)}) \\ &\stackrel{((G(Fr^k)^*)^n)^{-1}}{\longrightarrow} Hom_k(B/KB,k) \stackrel{G\phi^{-1}}{\longrightarrow} ZB \end{aligned}$$

is a mapping that differs from $\zeta_n^B$ by some central unit of $B$, and therefore the isomorphism induced by $G$ between the centres of $A$ and $B$ maps $T_nA^\perp$ to $T_nB^\perp$.

This concludes the proof of the theorem. ■

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*Note added in proof.* While this paper was in press, the method was applied in various examples. Moreover, the statement was generalised further. Holm and Skowroński used the invariant to clarify a delicate question on the derived equivalence classification of symmetric algebras of domestic type [3]. In Holm's derived equivalence classification of tame blocks, some cases for algebras with two simple modules remained open. Using the method outlined in the present paper, Holm and Zimmermann have been able to decide on some of these missing cases in [4]. Bessenrodt, Holm and Zimmermann show in [1] how to generalise the invariance statement of the ideals $T_n(A)^{\perp}$ to non-symmetric algebras by passing to trivial extension algebras. By interpreting $A/KA$ as the degree 0 Hochschild homology of $A$, and $Z(A)$ as the degree 0 Hochschild cohomology of $A$, Zimmermann, in [16], defines objects similar to Reynolds ideals for higher Hochschild homology and cohomology and proves their invariance under derived equivalence. The proof of Claim 2 actually shows the derived invariance of the $p$-power map on the degree 0 Hochschild homology. This point was raised by Mariano Suares-Alvarez in Luminy in September 2006 and by Burkhard Külshammer in an email to the author dated 15 March 2006.

## REFERENCES


[1] C. Bessenrodt, T. Holm and A. Zimmermann, *Generalized Reynolds ideals for non-symmetric algebras*, preprint (electronic), 9 pages, posted 8 March 2006 (see http://fr.arXiv.org/pdf/math.RT/0603189).

[2] S. Gelfand and Y. Manin, *Methods of homological algebra*, Springer Verlag, Berlin-Heidelberg, 1996.

[3] T. Holm and A. Skowroński, Derived equivalence classification of symmetric algebras of domestic type, *Journal of the Mathematical Society of Japan* **58** (2006), 1133–49.

[4] T. Holm and A. Zimmermann, 'Generalized Reynolds ideals and the scalar problem for derived equivalences of tame blocks of finite groups', working notes.

[5] L. Héthelyi, E. Horváth, B. Külshammer and J. Murray, Central ideals and Cartan invariants of symmetric algebras, *Journal of Algebra* **293** (2005), 243–60.

[6] B. Keller, A remark on tilting theory and *DG*-algebras, *Manuscripta Mathematica* **79** (1993), 247–53.

[7] B. Keller, Invariance and licalization for cyclic homology of *DG*-algebras, *Journal of Pure and Applied Algebra* **123** (1998), 223–73.

[8] S. König and A. Zimmermann, *Derived equivalences for group rings*; Lecture Notes in Mathematics 1685, Springer Verlag, Berlin–Heidelberg, 1998.

[9] B. Külshammer, Bemerkungen über die Gruppenalgebra als symmetrische Algebra I, II, III and IV, *Journal of Algebra* **72** (1981), 1-7; **75** (1982), 59–69; **88** (1984), 279–91 and **93** (1985), 310–23.

[10] B. Külshammer, Group-theoretical descriptions of ring theoretical invariants of group algebras, *Progress in Mathematics* **95** (1991), 425–41.

[11] S. Mac Lane, *Homology*, Springer Verlag, Berlin–Heidelberg, 1975.

[12] J. Rickard, Derived equivalences as derived functors, *Journal of the London Mathematical Society* **43** (1991), 37–48.

[13] R. W. Thomason and T. F. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift (a collection of papers to honor Grothendieck's 60th birthday), vol. 3, 247–435, Birkhäuser, Basel, 1990.

[14] C. Weibel, *Introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.

[15] A. Zimmermann, Tilted symmetric orders are symmetric orders, *Archiv der Mathematik* **73** (1999), 15–17.

[16] A. Zimmermann, Fine Hochschild invariants of derived categories for symmetric algebras, *Journal of Algebra* **308** (2007), 350–67.