BENDING FLUCTUATIONS OF AN ELASTIC LINE ON A CURVED SURFACE IN $\mathbb{R}^3_1$

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Abstract

In this work, we studied bending fluctuations of an elastic line on 3-dimensional Minkowski space. The path of bent elastic lines must satisfy a differential equation on the surface that is derived by variational methods. The bending energy per unit length scales with the square of the extent of bending (Hooke's Law) if the surface is a hyperbolic 2-space.

1. Introduction

In this section, we will outline some fundamental definitions and theorems.

Definition 1.1. Let α be a curve on a semi-Riemannian surface S in $\mathbb{R}^3_1$, parametrized by arc length s, $0 \leq s \leq l$. Let $\kappa(s)$ be the curvature of α at $\alpha(s)$. An elastic line of length l is defined as a curve with total arc length l and associated stress energy U,

$$U = \frac{1}{2} b K_2, \tag{1.1}$$

where b is the Hooke's Law bending constant and K_2 is the total square curvature

$$K_2 = \int_0^l \kappa^2 ds, \tag{1.2}$$

with s the arc length and $\kappa(s)$ the curvature along the curve in $\mathbb{R}^3_1$.

Definition 1.2. $\mathcal{F}(\mathbb{R}^{n+1}_\nu)$ denotes the set of all smooth, real-valued functions on $\mathbb{R}^{n+1}_\nu$.

Let $q \in \mathcal{F}(\mathbb{R}^{n+1}_\nu)$ as usual be the function $q(v) = \langle v, v \rangle$. Relative to natural coordinates

$$q = -\sum_{i=1}^{\nu} (u_i)^2 + \sum_{j=\nu+1}^{n+1} (u_j)^2.$$

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If P is the position vector field of $\mathbb{R}^{n+1}_\nu$, then $q = \langle P, P\rangle$. For $r > 0$ and $\varepsilon = \pm 1$, $Q = q^{-1}(\varepsilon r^2)$ is a semi-Riemannian hypersurface of $\mathbb{R}^{n+1}_\nu$ with unit normal $U = P/r$ and sign ε (see [3]).

Definition 1.3. Let W be a subspace of a Lorentz vector space V, and let g be the scalar product of V. There are three mutually exclusive possibilities for W :

(1) $g \mid W$ is positive definite; that is, W is an inner product space. Then W is said to be spacelike.

(2) $g \mid W$ is nondegenerate of index 1. Then W is timelike.

(3) $g \mid W$ is degenerate. Then W is lightlike.

W is called its causal character (see [3]).

Definition 1.4. Let S be a surface in $\mathbb{R}^3_1$. S is a spacelike surface if and only if

$$\langle N, N\rangle < 0.$$

Similarly, S is a timelike surface if and only if

$$\langle N, N\rangle > 0.$$

Here, N is a normal vector field of M (see [5]).

Definition 1.5. Let L be a 3-dimensional Lorentzian space. If (x_1, x_2, x_3) and (y_1, y_2, y_3) are the components of X and Y with respect to an allowable coordinate system, then

$$\langle X, Y\rangle \mid_L = -x_1y_1 + x_2y_2 + x_3y_3,$$

which is called a *Lorentzian inner product*. Furthermore, a Lorentzian exterior product $X \times Y$ is given by

$$X \times Y = (-x_2y_3 + x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

(see [6]).

Definition 1.6. Apart from the Frenet frame $\{T, n, b\}$, there also exists a second frame at every point of a curve α. At a point $\alpha(s)$ of α, let T denote the unit tangent vector to α, let N denote the unit normal to S, and let

$N \times T = \varepsilon Q(s)$, $\varepsilon = \pm 1$, respectively, with respect to which the inner product $\{T, Q, N\}$ gives an orthonormal basis in $\mathbb{R}^3_1$.

If S is a spacelike surface, $T \times Q = N$, $Q \times N = -T$, $N \times T = -Q$ (see [4]).

Definition 1.7. Let M be a semi-Riemannian surface, that is, a semi-Riemann manifold of dimension 2. For a coordinate system u, v in M, the components of the metric tensor are denoted by

$$E = g_{11} = \langle x_u, x_u\rangle, \quad F = g_{12} = g_{21} = \langle x_u, x_v\rangle, \quad G = g_{22} = \langle x_v, x_v\rangle,$$

and the line element is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

(see [3]).

Definition 1.8. Let S be a surface in $\mathbb{R}^3_1$ and let α be a curve on M. The function

$$k_g : I = [0, l] \subset R \quad \to R$$

defined for $s \in I$ by

$$k_g(s) =< T'(s), Q(s) >$$

is called the *geodesic curvature function* of α, and $k_g(s)$ is said to be the *geodesic curvature* of α at $\alpha(s)$ in $\mathbb{R}^3_1$ (see [4]).

Definition 1.9. Let S be a surface in $\mathbb{R}^3_1$. Let T denote the unit tangent vector to α. Let II denote the second fundamental form of S. Then $II(T,T) = \varepsilon \langle S(T), T \rangle N$ and $\langle N, N \rangle = \varepsilon$. The function

$$k_n : I \subset R \quad \to R$$

defined for $s \in I$ by

$$k_n(s) = \langle II(T(s), T(s)), N \rangle$$

is called the *normal curvature function* of α (see [3]).

Definition 1.10. Let M be a surface in $\mathbb{R}^3_1$ and α a curve on M. The function

$$\tau_g : I \subset R \quad \to R \ ,$$

defined for $s \in I$ by

$$\tau_g(s) =< Q'(s), N(s) >,$$

is called the *geodesic torsion function* of α, and $\tau_g(s)$ is said to be the *geodesic torsion* of α at $\alpha(s)$ in $\mathbb{R}^3_1$ (see [4]).

Remark 1.1. Let $x(u,v)$ be the spacelike surface, having parameter curves that are perpendicular to each other passing through the point $\alpha(s)$ of any curve α. If the curvature lines are chosen as parameter curves, the geodesic curvature is [4]

$$k_g = (k_g)_1 \cos\varphi + (k_g)_2 \sin\varphi + \tfrac{d\varphi}{ds} \qquad (\textit{Liouville's formula}), \tag{1.3}$$

where $(k_g)_1$is the geodesic curvature of the u-parameter curve at $\alpha(s)$, $(k_g)_2$ is the geodesic curvature of the v-parameter curve at $\alpha(s)$ and φ is the angle between the u-coordinate curve through $\alpha(s)$ and the curve α.

Also, the normal curvature is [4]

$$k_n = k_1 \cos^2\varphi + k_2 \sin^2\varphi \qquad (\textit{Euler's formula}), \tag{1.4}$$

where $k_1 = (k_n)_1$ and $k_2 = (k_n)_2$ are the principal curvatures of the surface at $\alpha(s)$.

And the geodesic torsion is [4]

$$\tau_g = (k_2 - k_1)\cos\varphi \sin\varphi. \tag{1.5}$$

Remark 1.2. Let $x(u, v)$ be the timelike surface, having parameter curves that are perpendicular to each other passing through point $\alpha(s)$ of any curve α. Then, the geodesic curvature is [4],

$$k_g = (k_g)_1 \cosh\varphi - (k_g)_2 \sinh\varphi - \tfrac{d\varphi}{ds} \qquad (\textit{Liouville's formula}). \tag{1.6}$$

Here,

$$(k_g)_1 = -\tfrac{1}{2}\tfrac{E_v}{|E||G|^{1/2}}, \qquad (k_g)_2 = \tfrac{1}{2}\tfrac{G_u}{|E|^{1/2}|G|}.$$

The normal curvature is [4]

$$k_n = k_1 \cosh^2\varphi - k_2 \sinh^2\varphi \qquad (\textit{Euler's formula}). \tag{1.7}$$

The geodesic torsion is [4],

$$\tau_g = (k_2 - k_1)\cosh\varphi \sinh\varphi. \tag{1.8}$$

Theorem 1.1. *Let S be a surface in $\mathbb{R}^3_1$ and let α be a curve on S. The analogue of the Frenet–Serret formulas is given by*

$$\begin{bmatrix} T' \\ Q' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_g & \varepsilon_3 k_n \\ -\varepsilon_1 k_g & 0 & \varepsilon_3 \tau_g \\ -\varepsilon_1 k_n & -\varepsilon_2 \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ N \end{bmatrix}, \tag{1.9}$$

where $< T, T >= \varepsilon_1$, $< Q, Q >= \varepsilon_2$, $< N, N >= \varepsilon_3$.

Theorem 1.2. *Let γ be a nonconstant geodesic of the pseudo-sphere $S^n_\nu(r) \subset \mathbb{R}^{n+1}_\nu$.*

(1) If γ is timelike, it is a parametrization of one branch of a hyperbola.

(2) If γ is null, it is a straight line.

(3) If γ is spacelike, it is a periodic parametrization of an ellipse.

This theorem holds also for the pseudo-hyperbolic space $H^n_\nu(r)$, provided the words spacelike and timelike are reversed [3].

Definition 1.11. Let h denote the second fundamental form of S in L^3. With respect to a Lorentzian frame field (e_1, e_2, e_3), h is represented by the matrix h_{ij}, where

$$h_{ij} = -\langle D_{e_i} e_j, e_3 \rangle\,, \tag{1.10}$$

D denoting covariant differentiation in L^3 [1].

2. The variational problem and its solution in $\mathbb{R}^3_1$

On a general surface, an elastic line is said to be relaxed if its energy U is a minimum, subject only to the constraint that the line be confined to the surface and only to boundary conditions at the initial point of the line. We do not want our definition of bending to include the bending necessary merely to stay on the surface in $\mathbb{R}^3_1$. We define the extent to which an elastic line is bent away from its relaxed position, and this definition must imply that the relaxed elastic line, in this sense, is not bent. Equation (2.1) gives a definition of what is meant by the bending of an elastic line through an angle α on a surface, such that the total geodesic curvature K_g equals α,

$$K_g = \int_0^l k_g ds = \alpha. \tag{2.1}$$

A relaxed elastic line is not generally congruent with a geodesic arc. In applications to the physics of bending away from the relaxed position, we therefore must be content with special cases for which the relaxed line does follow a geodesic path.

The main problem is to find a curve of length l that minimizes the energy functional, or equivalently, the total square curvature K_2 defined by Equation (1.2) subject to the constraint of total geodesic curvature fixed at the value α in Equation (2.1). In addition, the location on the surface of the initial point $s = 0$ is specified. Accordingly, we define an auxiliary functional J,

$$J = K_2 + aK_g, \tag{2.2}$$

where a is a constant Lagrange multiplier, and we seek stationary curves of J with no constraints other than fixed initial location and direction.

We consider the desired stationary curve, construct the orthonormal frame $[T(s), Q(s), N(s)]$ at each point along the curve and define a variational field of curves by displacements along the vector $Q(s)$, which is a surface tangent normal to the stationary curve.

Thus, α is expressed as

$$\alpha(s) = x\left(u(s), v(s)\right),\ 0 \le s \le l,$$

with

$$T(s) = \alpha'(s) = \frac{\partial x}{\partial u}\frac{du}{ds} + \frac{\partial x}{\partial v}\frac{dv}{ds}$$

and

$$Q(s) = p(s)x_u + q(s)x_v,$$

for suitable scalar functions $p(s)$ and $q(s)$.

Next, we must define variational fields for our problem. In order to obtain variational arcs of length l, it is generally necessary to extend α to an arc α^* defined for $0 \le s \le l^*$, with $l^* > l$, but sufficiently close to l so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \le s \le l^*$, be a scalar function of class C^2, not vanishing identically. Define

$$\eta(s) = \mu(s)p^*(s) \ \ , \ \xi(s) = \mu(s)q^*(s).$$

Then, along α

$$\eta(s)x_u + \xi(s)x_v = \mu(s)Q(s). \tag{2.3}$$

Assume also that

$$\mu(0) = 0, \mu'(0) = 0. \tag{2.4}$$

Now define

$$\beta(\sigma;t) = x\left(u(\sigma) + t\eta(\sigma), v(\sigma) + t\xi(\sigma)\right), \tag{2.5}$$

for $0 \le \sigma \le l^*$. For $|t| < \varepsilon$ (where $\varepsilon > 0$ depends upon the choice of α^* and of μ), the point $\beta(\sigma;t)$ lies in the coordinate patch. For fixed t, $\beta(\sigma;t)$ gives an arc with the same initial point and initial direction as α, because of (2.4). For $t = 0$, $\beta(\sigma;0)$ is the same as α^* and σ is arc length. For $t \ne 0$, the parameter σ is not arc length in general.

For fixed t, $|t| < \varepsilon$, let $L^*(t)$ denote the length of the arc $\beta(\sigma;t), 0 \le \sigma \le l^*$. Then

$$L^*(t) = \int_0^{l^*} \sqrt{\left|\left\langle \frac{\partial\beta}{\partial\sigma}(\sigma;t), \frac{\partial\beta}{\partial\sigma}(\sigma;t)\right\rangle\right|}d\sigma, \tag{2.6}$$

with

$$L^*(0) = l^* > l. \tag{2.7}$$

It is clear from (2.5) and (2.6) that $L^*(t)$ is continuous. In particular, it follows from (2.7) that

$$L^*(t) > \frac{l+l^*}{2} > l, \qquad (|t| < \varepsilon_1) \tag{2.8}$$

for a suitable ε_1 satisfying $0 < \varepsilon_1 \le \varepsilon$. Because of (2.8), we can restrict $\beta(\sigma;t)$, $0 \le |t| < \varepsilon_1$, to an arc of length l by restricting the parameter σ to an interval

$0 \leq \sigma \leq \lambda(t) \leq l^*$, by requiring

$$\int_0^{\lambda(t)} \sqrt{\left|\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle\right|} d\sigma = l. \tag{2.9}$$

Note that $\lambda(0) = l$. The function $\lambda(t)$ need not be determined explicitly, but we shall need

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \varepsilon_1 \int_0^l \mu k_g ds. \tag{2.10}$$

The proof of (2.10) and of other results below will depend on calculations from (2.5), such as

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0} = T, \qquad 0 \leq \sigma \leq l, \tag{2.11}$$

which gives

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0} = T' = \varepsilon_2 k_g Q + \varepsilon_3 k_n N. \tag{2.12}$$

Also, it follows from (2.3) that

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0} = \mu Q. \tag{2.13}$$

Using (2.3), the second differentiation of (2.13) gives

$$\frac{\partial^2\beta}{\partial t\partial\sigma} = -\varepsilon_1 \mu k_g T + \mu' Q + \varepsilon_3 \mu \tau_g N, \tag{2.14}$$

and the third differentiation of (2.13) gives

$$\begin{aligned} \left.\tfrac{\partial^3\beta}{\partial t\partial\sigma^2}\right|_{t=0} &= \left(-2\varepsilon_1 \mu' k_g - \varepsilon_1 \mu k_g' - \varepsilon_1\varepsilon_3 \mu \tau_g k_n\right) T \\ &+ \left(\mu'' - \varepsilon_1\varepsilon_2 \mu k_g^2 - \varepsilon_2\varepsilon_3 \mu \tau_g^2\right) Q \\ &+ \left(2\varepsilon_3 \mu' \tau_g - \varepsilon_1\varepsilon_3 \mu k_g k_n + \varepsilon_3 \mu \tau_g'\right) N. \end{aligned} \tag{2.15}$$

To prove (2.10), differentiate (2.9) with respect to t, remembering that l is constant, and evaluate at t=0 using (2.11) and (2.14), with $\lambda(0) = l$.

$$\frac{d\lambda}{dt}\Big|_{t=0}\sqrt{\left|\left\langle \frac{\partial\beta}{\partial\sigma}\Big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\Big|_{t=0}\right\rangle\right|}$$
$$+\int_0^l \left\langle \frac{\partial\beta}{\partial\sigma}\Big|_{t=0}, \frac{\partial^2\beta}{\partial\sigma\partial t}\Big|_{t=0}\right\rangle \frac{\sqrt{\left|\left\langle \frac{\partial\beta}{\partial\sigma}\big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\big|_{t=0}\right\rangle\right|}}{\left\langle \frac{\partial\beta}{\partial\sigma}\big|_{t=0}, \frac{\partial\beta}{\partial\sigma}\big|_{t=0}\right\rangle} ds = 0.$$

Here, $K_2(t)$ denotes the total square curvature of the arc $\beta(\sigma;t)$, $0 \le \sigma \le \lambda(t)$, $|t| < \varepsilon_1$. Since σ is not generally arc length for $t \neq 0$, the total square curvature is

$$K_2(t) = \int_0^{\lambda(t)} \frac{\left\langle \frac{\partial\beta}{\partial\sigma}(\sigma,t)\wedge\frac{\partial^2\beta}{\partial\sigma^2}(\sigma,t), \frac{\partial\beta}{\partial\sigma}(\sigma,t)\wedge\frac{\partial^2\beta}{\partial\sigma^2}(\sigma,t)\right\rangle}{\left|\left\langle \frac{\partial\beta}{\partial\sigma}(\sigma,t), \frac{\partial\beta}{\partial\sigma}(\sigma,t)\right\rangle\right|^3} \left|\left\langle \frac{\partial\beta}{\partial\sigma}(\sigma,t), \frac{\partial\beta}{\partial\sigma}(\sigma,t)\right\rangle\right|^{1/2} d\sigma.$$

We calculate the derivate $J'(t)$ and set t to 0. According to (2.2), we must compute both K_2' and K_g'. In calculating $K_2'(t)$, we give explicitly only those terms that do not vanish for $t = 0$. The omitted terms are those with a factor $\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial\beta}{\partial\sigma}\right\rangle$, which vanishes at $t = 0$, since $\langle T', T\rangle = 0$. Thus,

$$K_2'(t) = \frac{d\lambda}{dt}\left\{\left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-3/2}\left|-\left\langle\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|\right\}_{\sigma=\lambda(t)}$$
$$-3\int_0^{\lambda(t)}\left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-5/2}\left\langle\frac{\partial^2\beta}{\partial t\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle \frac{\left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|}{\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle}\left|\left\langle\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right| d\sigma$$
$$+2\int_0^{\lambda(t)}\left|\left\langle\frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma}\right\rangle\right|^{-3/2}\left\langle\frac{\partial^3\beta}{\partial t\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle \frac{\left|\left\langle\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right|}{\left\langle\frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2}\right\rangle} d\sigma + \ldots$$

Using (2.10), (2.11), (2.14) and (2.12), we find

$$\begin{aligned} K_2'(0) = \; & \varepsilon_1\int_0^l \mu k_g ds \left\{\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right|\right\}_{\sigma=\lambda(0)} \\ & +2\int_0^l k_g\left(\mu'' - \varepsilon_1\varepsilon_2\mu k_g^2 - \varepsilon_2\varepsilon_3\mu\tau_g^2\right)\frac{\left|\varepsilon_2 k_g^2+\varepsilon_3 k_n^2\right|}{\varepsilon_2 k_g^2+\varepsilon_3 k_n^2} ds \\ & +2\int_0^l k_n\left(2\varepsilon_3\mu'\tau_g - \varepsilon_1\varepsilon_3\mu k_g k_n + \varepsilon_3\mu\tau_g'\right)\frac{\left|\varepsilon_2 k_g^2+\varepsilon_3 k_n^2\right|}{\varepsilon_2 k_g^2+\varepsilon_3 k_n^2} ds \\ & +3\varepsilon_1\int_0^l \mu k_g\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right| ds. \end{aligned} \quad (2.16)$$

However, with integration by parts and (2.4),

$$2\int_0^l \mu'' k_g ds = 2\mu'(l)k_g(l) - 2\mu(l)k_g'(l) + 2\int_0^l \mu k_g'' ds, \tag{2.17}$$

and

$$4\int_0^l \mu' \tau_g k_n ds = 4\mu(l)k_n(l)\tau_g(l) - 4\int_0^l \mu k_n' \tau_g ds - 4\int_0^l \mu k_n \tau_g' ds. \tag{2.18}$$

We now calculate the derivative K_g' and then set t to 0, where

$$K_g'(0) = \begin{array}{l} \int_0^l \mu\{\varepsilon_1 k_g k_g(l) - \varepsilon_3 \tau_g^2 - \varepsilon_3 k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)\}ds \\ +\varepsilon_2(\mu'(l) - \mu'(0)). \end{array} \tag{2.19}$$

The final result can be written as

$$J'(0) = \begin{array}{l} \varepsilon_1 \int_0^l \mu k_g ds \left\{\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right|\right\}_{\sigma=\lambda(0)} \\ +2\int_0^l k_g \left(\mu'' - \varepsilon_1\varepsilon_2 \mu k_g^2 - \varepsilon_2\varepsilon_3 \mu \tau_g^2\right) \frac{\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right|}{\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2} ds \\ +2\int_0^l k_n \left(2\varepsilon_3 \mu' \tau_g - \varepsilon_1\varepsilon_3 \mu k_g k_n + \varepsilon_3 \mu \tau_g'\right) \frac{\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right|}{\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2} ds \\ +3\varepsilon_1 \int_0^l \mu k_g \left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right| ds \\ +a\int_0^l \mu\{\varepsilon_1 k_g k_g(l) - \varepsilon_3 \tau_g^2 - k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)\}ds \\ +a\varepsilon_2(\mu'(l) - \mu'(0)). \end{array} \tag{2.20}$$

2.1. Intrinsic equations for a relaxed elastic line on a timelike surface for a timelike arc α

If T is timelike, Q and N are spacelike, then

$$\begin{array}{l} < T, T >= \varepsilon_1 = -1 \\ < Q, Q >= \varepsilon_2 = 1 \\ < N, N >= \varepsilon_3 = 1, \end{array}$$

$$\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right| = k_g^2 + k_n^2. \tag{2.21}$$

Substituting (2.10), (2.11), (2.14), (2.17), (2.18) and (2.21) in (2.16), we find

$$K_2'(0) = \begin{array}{l} \int_0^l \mu\{2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g \\ +k_g\left(-k_g^2(l) - k_n^2(l) - k_g^2 - k_n^2 - 2\tau_g^2\right)\}ds \\ +2\mu'(l)k_g(l) - 2\mu(l)k_g'(l) + 4\mu(l)k_n(l)\tau_g(l), \end{array} \tag{2.22}$$

and from Equation (2.19)

$$K_g'(0) = \begin{array}{l} \int_0^l \mu\{-k_gk_g(l) - \tau_g^2 - k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)\}ds \\ +(\mu'(l) - \mu'(0)). \end{array} \tag{2.23}$$

So,

$$J'(0) = \begin{array}{l} \int_0^l \mu\{2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g[-k_g^2(l) - k_n^2(l) - k_g^2 - k_n^2 - 2\tau_g^2] \\ -a[k_gk_g(l) + \tau_g^2 + k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)]\}ds \\ -2\mu(k_g' - 2k_n\tau_g)\big|_0^l + \mu'(2k_g + a)\big|_0^l. \end{array} \tag{2.24}$$

From Equation (2.24) for all choices of the function $\mu(s)$ satisfying (2.4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $J'(0) = 0$, the given timelike arc α must satisfy two boundary conditions and a differential equation:

$$\begin{array}{ll} (1) & a = -2k_g(l), \\ (2) & k_g'(l) = 2k_n(l)\tau_g(l), \\ (3) & \begin{array}{l} 2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g[-k_g^2(l) - k_n^2(l) - k_g^2 - k_n^2 - 2\tau_g^2] \\ +2k_g(l)[k_gk_g(l) + \tau_g^2 + k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)] = 0. \end{array} \end{array} \tag{2.25}$$

2.2. Intrinsic equations for a relaxed elastic line on a timelike surface for a spacelike arc α

If Q is timelike, T and N are spacelike, then
(i) in the case of $k_g^2 < k_n^2$,

$$\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right| = -k_g^2 + k_n^2. \tag{2.26}$$

Substituting (2.10), (2.11), (2.14), (2.17), (2.18) and (2.26) in (2.16), we find $K_2'(0)$ can be written as

$$K_2'(0)=\begin{array}{l}\int\limits_0^l \mu\{2k_g''-2k_n\tau_g'-4k_n'\tau\\ +k_g\left(-k_g^2(l)+k_n^2(l)-k_g^2+k_n^2+2\tau_g^2\right)\}ds\\ +2\mu'(l)k_g(l)-2\mu(l)k_g'(l)+4\mu(l)k_n(l)\tau_g(l),\end{array}\tag{2.27}$$

and from Equation (2.19)

$$K_g'(0)=\int\limits_0^l \mu\{k_gk_g(l)-\tau_g^2-k_n(\frac{\mathcal{N}E}{G}u'^2+\frac{LG}{E}v'^2)\}ds-(\mu'(l)-\mu'(0)).\tag{2.28}$$

Therefore,

$$J'(0)=\begin{array}{l}\int\limits_0^l \mu\{2k_g''-2k_n\tau_g'-4k_n'\tau+k_g[-k_g^2(l)+k_n^2(l)-k_g^2+k_n^2+2\tau_g^2]\\ +a[k_gk_g(l)-\tau_g^2-k_n(\frac{NE}{G}u'^2+\frac{LG}{E}v'^2)]\}ds\\ +\mu'(2k_g(l)-a)\big|_0^l-2\mu(k_g'-2k_n\tau_g)\big|_0^l.\end{array}\tag{2.29}$$

From Equation (2.29), for all choices of the function $\mu(s)$ satisfying (2.4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $J'(0)=0$, the given arc α must satisfy two boundary conditions and a differential equation:

$$\begin{array}{ll}(1) & a=2k_g(l),\\ (2) & k_g'(l)=2k_n(l)\tau_g(l),\\ & 2k_g''-2k_n\tau_g'-4k_n'\tau\\ (3) & +k_g[-k_g^2(l)+k_n^2(l)-k_g^2+k_n^2+2\tau_g^2]\\ & +2k_g(l)[k_gk_g(l)-\tau_g^2-k_n(\frac{\mathcal{N}E}{G}u'^2+\frac{LG}{E}v'^2)]=0.\end{array}\tag{2.30}$$

(ii) In the case of $k_g^2>k_n^2$,

$$\left|\varepsilon_2k_g^2+\varepsilon_3k_n^2\right|=k_g^2-k_n^2.\tag{2.31}$$

Substituting (2.10), (2.11), (2.14), (2.17), (2.18) and (2.31) in (2.16), $K_2'(0)$ can be written as

$$\begin{array}{ll}K_2'(0) & =\int\limits_0^l \mu\{-2k_g''+2k_n\tau_g'+4k_n'\tau_g+k_g(k_g^2(l)-k_n^2(l)\\ & +k_g^2-k_n^2-2\tau_g^2)\}ds-2\mu'(l)k_g(l)+2\mu(l)k_g'(l)-4\mu(l)k_n(l)\tau_g(l).\end{array}\tag{2.32}$$

Hence,

$$J'(0) = \begin{array}{l} \int_0^l \mu\{-2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g + k_g[k_g^2(l) - k_n^2(l) \\ +k_g^2 - k_n^2 - 2\tau_g^2] + a[k_gk_g(l) - \tau_g^2 - k_n(\frac{\mathcal{N}E}{G}u'^2 + \frac{LG}{E}v'^2)]\}ds \\ -\mu'(2k_g + a)\big|_0^l + 2\mu(k_g' - 2k_n\tau_g)\big|_0^l . \end{array} \quad (2.33)$$

From Equation (2.33) for all choices of the function $\mu(s)$ satisfying (2.4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $J'(0) = 0$, the given arc α must satisfy two boundary conditions and a differential equation:

$$\begin{array}{ll} (1) & a = -2k_g(l), \\ (2) & k_g'(l) = 2k_n(l)\tau_g(l), \\ (3) & \begin{array}{l} -2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g + k_g[k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 - 2\tau_g^2)] \\ -2k_g(l)[k_gk_g(l) - \tau_g^2 - k_n(\frac{\mathcal{N}E}{G}u'^2 + \frac{LG}{E}v'^2)] = 0. \end{array} \end{array} \quad (2.34)$$

2.3. Intrinsic equations for a relaxed elastic line on a spacelike surface

If T, Q is spacelike and N is timelike,
(i) in the case of $k_g^2 < k_n^2$,

$$\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right| = k_n^2 - k_g^2. \quad (2.35)$$

Substituting (2.10), (2.11), (2.14), (2.17), (2.18) and (2.35) in (2.16), $K_2'(0)$ can be written as

$$K_2'(0) = \begin{array}{l} \int_0^l \mu\left\{-2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g\left(-k_g^2(l) + k_n^2(l) - k_g^2 + k_n^2 - 2\tau_g^2\right)\right\}ds \\ -2\mu'(l)k_g(l) + 2\mu(l)k_g'(l) + 4\mu(l)k_n(l)\tau_g(l), \end{array} \quad (2.36)$$

and from Equation (2.19),

$$K_g'(0) = \int_0^l \mu\{k_gk_g(l) + \tau_g^2 + k_n(\frac{\mathcal{N}E}{G}u'^2 + \frac{LG}{E}v'^2)\}ds + (\mu'(l) - \mu'(0)). \quad (2.37)$$

So,

$$J'(0) = \begin{array}{l} \int_0^l \mu\{-2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g[-k_g^2(l) + k_n^2(l) \\ -k_g^2 + k_n^2 - 2\tau_g^2] + a[k_gk_g(l) + \tau_g^2 + k_n(\frac{\mathcal{N}E}{G}u'^2 + \frac{LG}{E}v'^2)]\}ds \\ + 2\mu(k_g' + 2k_n\tau_g)\big|_0^l - \mu'(2k_g - a)\big|_0^l . \end{array} \quad (2.38)$$

From Equation (2.38), for all choices of the function $\mu(s)$ satisfying (2.4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $J'(0) = 0$, the given arc α must satisfy two boundary conditions and a differential equation:

$$\begin{array}{ll} (1) & a = 2k_g(l), \\ (2) & k_g'(l) = -2k_n(l)\tau_g(l), \\ (3) & \begin{array}{l} -2k_g'' - 2k_n\tau_g' - 4k_n'\tau_g + k_g[-k_g^2(l) + k_n^2(l) - k_g^2 + k_n^2 - 2\tau_g^2] \\ +2k_g(l)[k_gk_g(l) + \tau_g^2 + k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)] = 0. \end{array} \end{array} \tag{2.39}$$

(ii) In the case of $k_g^2 > k_n^2$,

$$\left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right| = k_g^2 - k_n^2. \tag{2.40}$$

Substituting (2.10), (2.11), (2.14), (2.17), (2.18) and (2.40) in (2.16), $K_2'(0)$ can be written as

$$K_2'(0) = \begin{array}{l} \int_0^l \mu\{2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g \\ +k_g\left(k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 + 2\tau_g^2\right)\}ds \\ +2\mu'(l)k_g(l) - 2\mu(l)k_g'(l) - 4\mu(l)k_n(l)\tau_g(l). \end{array} \tag{2.41}$$

Therefore,

$$J'(0) = \begin{array}{l} \int_0^l \mu\{2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g \\ +k_g[k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 + 2\tau_g^2] \\ +a[k_gk_g(l) + \tau_g^2 + k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)]\}ds \\ -2\mu(k_g' + 2k_n\tau_g)\big|_0^l + \mu'(2k_g + a)\big|_0^l. \end{array} \tag{2.42}$$

From Equation (2.42), for all choices of the function $\mu(s)$ satisfying (2.4), with arbitrary values of $\mu(l)$ and $\mu'(l)$, and $J'(0) = 0$, the given arc α must satisfy two boundary conditions and a differential equation:

$$\begin{array}{ll} (1) & a = -2k_g(l), \\ (2) & k_g'(l) = -2k_n(l)\tau_g(l), \\ (3) & \begin{array}{l} 2k_g'' + 2k_n\tau_g' + 4k_n'\tau_g \\ +k_g[k_g^2(l) - k_n^2(l) + k_g^2 - k_n^2 + 2\tau_g^2] \\ -2k_g(l)[k_gk_g(l) + \tau_g^2 + k_n(\frac{NE}{G}u'^2 + \frac{LG}{E}v'^2)] = 0. \end{array} \end{array} \tag{2.43}$$

Example 2.3.1. Let $-x^2 + y^2 + z^2 = -r^2, r > 0$ be a hyperbolic 2-space space. $H^2(r)$ is a spacelike surface. We consider this surface parametrized by

$$x(u,v) = (r\cosh u, r\sinh u\cos v, r\sinh u\sin v).$$

For the hyperbolic 2-space $E = r^2$, $G = r^2\sinh^2 u, \mathcal{L} = -\langle x_{u_u}, N\rangle = -r$, $\mathcal{N} = -\langle x_{vv}, N\rangle = -r\sinh^2 u$. From Equation (1.7), $k_n = -\frac{1}{r}$, $k_n^2 = c^2 = \frac{1}{r^2}$.

From the differential equation of (2.39),

$$2k_g'' + k_g^3 - k_g k_g^2(l) - 2c^2[k_g + k_g(l)] = 0; \tag{2.44}$$

with integrating factor k_g', a first integral is

$$2k_g'' + k_g^3 - k_g k_g^2(l) - 2c^2[k_g + k_g(l)] = 0; \tag{2.45}$$

with integrating factor k_g', a first integral is

$$(k_g')^2 + \frac{1}{4}k_g^4 - \frac{1}{2}k_g^2 k_g^2(l) - c^2 k_g[k_g + 2k_g(l)] = cons. \tag{2.46}$$

From (2) of Equation (2.39), $k_g'(l) = 0$ due to $\tau_g = 0$, the constant is evaluated by setting $s = l$ and so,

$$(k_g')^2 + \frac{1}{4}[k_g^2 - k_g^2(l)]^2 - c^2[k_g + k_g(l)]^2 = 0. \tag{2.47}$$

Each of the three nonnegative terms on the left hand side of Equation (2.47) must vanish. Thus, $k_g(s)$ must have the constant value $\pm k_g(l)$ along its entire length. From Equation (2.1), $k_g(s) = \frac{\alpha}{l}$.

The definition of the energy ΔU of a bending fluctuation on the pseudosphere is given as

$$\Delta U = U(\alpha) - U(0), \tag{2.48}$$

where U is the energy of the elastic line defined by Equation (1.1) and $U(\alpha)$ is the energy when the elastic line coincides with the stationary curve. $U(0)$ is the energy when the elastic line is not bent. The vanishing of the bending angle α implies vanishing geodesic curvature, ellipses and hyperbolas for the pseudo-sphere from Theorem 1.2. We know that ellipses and hyperbolas are relaxed positions for elastic lines. $U(\alpha)$ is given by

$$U(\alpha) = \frac{1}{2}b\int_0^l \left|\varepsilon_2 k_g^2 + \varepsilon_3 k_n^2\right| ds. \tag{2.49}$$

In the case of $k_g^2 < k_n^2$ and $k_g^2 > k_n^2$,

$$\Delta U = \pm\frac{1}{2}b\frac{\alpha^2}{l} \tag{2.50}$$

follows for the hyperbolic 2-space.

Hooke's Law holds for the hyperbolic 2-space.

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