# THE CONTRACTED $l^1$-ALGEBRA OF A McALISTER MONOID

By

M.J. Crabb*

and

W.D. Munn
Department of Mathematics, University of Glasgow

[Accepted 21 June 2006. Published 20 March 2007.]

## Abstract

Let $M_X$ denote the McAlister monoid on a set $X$ with at least two elements. It is shown that the contracted $l^1$-algebra of $M_X$ is: (a) *-primitive if and only if $X$ is countably infinite; (b) prime if and only if $X$ is infinite.

## 1. Introduction

The McAlister monoid $M_X$ on a set $X$ with at least two elements is the monoid obtained from the free inverse monoid on $X$ by factoring out the ideal generated by all elements $xy^{-1}$ and $x^{-1}y$ for $x, y \in X$ and $x \neq y$. Thus, $M_X$ is an inverse monoid with zero and it inherits several of the properties of the free inverse monoid on $X$. We consider here the Banach algebra $l^1_0(M_X)$, called the contracted $l^1$-algebra of $M_X$, obtained from the standard $l^1$-algebra of $M_X$ by identifying the zero of $M_X$ with that of its algebra. This has an involution *, induced by inversion ($x \mapsto x^{-1}$) in $M_X$. The main result, Theorem 2, concerns the algebraic property of *-primitivity, a strong form of primitivity defined below, and shows that $l^1_0(M_X)$ has this property if and only if $X$ is countably infinite. In Theorem 3, it is proved that $l^1_0(M_X)$ is prime if and only if $X$ is infinite. Analogous results hold if we replace $l^1_0(M_X)$ by the usual contracted semigroup algebra $\mathbb{C}_0[M_X]$ of $M_X$ over the complex field $\mathbb{C}$. A more detailed discussion of the various concepts mentioned above now follows.

## 2. Discussion

By a *-*module* for an algebra $R$ over $\mathbb{C}$ with an involution * we mean a (left) module $V$ for $R$, under the action $\circ$, which admits an inner product $\langle \ | \ \rangle$ such that, for all $u, v \in V$ and all $a \in R$,

$$\langle a \circ u |\, v\rangle = \langle u |\, a^* \circ v\rangle.$$

As in [4], we say that $R$ is *-*primitive* if and only if it has a faithful irreducible *-module. Clearly, if $R$ is *-primitive then it is primitive.

---

*Corresponding author, e-mail: m.crabb@maths.gla.ac.uk

*Mathematical Proceedings of the Royal Irish Academy*, **107**A (1), 31–41 (2007) © Royal Irish Academy

The $l^1$-*algebra* $l^1(S)$ *of a semigroup* $S$ consists of all functions $a: S \to \mathbb{C}$ of finite or countably infinite support and such that $\sum_{x\in S}|a(x)| < \infty$, where addition and scalar multiplication are defined pointwise and multiplication is convolution. Then $l^1(S)$ is a Banach algebra, with norm $\|\ \|$ defined by $\|a\| = \sum_{x\in S}|a(x)|$. By identifying each $x \in S$ with its characteristic function, we can write a typical element $a$ of $l^1(S)$ as $\sum_{x\in S}\alpha_x x$ $(\alpha_x \in \mathbb{C})$, where $\sum_{x\in S}|\alpha_x| < \infty$.

The semigroup algebra $\mathbb{C}[S]$ of $S$ over $\mathbb{C}$ is the subalgebra of $l^1(S)$ consisting of all elements of finite support. In the case where $S$ is a nontrivial semigroup with zero $z$, it is frequently convenient to replace $\mathbb{C}[S]$ by $\mathbb{C}[S]/\mathbb{C}z$, where $\mathbb{C}z$ is the ideal $\{\alpha z : \alpha \in \mathbb{C}\}$—thereby, in effect, identifying $z$ with the zero of the algebra. The quotient $\mathbb{C}[S]/\mathbb{C}z$ is called the 'contracted semigroup algebra' of $S$ over $\mathbb{C}$ and is denoted by $\mathbb{C}_0[S]$. Bearing in mind that $\mathbb{C}z$ is a closed ideal of $l^1(S)$, we call the Banach algebra $l^1(S)/\mathbb{C}z$ the *contracted* $l^1$-*algebra of* $S$ and denote it by $l^1_0(S)$. Taking $S'$ to be the set of all nonzero elements of $S$, we can write a typical element $a$ of $l^1_0(S)$ as $\sum_{x\in S'}\alpha_x x$ $(\alpha_x \in \mathbb{C})$, where $\sum_{x\in S'}|\alpha_x| < \infty$; and we define the *support of* $a$, $\operatorname{supp}(a)$, to be $\{x \in S' : \alpha_x \neq 0\}$.

We shall be concerned with the case in which $S$ is an inverse semigroup; that is, a semigroup in which to each $x \in S$ there corresponds a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. For an account of such semigroups, see [6]. In particular, inversion $(x \mapsto x^{-1})$ is an involution on $S$ and this extends to an involution $*$ on $l^1(S)$ by the rule that

$$\left(\sum_{x\in S}\alpha_x x\right)^* = \sum_{x\in S}\overline{\alpha_x}x^{-1} \qquad (\alpha_x \in \mathbb{C}),$$

where $\overline{\alpha_x}$ denotes the complex conjugate of $\alpha_x$. This induces an involution on the semigroup algebra $\mathbb{C}[S]$. For the case in which $S$ is a nontrivial inverse semigroup with zero, inversion in $S$ induces an involution $*$ on $l^1_0(S)$ by a similar rule; and this, in turn, induces an involution on $\mathbb{C}_0[S]$.

In preparation for a discussion of McAlister monoids, we introduce further notation. First, the cardinal of a set $S$ will be denoted by $|S|$, singleton sets will be identified with their elements and, for two sets $S$ and $T$, $S\backslash T := \{x \in S : x \notin T\}$. The set of positive integers is denoted by $\mathbb{N}$ and we write $\aleph_0 := |\mathbb{N}|$. Let $X$ be a fixed nonempty set. The free group on $X$ will be denoted by $G$, its elements consisting of the reduced words in the alphabet $X \cup X^{-1}$, where $X^{-1}$ is the set of formal inverses of the elements of $X$. The identity of $G$ is the empty word, denoted by 1. We define the *length* $l(w)$ and the *content* $\operatorname{con}(w)$ of $w \in G$ as follows: if, in reduced form, $w = y_1y_2\cdots y_n$ $(y_i \in X \cup X^{-1})$, then $l(w) = n$ and $\operatorname{con}(w) = \{x_1, x_2, \ldots, x_n\}$, where $x_i = y_i$ if $y_i \in X$ and $x_i = y_i^{-1}$ if $y_i \in X^{-1}$; also $l(1) = 0$ and $\operatorname{con}(1) = \emptyset$. For a subset $H$ of $G$, we write

$$\operatorname{con}(H) := \cup_{h\in H}\operatorname{con}(h); \qquad H^{-1} := \{g \in G : g^{-1} \in H\}.$$

For $w \in G$, we define $\overline{w} \subseteq G$ by

$$\overline{w} := \{g \in G : g \text{ is a prefix of } w \text{ (including 1 and } g)\},$$

and for $H \subseteq G$ we write $\overline{H} := \cup_{h \in H} \overline{h}$. A subset $A$ of $G$ is termed *left-closed* if and only if $\overline{A} = A \neq \emptyset$.

Now let $\mathcal{E}$ denote the set of all finite left-closed subsets of $G$ and let

$$I_X := \{(A, g) \in \mathcal{E} \times G : \ g \in A\}.$$

For $(A, g),\ (B, h) \in I_X$ it can be shown that $A \cup gB \in \mathcal{E}$; thus, since $gh \in A \cup gB$, it follows that $(A \cup gB, gh) \in I_X$. We can therefore define a multiplication in $I_X$ by the rule that

$$(A, g)(B, h) = (A \cup gB, gh).$$

With this multiplication, $I_X$ is *the free inverse monoid on* $X$ [6, chapter 6]. Note that its identity is $(1, 1)$ and that, for all $(A, g) \in I_X$,

$$(A, g)^{-1} = (g^{-1}A, g^{-1}), \qquad (A, g)^2 = (A, g) \Leftrightarrow g = 1.$$

For $x \in X$, write $[x] := (\overline{x}, x) = (\{1, x\}, x)$. Then the mapping $x \mapsto [x]$ embeds $X$ in $I_X$.

*For the remainder of the paper we shall assume that* $X$ *is a set with* $|X| \geq 2$.

Let $N$ denote the two-sided ideal of $I_X$ generated by all elements of the form $[x][y]^{-1}$ or of the form $[x]^{-1}[y]$, where $x, y \in X$ and $x \neq y$. *The McAlister monoid* $M_X$ *on* $X$ is defined to be the Rees quotient

$$I_X/N.$$

This concept, starting from a different definition, was introduced and studied by Lawson [6, section 9.4], the title reflecting the fact that it is closely related to a semigroup that first appeared in [7]. An alternative treatment, based on the representation of the elements of $I_X$ by birooted word trees, is given in [9]. Observe that $M_X$ inherits certain key properties of $I_X$: in particular, it is combinatorial and completely semisimple, each nonzero principal factor being a finite semigroup of matrix units [6].

It is convenient in this paper to give a description of $M_X$ that is derived directly from the description of $I_X$ given above. First, we denote by $P$ the set of all elements of $G$ that (in reduced form) have no negative exponents. Thus $P$ is isomorphic to the free monoid on $X$. Now write

$$\mathcal{L} := \{\overline{u} \cup \overline{v^{-1}} : \ u, v \in P\}$$

and note that $\mathcal{L} \subset \mathcal{E}$. The following theorem is readily deduced from the description in [9] of the elements of $M_X$ in terms of linear birooted word trees. However, we give a direct proof below.

**Theorem 1.** $M_X = \{(A,g) \in \mathcal{L} \times G : \ g \in A\} \cup \{0\}$ *with multiplication according to the rule that*

$$(A,g)(B,h) = \begin{cases} (A \cup gB, gh) & \text{if } A \cup gB \in \mathcal{L}, \\ 0 & \text{otherwise,} \end{cases}$$

*and* $0^2 = 0 = 0(A,g) = (A,g)0$.

Proof. It suffices to show that

$$I_X \backslash N = \{(A,g) \in \mathcal{L} \times G : \ g \in A\}. \tag{1}$$

Note that, for $(A,g) \in I_X$, $(A,g) \in N$ if and only if $(A,1) \in N$. Note also that $N$ is the ideal of $I_X$ generated by all elements of the form $(\overline{xy^{-1}}, 1)$ or $(\overline{x^{-1}y}, 1)$ for $x, y \in X$ with $x \neq y$.

First, observe that $A \in \mathcal{E} \backslash \mathcal{L}$ if and only if, for some $x, y \in X$ with $x \neq y$, one of the following holds:
(a) $wxy^{-1} \in A$ for some $w \in P$,
(b) $wx^{-1}y \in A$ for some $w \in P^{-1}$,
(c) $wx, wy \in A$ for some $w \in P$,
(d) $wx^{-1}, wy^{-1} \in A$ for some $w \in P^{-1}$.
If (a) holds then $(A,1) = (A,w)(\overline{xy^{-1}},1)(A,w)^{-1} \in N$. Similarly, if (b) holds then $(A,1) \in N$. If (c) holds then $(A,1) = (A \cup wx \cup wy, 1) = (A,wx)(\overline{x^{-1}y},1)(A,wx)^{-1} \in N$. Similarly, if (d) holds then $(A,1) \in N$. Thus

$$A \in \mathcal{E} \backslash \mathcal{L} \Rightarrow (A,1) \in N.$$

Conversely, assume that $(A,1) \in N$. Then there exist $w \in A$ and $x, y \in X$ with $x \neq y$, such that either (e) $wx, wxy^{-1} \in A$ or (f) $wx^{-1}, wx^{-1}y \in A$. Assume that (e) holds and suppose that $A \in \mathcal{L}$. Then $w \in P \cup P^{-1}$. But, by (a), $w \notin P$ and so $w \in P^{-1}$. Now $w = vz^{-1}$, with $v \in P^{-1}$ and $z \in X \backslash x$, is impossible by (b); and $w = vx^{-1}$, with $v \in P^{-1}$, is impossible by (d). Hence $A \notin \mathcal{L}$. Similarly, if (f) holds then $A \notin \mathcal{L}$. Thus,

$$(A,1) \in N \Rightarrow A \in \mathcal{E} \backslash \mathcal{L}.$$

Consequently, for $(A,g) \in I_X$, $(A,g) \notin N$ if and only if $A \in \mathcal{L}$, and so (1) holds. ■

Denote the set of all idempotents of $M_X$ by $E$. Recall that $E$ is a commutative subsemigroup of $M_X$ (called the *semilattice* of $M_X$) and is partially ordered by the rule that

$$e \geq f \Leftrightarrow ef = f.$$

From Theorem 1, the following results are easily deduced:

$$E = \{(\overline{u} \cup \overline{v^{-1}}, 1) : \ u, v \in P\} \cup \{0\};$$

and, for $e = (\overline{u_1} \cup \overline{v_1^{-1}}, 1)$ and $f = (\overline{u_2} \cup \overline{v_2^{-1}}, 1)$ $\quad (u_1, u_2, v_1, v_2 \in P)$,

$$e \geq f \Leftrightarrow u_1 \text{ is a prefix of } u_2 \text{ and } v_1 \text{ is a suffix of } v_2.$$

For $e, f \in E$ we write $e \succ f$ ('$e$ covers $f$') to mean that $e > f$ and that there is no $g \in E$ for which $e > g > f$. Note that, for all $f \in E$,

$$(1,1) \succ f \quad \Leftrightarrow \quad f \in \{(\overline{x}, 1) : \ x \in X\} \cup \{(\overline{x^{-1}}, 1) : \ x \in X\}.$$

We now give four lemmas. The first of these is a general lemma on Banach algebras and is adapted from a result of Duncan [5]. (See also [8].) A proof is provided for convenience.

**Lemma 1.** *Let $R$ be a Banach algebra with norm $\| \ \|$, let $V$ be a Banach space with norm $\| \ \|_V$ and let $\circ$ be a left action of $R$ on $V$ such that*

$$(\forall\, a \in R)(\forall\, v \in V) \quad \|a \circ v\|_V \leq \kappa \|a\|.\|v\|_V,$$

*where $\kappa$ is a positive constant. If there exists a cyclic vector $v_0$ in $V$ and if, for all $v \in V \backslash 0$, there exists a sequence $(a_n)$ in $R$ such that $a_n \circ v \to v_0$, then $V$ is irreducible.*

PROOF. Let $v_0$ be a cyclic vector in $V$. Since the mapping $f : R \to V$ defined by $f(a) := a \circ v_0$ is continuous, the open mapping theorem shows that, for some positive real number $\delta$,

$$\{v \in V : \ \|v\|_V < \delta\} \subseteq \{f(a) : \ a \in R \text{ and } \|a\| < 1\}.$$

Let $v \in V \backslash 0$ and let $(a_n)$ be a sequence in $R$ such that $a_n \circ v \to v_0$. Then there exists $n \in \mathbb{N}$ such that $\|v_0 - a_n \circ v\|_V < \delta$. Hence, there exists $a \in R$ with $\|a\| < 1$, such that

$$v_0 - a_n \circ v = a \circ v_0.$$

Define $c \in R$ by $c := -\sum_{r=1}^{\infty} a^r$. Then $a + c - ca = 0$. Hence,

$$\begin{aligned}(a_n - ca_n) \circ v &= (v_0 - a \circ v_0) - c \circ (v_0 - a \circ v_0)\\ &= v_0 - (a + c - ca) \circ v_0 = v_0.\end{aligned}$$

Since $v_0$ is cyclic, it follows that $v$ is cyclic. Thus $V$ is irreducible. ■

The next three lemmas involve different restrictions on the cardinality of $X$.

**Lemma 2.** *Let $X$ be finite. Then $l_0^1(M_X)$ is not prime.*

PROOF. Let $E$ denote the set of idempotents of $M_X$ and, for brevity, let $e$ denote the identity $(1, 1)$. Since $X$ is finite, $e$ covers exactly $2n$ elements of $E \backslash 0$, where

$n = |X|$. Define $\sigma(e) \in l_0^1(M_X)$ by

$$\sigma(e) := \prod_{f \in E,\, e \succ f} (e - f).$$

Let $a \in l_0^1(M_X)\backslash 0$ and let $p \in \text{supp}(a)$. First, suppose that $e > pp^{-1}$. Then there exists $f \in E$ such that $e \succ f \geq pp^{-1}$, and so $\sigma(e)pp^{-1} = 0$. Thus $\sigma(e)p = 0$. Next, suppose that $e = pp^{-1}$. Now $p = (A, g)$ for some $A \in \mathcal{L}$ and $g \in A$. Since $pp^{-1} = (A, 1)$, we have that $A = 1$ and so $g = 1$. Thus $p = e$ and therefore $\sigma(e)p = \sigma(e)$. In either case, $\sigma(e)p(e - \sigma(e)) = 0$. Hence, $\sigma(e)a(e - \sigma(e)) = 0$. But $\sigma(e) \neq 0$ and $e - \sigma(e) \neq 0$ in $l_0^1(M_X)$. Thus $l_0^1(M_X)$ is not prime. ■

**Lemma 3.** *Let $X$ be infinite, let $I$ be a two-sided ideal of $l_0^1(M_X)$ and let $\epsilon$ be a positive real number. Then there exist $e = e^2 \in M_X\backslash 0$ and $r \in l_0^1(M_X)$, such that $e + r \in I$ and $\|r\| < \epsilon$. In particular, $e$ can be taken to be either $(\overline{w}, 1)$ or $(\overline{w^{-1}}, 1)$ for some $w \in P$.*

Proof. Let $a \in I\backslash 0$. Consider $(A, g) \in \text{supp}(a)$, with $|A|$ minimal in $\{|B| : (B, h) \in \text{supp}(a)\}$. Let $\alpha \in \mathbb{C}\backslash 0$ be the coefficient of $(A, g)$ in $a$. Then $a = b + c$, where $b, c \in l_0^1(M_X)$, $(A, g) \in \text{supp}(b)$, $|\text{supp}(b)| < \infty$, $\text{supp}(b)\cap\text{supp}(c) = \emptyset$ and $\|c\| < \epsilon|\alpha|$.

Let $Y := \cup_{w \in \,\text{supp}(b)}\text{con}(w)$. Then, since $|\text{supp}(b)| < \infty$, $|Y| < \infty$. Choose $x \in X\backslash Y$ and define $D \in \mathcal{L}$ by $D := \overline{ux} \cup \overline{(xv)^{-1}}$, where $A = \overline{u} \cup \overline{v^{-1}}$ $\quad (u, v \in P)$. Then $A \subset D$. We show that

$$(D, 1)b(A, g)^{-1} = \alpha(D, 1). \tag{2}$$

Let $(B, h) \in \text{supp}(b)$. Then, with $T := A \cup B \cup hg^{-1}A$, we have that

$$(A, 1)(B, h)(A, g)^{-1} = \begin{cases} (T, hg^{-1}) & \text{if } T \in \mathcal{L}, \\ 0 & \text{otherwise.} \end{cases}$$

Now $A \subseteq T$. First, assume that $A = T$. Then $B \subseteq A$. Hence, since $|B| \geq |A|$, we have that $B = A$. Also, since $hg^{-1}A \subseteq A$, all powers of $hg^{-1}$ lie in $A$ and so $h = g$. Thus $(B, h) = (A, g)$. Next, assume that $A \subset T$ and that $T \in \mathcal{L}$. Then $T = \overline{u_1} \cup \overline{v^{-1}}$ for some $u_1, v_1 \in P$, and so either $l(u_1) > l(u)$ or $l(v_1) > l(v)$. Hence, either $l(u_1) \geq l(ux)$ or $l(v_1) \geq l(xv)$. Thus, since $x \notin \text{con}(T)$, $D \cup T \notin \mathcal{L}$ and so $(D, 1)(T, hg^{-1}) = 0$. Since $(D, 1)(A, 1) = (D, 1)$, we have established (2). Consequently,

$$(D, 1)a(A, g)^{-1} = \alpha(D, 1) + (D, 1)c(A, g)^{-1}.$$

Write $e := (D, 1)$ and $r := \alpha^{-1}(D, 1)c(A, g)^{-1}$. Then $e = e^2 \in M_X\backslash 0$, $e + r \in I$ and $\|r\| \leq |\alpha|^{-1}\|c\| < \epsilon$.

For the last part, define $p, q \in M_X$ by taking $p := (\overline{t}, t)$, $q := (\overline{s}, s)$, where $s := ux$, $t := xv$, and so $e = (\overline{s} \cup \overline{t^{-1}}, 1)$. It is easily verified that $pep^{-1} = (\overline{ts}, 1)$ and $q^{-1}eq = (\overline{(ts)^{-1}}, 1)$. Then $p(e + r)p^{-1} = e_1 + r_1 \in I$, where $e_1 := (\overline{ts}, 1)$ and

$r_1 := prp^{-1}$; also, $q^{-1}(e+r)q = e_2+r_2 \in I$, where $e_2 := (\overline{(ts)^{-1}}, 1)$ and $r_2 := q^{-1}rq$. Further, $\|r_i\| \leq \|r\| < \epsilon \quad (i = 1, 2)$. ■

**Lemma 4.** *Let $X$ be uncountably infinite and let $f$ be a positive linear functional on $l^1_0(M_X)$ with $f((1,1)) \neq 0$. Then $f$ vanishes on a nonzero, two-sided ideal of $l^1_0(M_X)$.*

Proof. By a projection in $l^1_0(M_X)$ we mean an element $a$ such that $a = a^2 = a^*$. Note that every idempotent in $M_X$ is a projection in $l^1_0(M_X)$, since $e = e^2 \in M_X$ implies $e^{-1} = e$. Note also that if $a$ is a projection in $l^1_0(M_X)$, then $f(a) = f(a^*a) \geq 0$.

Let $S$ be a nonempty set of pairwise-orthogonal projections in $l^1_0(M_X)$. We show first that $|\{a \in S : \; f(a) > 0\}| \leq \aleph_0$. Let $a_1, a_2, \ldots, a_n$ be finitely many distinct elements of $S$, and let $s := a_1+a_2+\cdots+a_n$. Then $((1,1)-s)^2 = (1,1)-s = ((1,1)-s)^*$, and so $f((1,1)-s) \geq 0$. For all $n \in \mathbb{N}$, let $S_n := \{a \in S : \; f(a) > \frac{1}{n} f((1,1))\}$. By the preceding argument, $|S_n| \leq n$. Suppose that $a \in S$ is such that $f(a) > 0$. Then there exists $n \in \mathbb{N}$ such that $a \in S_n$. Thus

$$\{a \in S : \; f(a) > 0\} \subseteq \cup_{n=1}^{\infty} S_n,$$

and so $|\{a \in S : \; f(a) > 0\}| \leq \aleph_0$, as required.

For each pair $(m, n)$ of nonnegative integers, define a subset $\mathcal{L}_{m,n}$ of $\mathcal{L}$ by

$$\mathcal{L}_{m,n} := \{\overline{u} \cup \overline{v^{-1}} : \; u, v \in P, \; l(u) = m, \; l(v) = n\}.$$

Now for all $u_1, u_2, v_1, v_2 \in P$ with $l(u_1) = l(u_2)$, $l(v_1) = l(v_2)$ and either $u_1 \neq u_2$ or $v_1 \neq v_2$, we have that

$$(\overline{u_1} \cup \overline{v_1^{-1}}, 1)(\overline{u_2} \cup \overline{v_2^{-1}}, 1) = 0.$$

Hence, if $m$ and $n$ are not both zero, $\{(A, 1) : \; A \in \mathcal{L}_{m,n}\}$ is a set of pairwise-orthogonal projections in $l^1_0(M_X)$ and so, by the first part of the proof,

$$|\{A \in \mathcal{L}_{m,n} : \; f((A, 1)) > 0\}| \leq \aleph_0.$$

Write $Y_{m,n} := \cup_{A \in \mathcal{L}_{m,n}} \{\mathrm{con}(A) : \; f((A,1)) > 0\}$ and let $Y$ denote the union of all such sets $Y_{m,n}$. Thus, for each $(m,n)$, $|Y_{m,n}| \leq \aleph_0$ and so $|Y| \leq \aleph_0$. Choose $x \in X \backslash Y$. Then $f((A,1)) = 0$ for all $A \in \mathcal{L}$ with $x \in \mathrm{con}(A)$. Let $(A, g) \in M_X$, with $x \in \mathrm{con}(A)$. By the Cauchy–Schwarz inequality,

$$|f((A,g))|^2 = |f((A,g)(1,1))|^2 \leq f((A,g)(A,g)^{-1}) f((1,1)) = 0,$$

and so $f((A,g)) = 0$. Thus $f$ vanishes on the ideal of $\mathbb{C}_0[M_X]$ spanned by $\{(A,g) \in M_X : \; x \in \mathrm{con}(A)\}$. Hence, by continuity [1, §37, corollary 9] $f$ vanishes on its closure, a nonzero ideal of $l^1_0(M_X)$. ■

The main result now follows.

**Theorem 2.** *$l^1_0(M_X)$ is $^*$-primitive if and only if $X$ is countably infinite.*

Proof. Assume that $|X| = \aleph_0$. Then $|P| = \aleph_0$ and so the elements of $P\backslash 1$ can be written as a sequence $(w_i)$. Let $(x_i)$ be the sequence of elements of $X$ obtained by taking, in order, all the letters of $w_1$, followed by those of $w_2$, followed by those of $w_3$, and so on. Define $H \subset G$ by writing

$$H := \{h_i : \ i = 0, 1, 2, \dots \},$$

where $h_0 = 1$ and, for $i \geq 1$, $h_i = x_1 x_2 \cdots x_i$. Let $V$ denote the Banach space $l^1(H^{-1})$; thus a typical element of $V$ is of the form $\sum_{i=0}^{\infty} \alpha_i h_i^{-1}$ for some sequence $(\alpha_i)_{i=0}^{\infty}$ of complex numbers with $\sum_0^\infty |\alpha_i| < \infty$. We shall define a left action $\circ$ of $l^1_0(M_X)$ on $V$.

First, for all $(A, g) \in M_X\backslash 0$ and all $f \in H$, write

$$(A, g) \circ f^{-1} = \begin{cases} gf^{-1} & \text{if } A \subseteq gf^{-1}H, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $A \subseteq gf^{-1}H$, then $1 \in gf^{-1}H$ and so $gf^{-1} \in H^{-1}$. Now, for all $(A, g), (B, h) \in M_X\backslash 0$ and all $f \in H$,

$$(B, h) \circ [(A, g) \circ f^{-1}] = \begin{cases} hgf^{-1} & \text{if } A \subseteq gf^{-1}H \text{ and } B \subseteq hgf^{-1}H, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and

$$[(B, h)(A, g)] \circ f^{-1} = \begin{cases} hgf^{-1} & \text{if } B \cup hA \in \mathcal{L} \text{ and } B \cup hA \subseteq hgf^{-1}H, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We next show that

$$B \cup hA \subseteq hgf^{-1}H \quad \Rightarrow \quad B \cup hA \in \mathcal{L}. \tag{5}$$

For all $n \in \mathbb{N}$, write $H_n := \overline{h_n}$. Assume that $B \cup hA \subseteq hgf^{-1}H$. Then $B \cup hA \subseteq k^{-1}H_n$ for some $n$, where $k := fg^{-1}h^{-1}$. Now $H_n \in \mathcal{L}$ and, since $1 \in B$, $k \in H_n$. Hence $(H_n, k) \in M_x\backslash 0$. Thus, $(k^{-1}H_n, k^{-1}) = (H_n, k)^{-1} \in M_X\backslash 0$ and so $k^{-1}H_n \in \mathcal{L}$. But, in $I_X$,

$$(B \cup hA, 1)(k^{-1}H_n, 1) = (k^{-1}H_n, 1).$$

Hence, if $(B \cup hA, 1) \in N$ then $(k^{-1}H_n, 1) \in N$, which is false. Thus, $B \cup hA \in \mathcal{L}$, which establishes (5).

Since $A \subseteq gf^{-1}H$ and $B \subseteq hgf^{-1}H$ if and only if $B \cup hA \subseteq hgf^{-1}H$, it follows from (3),(4) and (5) that

$$(B, h) \circ [(A, g) \circ f^{-1}] = [(B, h)(A, g)] \circ f^{-1}.$$

Now $\|(A, g) \circ f^{-1}\|_V \leq 1$, where $\| \ \|_V$ denotes the norm on $V$. Hence, by linearity and continuity, we can extend $\circ$ to a left action of $l_0^1(M_X)$ on $V$ with the property that

$$(\forall\, a \in l_0^1(M_X))(\forall\, v \in V) \qquad \|a \circ v\|_V \leq \|a\|.\|v\|_V. \tag{6}$$

First, we show that the module $V$ is faithful. Denote the ideal $\{a \in l_0^1(M_X) : a \circ V = 0\}$ by $\operatorname{ann}(V)$. Suppose that $\operatorname{ann}(V) \neq 0$. Then, by Lemma 3, there exists $w \in P$ and $r \in l_0^1(M_X)$, such that $(\overline{w}, 1) + r \in \operatorname{ann}(V)$ and $\|r\| < \frac{1}{2}$. Now there exists $v \in H$ such that $v, vw \in H$. Hence, $(\overline{w}, 1) \circ v^{-1} = v^{-1}$ and so, by (6),

$$\|((\overline{w}, 1) + r) \circ v^{-1}\|_V \geq 1 - \|r \circ v^{-1}\|_V \geq 1 - \|r\| > \frac{1}{2}.$$

But $((\overline{w}, 1) + r) \circ v^{-1} = 0$, which gives a contradiction. Hence, $\operatorname{ann}(V) = 0$ and so $V$ is faithful.

We now establish the irreducibility of $V$. To begin with, we prove that 1 is a cyclic vector in $V$. Consider a typical element $v = \sum_0^\infty \alpha_i h_i^{-1} \in V$, where $(\alpha_i)$ is a sequence of complex numbers with $\sum_0^\infty |\alpha_i| < \infty$. For all $i$, $(H_i, h_i) \in M_X\backslash 0$ and $(H_i, h_i)^{-1} \circ 1 = (h_i^{-1}H_i, h_i^{-1}) \circ 1 = h_i^{-1}$. Thus, $\left(\sum_0^\infty \alpha_i (H_i, h_i)^{-1}\right) \circ 1 = v$ and so 1 is cyclic in $V$. Next, we show that, for $v \in V\backslash 0$, there exists a sequence $(a_n)$ of elements of $l_0^1(M_X)$ such that $a_n \circ v \to 1$ as $n \to \infty$. Again, take $v$ as above and assume that $\alpha_j \neq 0$. Then $(H_j, h_j) \circ h_j^{-1} = 1$ and so

$$(H_j, h_j) \circ v = \sum_{i=0}^{\infty} \beta_i h_i^{-1} \tag{7}$$

for some coefficients $\beta_i$ with $\sum_0^\infty |\beta_i| < \infty$ and $\beta_0 (= \alpha_j) \neq 0$. Let $n \in \mathbb{N}$. Suppose that there exists $r \in \{1, 2, \ldots, n\}$ such that, for all $q \in \mathbb{N}$, $h_r H_q \subseteq H$. Then $x_i = x_{i+r}$ for all $i \in \mathbb{N}$, which is impossible from the definition of $H$. Hence, for all $r \in \{1, 2, \ldots, n\}$, there exists $q_r \in \mathbb{N}$ such that $h_r H_{q_r} \nsubseteq H$. Let $m_n = \max\{q_1, q_2, \ldots, q_n\}$. Then $h_r H_{m_n} \nsubseteq H$, since $H_{q_r} \subseteq H_{m_n}$; that is, $(H_{m_n}, 1) \circ h_r^{-1} = 0$ $(r = 1, 2, \ldots, n)$. Also, $(H_{m_n}, 1) \circ 1 = 1$ and so, from (7),

$$[(H_{m_n}, 1)(H_j, h_j)] \circ v = \beta_0 1 + \sum_{i \in S} \beta_i h_i^{-1}$$

for some $S \subseteq \{i \in \mathbb{N} :\ i > n\}$. Write $a_n := \beta_0^{-1}(H_{m_n}, 1)(H_j, h_j)$. Then $a_n \circ v \to 1$ as $n \to \infty$. Since 1 is cyclic, Lemma 1 shows that $V$ is irreducible.

To complete the proof that $l_0^1(M_X)$ is *-primitive, we now show that $V$ is a *-module. Let $\langle\ |\ \rangle$ denote the inner product on $V$ induced by taking $H^{-1}$ to be an orthonormal basis. Let $h, k \in H^{-1}$ and let $(A, g) \in M_X\backslash 0$. Then $(A, g)^{-1} \in M_X\backslash 0$ and

$$\begin{aligned}(A, g) \circ h = k &\Leftrightarrow A \subseteq ghH \text{ and } gh = k \\ &\Leftrightarrow g^{-1}A \subseteq g^{-1}kH \text{ and } g^{-1}k = h \\ &\Leftrightarrow (g^{-1}A, g^{-1}) \circ k = h.\end{aligned}$$

Thus,

$$\begin{aligned}\langle (A,g)\circ h|k\rangle &= \begin{cases} 1 & \text{if } (A,g)\circ h = k,\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} 1 & \text{if } (A,g)^{-1}\circ k = h,\\ 0 & \text{otherwise}\end{cases}\\ &= \langle h|(A,g)^{-1}\circ k\rangle.\end{aligned}$$

Hence, by linearity and continuity, $\langle a\circ u|v\rangle = \langle u|a^*\circ v\rangle$ for all $a\in l_0^1(M_X)$ and all $u,v\in V$. Consequently, $V$ is a *-module and so $l_0^1(M_X)$ is *-primitive.

Conversely, assume that $l_0^1(M_X)$ is *-primitive. We show that $X$ must be countably infinite. First, we note that $l_0^1(M_X)$ is primitive and therefore prime. Hence, by Lemma 2, $X$ is infinite. Suppose that $X$ is uncountable. Let $V$ be a faithful irreducible *-module for $l_0^1(M_X)$, with left action $\circ$, and let $\langle\ |\ \rangle$ be an inner product on $V$ such that, for all $a\in l_0^1(M_X)$ and all $u,v\in V$, $\langle a\circ u|v\rangle = \langle u|a^*\circ v\rangle$. Since $V$ is faithful, there exists $u\in V\backslash 0$ such that $(1,1)\circ u\neq 0$. Define a positive linear functional $f$ on $l_0^1(M_X)$ by the rule that, for all $a\in l_0^1(M_X)$,

$$f(a) = \langle a\circ u|u\rangle.$$

Observe that $f((1,1))\neq 0$. Hence, by Lemma 4, $f$ vanishes on a nonzero ideal $I$ of $l_0^1(M_X)$. Let $a\in I\backslash 0$. Then, for all $b,c\in l_0^1(M_X)$,

$$0 = f(bac) = \langle (bac)\circ u|u\rangle. \tag{8}$$

Let $v\in V\backslash 0$. Since $V$ is irreducible, there exists $c\in l_0^1(M_X)$ such that $c\circ u = v$. Suppose that $a\circ v\neq 0$. Then there exists $b\in l_0^1(M_X)$ such that $b\circ(a\circ v) = u$. Thus $(bac)\circ u = u$ and so, from (8), $0 = \langle u|u\rangle$, which is false. Hence $a\circ v = 0$. This shows that $V$ is not faithful, contrary to our hypothesis. Consequently, $X$ cannot be uncountable. We have thus shown that if $l_0^1(M_X)$ is *-primitive, then $X$ is countably infinite. This completes the proof. ■

Finally, we have

**Theorem 3.** *$l_0^1(M_X)$ is prime if and only if $X$ is infinite.*

PROOF. By Lemma 2, if $l_0^1(M_X)$ is prime then $X$ is infinite. Now assume that $X$ is infinite. Let $I_1$ and $I_2$ be nonzero, two-sided ideals of $l_0^1(M_X)$. By Lemma 3, there exist $w_1,w_2\in P$ and $r_1,r_2\in l_0^1(M_X)$, such that

$$(\overline{w_1},1)+r_1\in I_1,\quad (\overline{w_2^{-1}},1)+r_2\in I_2,\quad \|r_1\|<\frac{1}{3},\quad \|r_2\|<\frac{1}{3}.$$

Then

$$((\overline{w_1},1)+r_1)((\overline{w_2^{-1}},1)+r_2) = (\overline{w_1}\cup\overline{w_2^{-1}},1)+t\in I_1I_2, \tag{9}$$

where $t := r_1(\overline{w_2^{-1}}, 1) + (\overline{w_1}, 1)r_2 + r_1r_2$. But $\|(\overline{w_1} \cup \overline{w_2^{-1}}, 1)\| = 1$ and

$$\|t\| \leq \|r_1\| + \|r_2\| + \|r_1\|.\|r_2\| < 1.$$

Hence, from (9), $I_1I_2 \neq 0$. This shows that $l_0^1(M_X)$ is prime. ■

## 3. Concluding remarks

1. Exact counterparts of Theorems 2 and 3 hold for the contracted semigroup algebra $\mathbb{C}_0[M_X]$: this algebra is *-primitive if and only if $X$ is countably infinite, and it is prime if and only if $X$ is infinite. The method of proof is similar to that for $l_0^1(M_X)$, but simpler in various places—for example, Lemma 1 is not required.

2. Parts of the argument above can also be adapted to establish results on the contracted semigroup algebra $F_0[M_X]$ of $M_X$ over an arbitrary field $F$. Thus, it can be shown that (a) if $X$ is countably infinite then $F_0[M_X]$ is primitive; and (b) $F_0[M_X]$ is prime if and only if $X$ is infinite. These results were announced in [9], where an alternative proof of (a) is obtained.

3. The techniques in this paper have also been used by one of us to provide an example of a prime $C^*$-algebra that is not primitive [2], and to show that $l^1(I_X)$ is *-primitive if and only if $X$ is infinite [3].

## References

[1] F.F. Bonsall and J. Duncan, *Complete normed algebras*, Springer–Verlag, Berlin–Heidelberg–New York, 1973.

[2] M.J. Crabb, A new prime $C^*$-algebra that is not primitive, *Journal of Functional Analysis* **236** (2006), 630–3.

[3] M.J. Crabb, The $l^1$-algebra of a free inverse monoid, preprint, University of Glasgow.

[4] M.J. Crabb and W. D. Munn, On the contracted $l^1$-algebra of a 0-bisimple inverse semigroup, *Proceedings of the Royal Society of Edinburgh* **135A** (2005), 285–95.

[5] J. Duncan, Dual representations of Banach algebras, unpublished PhD thesis, University of Newcastle-upon-Tyne, 1964.

[6] M.V. Lawson, *Inverse semigroups*, World Scientific, Singapore, 1998.

[7] D.B. McAlister, Inverse semigroups separated over a subsemigroup, *Transactions of the American Mathematical Society* **172** (1973), 85–117.

[8] C.M. McGregor, A representation for $l^1(S)$, *Bulletin of the London Mathematical Society* **8** (1976), 156–60.

[9] W.D. Munn, On McAlister's monoid and its contracted algebra, *Proceedings of the Conference on Semigroups and Languages*, Lisbon, 2005.