GENERALIZED WEYL'S THEOREM FOR POSINORMAL OPERATORS

By S. Mecheri
King Saud University, College of Science, Department of Mathematics
P.O. Box 2455, Riyadh 11451, Saudi Arabia

[Accepted 14 December 2006. Published 20 April 2007.]

Abstract

Let A be a bounded linear operator acting on infinite dimensional separable Hilbert space H. In this paper, we prove that the generalized Weyl's theorem holds for $f(A)$ if A is conditionally totally posinormal or totally posinormal, where f is a function analytic in an open neighborhood of $\sigma(A)$. Other related results are also given.

1. Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on infinite dimensional separable Hilbert space H. If $A \in B(H)$ we shall write $N(A)$ and $R(T)$ for the null space and the range of A, respectively. Also, let $\alpha(A) := dimN(A)$, $\beta(A) := dimN(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of A, respectively.

An operator $A \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space and its range has finite co-dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(A) - \beta(A).$$

A is called Weyl if it is of index zero, and Browder if it is Fredholm of finite ascent and descent; equivalently [12, theorem 7.9.3], if A is Fredholm and $A - \lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in \mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by [11; 12]:

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\},$$

$$\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\},$$

$$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Browder}\},$$

respectively. Evidently,

$$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) = \sigma_e(A) \cup acc\sigma(A),$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$. If we write $isoK =$

---

*E-mail: mecherisalah@hotmail.com

$K \setminus accK$, then we let

$$\pi_{00}(A) := \{\lambda \in iso\sigma A : 0 < \alpha(A-\lambda) < \infty\},$$

$$p_{00}(A) := \sigma(A) \setminus \sigma_b(A).$$

We say that Weyl's theorem holds for A if

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A).$$

More generally, Berkani in [1] says that the generalized Weyl's theorem holds for A provided:

$$\sigma(A) \setminus \sigma_{Bw}(A) = E(A),$$

where $E(A)$ and $\sigma_{Bw}(A)$, respectively, denote the isolated point of the spectrum that are eigenvalues (no restriction on multiplicity) and the set of complex numbers λ for which $A-\lambda I$ fails to be Weyl. Let X be a Banach space. An operator $A \in B(X)$ is called B-Fredholm by Berkani [1] if there exists $n \in \mathbb{N}$ for which the induced operator

$$A_n : A^n(X) \to A^n(X)$$

is Fredholm in the usual sense, and B-Weyl if, in addition, A_n has index zero. Note that, if the generalized Weyl's theorem holds for A, then so does Weyl's theorem [1]. We say that Browder's theorem holds for A if

$$\sigma(A) \setminus \sigma_w(A) = p_{00}(A).$$

We say that $A \in B(H)$ has the single valued extension property (SVEP) if, for every open set $U \subseteq \mathbb{C}$, the only analytic function $f : U \to H$ that satisfies the equation $(A-\lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$. For a $A \in B(H)$, let

$$H_0(A) = \{x \in H : \lim_{n\to\infty} ||A^n x||^{\frac{1}{n}} = 0\}$$

denote the quasi-nilpotent part of the operator A.

A is said to have finite ascent if $\ker A^m = \ker A^{m+1}$ for some positive integer m, and finite descent if $R(A^n) = R(A^{n+1})$ for some positive integer n. Laursen [15, proposition 1.8] proved that if $A - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, then A has the single valued extension property.

An operator $A \in B(H)$ is said to be posinormal (the word 'posinormal' stands for 'positive-normal'), if there exists a $P \geq 0$ in $B(H)$ such that $AA^* = A^*PA$. Or equivalently, $A \in B(H)$ is posinormal if there exists a co-isometry $V^* \in B(H)$ and a positive operator $P \in B(H)$ such that $A = A^*PV^*$.

Rhaly [19] introduced posinormal operators and proved many interesting properties of posinormal operators; and such operators have since been considered by Jeon *et al.* [13].

The class of posinormal operators contains, in particular, the classes consisting of hyponormal operators $(A \in B(H) : AA^* \leq A^*A)$, M-hyponormal $(A \in B(H) :$

$|(A-\lambda I)^*|^2 \leq M|(A-\lambda I)|^2$ for some real number $M > 0$) and dominant operators ($A \in B(H)$: $|(A-\lambda I)^*|^2 \leq M_\lambda |(A-\lambda I)^2$ for some real number $M_\lambda > 0$ and all complex numbers λ). A posinormal operator A is said to be conditionally totally posinormal (resp., totally posinormal), shortened to $A \in CTP$ (resp., $A \in TP$), if to each complex number λ there corresponds a positive P_λ such that $|(A-\lambda I)^*|^2 = |P_\lambda^{\frac{1}{2}}(A-\lambda)|^2$ (resp., if there exists a positive operator P such that $|(A-\lambda I)^*|^2 = |P^{\frac{1}{2}}(A-\lambda)|^2$ for all λ). A is called algebraically posinormal if there exists a nonconstant polynomial $q(z)$ such that $q(A)$ is posinormal.

In [24], Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators [5], and to several classes of operators including semi-normal operators [3; 4]. Recently, Lee and Lee [16] showed that Weyl's theorem holds for algebraically hyponormal operators. R. Curto and Y.M. Han [6] have extended Lee and Lee's results to the algebraically paranormal operator $A \in B(H)$, where H is a separable Hilbert space. In [17], the author showed that Weyl's theorem holds for algebraically (p, k)-quasi-hyponormal $A \in B(H)$, where H is a general Hilbert space. Berkani [1] showed that if A is a hyponormal operator, then A satisfies the generalized Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the B-Weyl spectrum $\sigma_{Bw}(A)$ of A satisfies the spectral mapping theorem. Han and Lee [10] showed that Weyl's theorem holds for algebraically hyponormal operators.

Recently H. Zguitti [25] showed that if A is isoloid, and if the generalized Weyl's theorem holds for algebraically paranormal operators, then the generalized Weyl's theorem holds for $f(A)$, where f is a function analytic in an open neighborhhod of $\sigma(A)$. Duggal and Kubrusky [8] showed that Weyl's theorem holds for $f(A)$, where f is a function analytic in an open neighborhhod of $\sigma(A)$ in the case where A is a conditionally totally posinormal operator or totally posinormal with additional conditions. In this paper we show that the generalized Weyl's theorem holds for $f(A)$ if A is a conditionally totally posinormal operator or totally posinormal with some additional conditions, where f is a function analytic in an open neighborhood of $\sigma(A)$.

2. Main results

Lemma 2.1. *Let $A \in B(H)$ be CTP or TP . Then $A-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$. In particular, T has the single valued extension property.*

PROOF. It is easy to see that if $A \in CTP$, then $\ker(A-\lambda) \subseteq \ker(A-\lambda)^*$. Hence $A-\lambda$ has *ascent* ≤ 1. In particular, A has SVEP. ■

Theorem 2.1. *Let $A \in B(H)$ be CTP or TP . Let $\lambda \in \sigma(A)$ be an isolated point of $\sigma(A)$. Then*

$$H_o(A) = E_\lambda H$$

where E_λ denotes the Riesz idempotent for λ.

Proof. Since A has the single valued extension property by Lemma 2.1, the equality follows from [20, p. 424]. ■

Proposition 2.1. *Let $A \in B(H)$ be CTP, respectively TP, and let $\mathcal{M} \subset H$ be an invariant subspace of A. Then the restriction $A|_{\mathcal{M}}$ is also CTP, respectively TP.*

Proof. Let P be the orthogonal projection on $\mathcal{M}$. Then for all $z \in \mathbb{C}$ and for all $x \in \mathcal{M}$,

$$||(A - zI|_{\mathcal{M}})^* x|| = |P(A - Z)^* x|| = ||(A - zI)^* x|| = \mathcal{M}_z ||(A|_{\mathcal{M}} - zIx)||.$$

■

Before proving the following lemma, we need the following notations and definitions. Let $A = U|A|$ be the polar decomposition of $A \in B(H)$. A is said to be class $\mathcal{Y}_\alpha$ for $\alpha \geq 1$ if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A - \lambda)^* (A - \lambda) \text{ for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \cup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator A is M-hyponormal, i.e. there exists a positive number M such that

$$(A - \lambda)(A - \lambda)^* \leq M^2 (A - \lambda)^* (A - \lambda) \text{ for all } \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_2$ (see [23]).

Lemma 2.2. *Let $A \in B(H)$ be a totally posinormal operator. If $\sigma(A - \lambda I) = \{0\}$, then $A - \lambda I = 0$.*

Proof. By the same argument used in the proof of [23, lemma 10], we can show that if A is totally posinormal, then $A \in \mathcal{Y}_2 \subseteq \mathcal{Y}$. and it is known that if $A \in \mathcal{Y}$ and if $\sigma(A) = \{0\}$, then $A = 0$ (see [23, lemma 14]). Therefore if A is totally posinormal, then $A - \lambda I$ is also totally posinormal and $\sigma(A - \lambda I) = \{0\}$. Hence $A - \lambda I = 0$. ■

Lemma 2.3. *Let A be a quasinilpotent, algebraically totally posinormal operator. Then A is nilpotent.*

Proof. Assume that $p(A)$ is totally posinormal for some nonconstant polynomial p. Since $\sigma(p(A)) = p(\sigma(A))$, the operator $p(A) - p(0)$ is quasinilpotent. Thus Lemma 2.2 would imply that

$$cA^m (A - \lambda_1) ... (A - \lambda_n) \equiv p(A) - p(0) = 0,$$

where $m \geq 1$. Since $A - \lambda_i$ is invertible for every $\lambda \neq 0$, we must have $A^m = 0$. ■

It is known [16, theorem 3.36] that SVEP is stable under the functional calculus, i.e. if $A \in B(H)$ has SVEP, then so does $f(A)$ for each f analytic in an open neighborhood of $\sigma(A)$. The following lemma is immediate:

Lemma 2.4. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then $f(A)$ has SVEP for each analytic function f on a neighborhood of $\sigma(A)$*

Theorem 2.2. *Let A be a totally posinormal operator. Then generalized Weyl's theorem holds for A.*

PROOF. Assume that $\lambda \in \sigma(A) \backslash \sigma_{Bw}(A)$. Then $A-\lambda I$ is B-Weyl and not invertible. We claim that $\lambda \in \partial\sigma(A)$. Assume to the contrary that λ is an interior point of $\sigma(A)$. Then there exists a neigborhood U of λ such that $dim(A-\mu) > 0$ for all $\mu \in U$. It follows from [9, theorem 10], that A does not have SVEP. On the other hand, since A is totally posinormal, it follows from Lemma 2.4 above that A has SVEP, which is a contradiction. Therefore $\lambda \in \partial\sigma(A)$. Conversely, assume that $\lambda \in E(A)$, then λ is isolated in $\sigma(A)$. From [14, theorem 7.1], we have $X = M \oplus N$, where M, N are closed subspaces of X, $U = (A-\lambda I)|_M$ is an invertible operator and $V = (A-\lambda I)|_N$ is a quasinilpotent operator. Since A is totally posinormal, V is also totally posinormal, and from Lemma 2.2 V is nilpotent. Therefore $A-\lambda I$ is Drazin invertible [21, proposition 6] and [15, corollary 2.2]. By [2, lemma 4.1], $A-\lambda I$ is a B-Fredholm operator of index 0. ■

Theorem 2.3. *Let A be a totally posinormal operator. Then $f(A)$ obeys the generalized Weyl's theorem for every function f analytic in a neigberhood of $\sigma(A)$.*

PROOF. Since the operator A satisfies the generalized Weyl's theorem and it is isoloid, it follows from [1, lemma 2.9] that $f(A)$ obeys the generalized Weyl's theorem. ■

Theorem 2.4. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then $f(A)$ satisfies Browder's theorem for each function f analytic in a neighborhood of $\sigma(A)$*

PROOF. It is known that operators with SVEP satisfy Browder's theorem [7]. Then $f(A)$ satisfies Browder's theorem. This completes the proof. ■

Before proving the following theorems we need some notations and definitions. Let $A \in B(H)$, let n be a nonnegative integer and define $A_{[n]}$ to be the restriction of A to $R(A^n)$ viewed as a map from $R(A^n)$ to $R(A^n)$ (in particular $A_{[0]} = A$). If, for some integer n, the range space $R(A^n)$ is closed and $A_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then A is called an upper (resp. lower) semi-B-Fredholm operator. Moreover, if $A|_{[n]}$ is a Browder operator, then A is called a B-Browder operator. Similarly, we can define the B-Fredholm's spectrum $\sigma_{BF}(A)$ and the B-Browder's spectrum $\sigma_{BB}(A)$. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator.

It is known that Weyl's theorem holds for M-hyponormal operators, but it does not hold for dominant operators that are equivalent to CTP operators. Hence, it is an interesting problem to seek a condition that implies Weyl's theorem for CTP operators. Restricting themselves to only those $A \in CTP$ for which the spectrum $\sigma((A - \lambda I)|_M) = \{0\} \Rightarrow (A - \lambda I)|_M = \{0\}$ for every $M \in Lat(A)$, Jeon *et al.* [13, proposition 3.5] have shown that A satisfies Weyl's theorem. In the following theorems we can expand on this.

Theorem 2.5. *Let $A \in B(H)$ be a conditionally totally posinormal operator such that ($\sigma((A - \lambda I)|_M) = \{0\} \Rightarrow (A - \lambda)|_M = 0$ for every $M \in Lat(A)$). Then the generalized Weyl's theorem holds for A.*

Proof. We will show that $\sigma(A) \setminus \sigma_{Bw}(A) \subset E(A)$. For this, assume that $\lambda \in \sigma(A) \setminus \sigma_{Bw}(A)$. Then $A - \lambda$ is a B-Fredholm operator of index zero, and there exists a direct sum decomposition $H = H_1 \oplus H_2$, such that $A_1 = (A - \lambda)|_{H_1}$ is a Fredholm operator of index zero, $A_2 = (A - \lambda)|_{H_2}$ is nilpotent and $A - \lambda = A_1 \oplus A_2$ [2, lemma 4.1]. We have two possibilities: either $\lambda \in \sigma(A|_{H_1})$ or $\lambda \notin \sigma(A_{H_1})$.

Assume that $\lambda \in \sigma(A|_{H_1})$. Since A is conditionally totally posinormal, $A|_{H_1}$ is also conditionally totally posinormal. Hence, [13, proposition 3.5] implies that $A|_{H_1}$ satisfies Weyl's theorem. Therefore if $\lambda \in \sigma(A_{H_1})$, then $\lambda \in \pi_{00}(A|_{H_1})$. Hence, $\lambda \in iso\sigma(A|_{H_1})$. Now, since $A - \lambda = (A|_{H_1} - \lambda I|_{H_1}) \oplus A_2$, and A_2 is nilpotent, we have $\sigma(A_1) \setminus \{0\} = \sigma(A - \lambda) \setminus \{0\}$ and $\lambda \in iso\sigma(A)$. This implies that $\lambda \in \pi_{00} \subset E(A)$. Conversely, let $\lambda \in E(A)$. Then λ is an isolated point of $\sigma(A)$. Therefore λ is an isolated point of $\sigma(A)$. Let P be the spectral projection

$$P = \int_{\partial B_0} (\lambda_0 I - A)^{-1} d\lambda_0,$$

where B_0 is an open disk centred at λ that contains no other points of $\sigma(A)$. Then A can be represented as the direct sum

$$A = A_1 \oplus A_2, \text{where}\, \sigma(A_1) = \{\lambda\} \text{ and}\, \sigma(A_2) = \sigma(A) \setminus \{\lambda\}).$$

Then $\lambda I - A_2$ is invertible. We have to consider two cases.

Case I where $\lambda = 0$. Assume that $\lambda = 0$. Then $\sigma(A_1) = \{0\}$. Since $A_1 \in$ CTP such that $\sigma(A) = 0 \Rightarrow A = 0$, it follows that $A_1 = 0$. Therefore $\lambda I - A = 0 \oplus \lambda I - A_2$.

Case II where $\lambda \neq 0$. Since A_1 is an invertible CTP operator, it follows that A_1^{-1} is a CTP operator. Then $||A_1|| = |\lambda|$ and $||A_1^{-1}|| = \frac{1}{\lambda}$. Therefore, for any $x \in R(P)$, we have:

$$||x|| \leq ||A_1^{-1}||||A_1 x|| = \frac{1}{|\lambda|}||A_1 x|| \leq \frac{1}{|\lambda|}|\lambda||x| = ||x||.$$

Hence $\frac{1}{\lambda}A_1$ is unitary. Therefore A_1 is normal and $\lambda I - A_1$ is also normal. Since $\lambda I - A_1$ is quasinilpotent and the only normal quasinilpotent operator is zero, it follows that $\lambda I - A = 0 \oplus \lambda I - A_2$. Now since $\lambda I - A_2$ is invertible, it is known that $\lambda I - A$ has finite acsent and descent. Suppose that $ascent(\lambda_0 I - A) = descent(\lambda_0 -$

$A) = p$. Then $H = N[(\lambda_0 I - A)^p] \oplus R[(\lambda_0 I - A)^p]$. Therefore $(\lambda_0 I - A)|_{[p]}$ is Weyl, hence $\lambda_0 \in \sigma(A) \setminus \sigma_{Bw}(A)$. ■

Remark 2.1. Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A - \lambda I)|_M) = \{0\} \Rightarrow (A - \lambda I)|_M = 0$. If A is quasinilpotent, then Lemma 2.4 would imply that A is nilpotent. By using the same techniques used in the proof of [6, lemma 2.3], it is easy to see that A is isoloid.

Theorem 2.6. *Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A - \lambda I)|_M) = \{0\} \Rightarrow (A - \lambda I)|_M = 0$. Then $f(A)$ satisfies the generalized Weyl's theorem for every function f analytic in a neighborhood of $\sigma(A)$. In particular, Weyl's theorem holds for $f(A)$.*

Proof. By the previous theorem, A satisfies the generalized Weyl's theorem and since A is isoloid by Remark 2.1, [1, lemma 2.9] would imply that the generalized Weyl's theorem holds for $f(A)$. ■

The essential approximate point spectrum $\sigma_{ea}(A)$ is defined by:

$$\sigma_{ea}(A) = \cap\{\sigma_a(A + K) : K \text{ is a compact operator}\},$$

where $\sigma_a(A)$ is the approximate point spectrum of A. We consider the set

$$\Phi_+^-(H) = \{A \in B(H) : A \text{ is left semi-Fredholm and } \operatorname{ind} A \leq 0\}.$$

V. Rakočević [18] proved that

$$\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+^-(H)\},$$

and the inclusion $\sigma_{ea}(f(A)) \subset f(\sigma_{ea}(A))$ holds for all functions $f(z)$ that are analytic on some open neighborhood of $\sigma(A)$ with no restriction on A. The next theorem shows the spectral mapping theorem on the essential approximate point spectrum of conditionally totally posinormal operators or totally posinormal operators.

Lemma 2.5. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$. If $A - \lambda$ is semi-Fredholm and it has finite ascent, then ind $(A - \lambda) \leq 0$.*

Proof. If $A - \lambda$ has finite descent, then i*nd* $(A - \lambda) = 0$ by [22, theorem V 6.2]. If $A - \lambda$ does not have finite descent, then

$$n \; ind \; (A - \lambda) = \dim N(A - \lambda)^n - \dim R((A - \lambda)^n)^\perp \to -\infty.$$

Hence i*nd* $(A - \lambda) < 0$. ■

Corollary 2.1. *Let $A \in B(H)$ be conditionally totally posinormal operators or totally posinormal operators. If $A - \lambda$ is semi-Fredholm for some $\lambda \in \mathbb{C}$, then ind $(A - \lambda) \leq 0$.*

Theorem 2.7. *Let $A \in B(H)$ be conditionally totally posinormal operators or totally posinormal operators. Then*

$$\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for every function $f(z)$ that is analytic on some open neighborhood G of $\sigma(A)$.

Proof. It suffices to show that $f(\sigma_{ea}(A)) \subseteq \sigma_{ea}(f(A))$.

We may assume that f is nonconstant. Let $\lambda \notin \sigma_{ea}(f(A))$ and let

$$f(z) - \lambda = g(z)\Pi_{j=1}^{n}(z - \lambda_j),$$

where $\lambda_j \in G$ and $g(z) \neq 0$ for all $z \in G$. Then

$$f(A) - \lambda = g(T)\Pi_{j=1}^{n}(A - \lambda_j).$$

Since $\lambda \notin \sigma_{ea}(f(A))$ and all operators on the right side of above equality commute, each $(A - \lambda_j)$ is left semi-fredholm and ind $(A - \lambda_j) \leq 0$ by the previous corollary. Thus $\lambda_j \notin \sigma_{ea}(A)$ and $\lambda \notin f(\sigma_{ea}(A))$. ■

Acknowledgement

This research was supported by the K.S.U. research center project, no. Math/2007/10.

References

[1] M. Berkani and A. Arroud, Generalized Weyl's theorem and Hyponormal operators, *Journal of the Australian Mathematical Society* **76** (2004), 291–302.

[2] M. Berkani, Index of B-Fredholm operators and poles of the resolvant, *Journal of Mathematical Analysis and Applications* **272** (2002), 596–603.

[3] S.K. Berberian, An extension of Weyl's theorem to a class of not necessarily normal, *Michigan Mathematical Journal* **16** (1969), 273–9.

[4] S.K. Berberian, The Weyl spectrum of an operator, *Indiana University Mathematical Journal* **20** (1970), 529–44.

[5] L.A. Coburn, Weyl's theorem for nonnormal operators, *Michigan Mathematical Journal* **13** (1966), 285–8.

[6] R.E. Curto and Y.M. Han, Weyl's theorem for algebraically paranormal operators, *Integral Equations Operator Theory* **47** (2003), 307–14.

[7] B.P. Duggal and S.V. Djordjevic, Dunfort's Property (C) and Weyl's theorem, *Integal Equations Operator Theory* **43** (2002), 290–7.

[8] B.P. Duggal and C. Kubrusky, Weyl's theorem for posinormal operators, *Journal of the Korean Mathematical Society* **42**(2005), 529–41.

[9] J.K. Finch, The single valued extension property on a Banach space, *Pacific Journal of Mathematics* **58** (1975), 61–9.

[10] Y.M. Han and W.Y. Lee, Weyl's theorem holds for algebraically hyponormal operators, *Proceedings of the American Mathematical Society* **128** (2000),2291–6.

[11] R.E. Harte, Fredholm, Weyl and Browder theory, *Mathematical Proceedings of the Royal Irish Academy* **85**A (1985), 151–76.

[12] R.E. Harte, *Invertibility and singularity for bounded linear operators*, Dekker, New York, 1988.

[13] In Ho Jeon, Se Hee Kim, Eungil Ko and Ji Eum Park, On positive-normal operators, *Bulletin of the Korean Mathematical Society* **39** (2002), 33–41.

[14] J.J. Koliha, A generalized Drazin inverse, *Glasgow Mathematical Journal* **38** (1996), 367–81.

[15] K.B. Laursen, Operators with finite ascent, *Pacific Journal of Mathematics* **152** (1992), 323–36.

[16] S.H. Lee and W.Y. Lee, A spectral mapping theorem for the Weyl spectrum, *Glasgow Mathematical Journal* **38** (1996), 61–4.

[17] S. Mecheri, Weyl's thoeroem for algebraically (p, k)-quasihyponormal, *Georgian Mathematical Journal* **13** (2006), 1998–2007

[18] V. Rakočević, On the essential approximate point spectrum II, *Matematicki Vesnik* **36** (1984), 89–97.

[19] H.C. Rhaly Jr., Posinormal operators, *Journal of the Mathematical Society of Japan* **46** (1994), 587–607.

[20] F. Riesz and B. Sz. Nagy, *Functional analysis*, Frederick Ungar, New York, 1955.

[21] S. Roch and B. Silbermann, Continuity of Generalized inverses in Banach Algebras, *Studia Mathematica* **136** (1999), 197–227.

[22] A.E. Taylor and D.C. Lay, *Introduction to functional analysis* 2nd edn, Krieger, Florida, 1980.

[23] A. Uchiyama and T. Yoshino, On the class $\mathcal{Y}$ operators, *Nihonkai Mathematical Journal* **8** (1997), 179-1-94.

[24] H. Weyl, Uber beschrankte quadratishe formen, deren Differenz vollsteig ist, *Rendiconti del Circolo Matematico di Palermo* **27** (1909), 373–92.

[25] H. Zguitti, A note on generalized Weyl's theorem, *Journal of Mathematical Analysis and Applications* **316** (2006), 373–81.