CONVERGENCE OF FUNCTIONS OF OPERATORS IN THE STRONG OPERATOR TOPOLOGY

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Abstract

In this note, we show that if a net $f_\alpha(T)$ converges pointwise to $f(T)$, then there is a subsequence f_{α_n} uniformly convergent to f on the compact subsets of some part of the spectrum of T. This extends a well known result in the case of uniform convergence of functions of operators. We give some applications of the result.

1. Introduction

Throughout this paper, $\mathcal{H}$ will denote an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$. In general, we will follow the notation in [5] and [7].

For an operator T in $\mathcal{B}(\mathcal{H})$ we will use $\sigma(T)$ and $\rho(T)$ for the spectrum and the resolvent set of T. We denote by $\sigma_p(T)$ the point spectrum of T, and we denote by $\sigma_c(T)$ the compression spectrum of T.

If K is a compact subset of the complex plane, $\widehat{K}$ will denote the *polynomially convex hull* of K. If G is an open subset of $\mathbf{C}$ and $g : G \longrightarrow \mathbf{C}$ is an analytic function, then we will denote by g^* the analytic function $g^* : \overline{G} \longrightarrow \mathbf{C}$ given by $g^*(z) = \overline{g(\overline{z})}$. Here and hereafter the bar denotes complex conjugate. Recall that if T is an operator and g is an analytic function defined in a neighborhood of $\sigma(T)$, then $g(T)^* = g^*(T^*)$.

Let T be an operator and let f_n, f be analytic functions defined in a neighborhood of $\sigma(T)$. The spectral mapping theorem (see [5, theorem 4.10, p. 204]) and the fact that the spectral radius is less than or equal to the norm imply that

$$\sup_{z\in\sigma(T)} |f_n(z) - f(z)| \leq ||f_n(T) - f(T)||.$$

If $f_n(T) \to f(T)$ in the norm topology, we obtain from the previous inequality that $f_n \to f$ uniformly on $\sigma(T)$.

Let f_α be a net of functions that are defined and analytic in a neighborhood of $\sigma(T)$. If $f_\alpha(T) \to f(T)$ in the strong operator topology (that is, in the topology of pointwise convergence), it is easy to see that $f_\alpha(z) \to f(z)$ for every $z \in \sigma_p(T)$.

Let $\lambda \in \sigma_c(T)$. Then $\overline{\lambda} \in \sigma_p(T^*)$. Let x be a norm one eigenvector of T^*

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corresponding to $\overline{\lambda}$. Then $f^*(T^*)(x) = f^*(\overline{\lambda})x = \overline{f(\lambda)}x$ and the same is true for each f_α. Thus,

$$\begin{aligned} f_\alpha(\lambda) = f_\alpha(\lambda)||x||^2 = f_\alpha(\lambda)\langle x, x\rangle = \langle x, \overline{f_\alpha(\lambda)}x\rangle = \langle x, f_\alpha^*(T^*)x\rangle = \langle f_\alpha(T)x, x\rangle \\ \rightarrow \quad \langle f(T)x, x\rangle = \langle x, f^*(T^*)x\rangle = \langle x, \overline{f(\lambda)}x\rangle = f(\lambda)\langle x, x\rangle = f(\lambda). \end{aligned}$$

Therefore $f_\alpha(z) \rightarrow f(z)$ for every $z \in \sigma_p(T) \cup \sigma_c(T)$.

It seemed natural to us to ask if this convergence is somehow, somewhere uniform. We will show that there is Ω, an open subset of $\sigma(T)$, and there is an increasing subsequence of indices α_n, such that f_{α_n} converges to f uniformly on the compact subsets of Ω.

2. Apostol triangular representation

This section contains an account of the upper triangular representation of a general operator due to Constantin Apostol, as well as some consequences of this representation.

For an operator T we will denote by $\rho_{SF}(T)$ the semi-Fredholm domain of T, and we will denote by by $\mathcal{P}_\pm(T)$ its subset on which the Fredholm index is non zero. The symbol $\rho(T)$ stands for the resolvent set of T.

If T is an operator, then the function $min.ind_T : \rho_{SF}(T) \longrightarrow \mathbf{N}$, given by $min.ind_T(\lambda) = \min\{\dim\ker(\lambda - T), \dim\ker(\lambda - T)^*\}$, is called *the minimal index of* T.

If T is an operator, then the map $k_T : \rho_{SF}(T) \longrightarrow \mathcal{B}(\mathcal{H})$, given by $k_T(\lambda) = P_{\ker(\lambda-T)}$, is called the *kernel function* of T. Here, $P_{\ker(\lambda-T)}$ denotes the orthogonal projection onto $\ker(\lambda - T)$. The points where k_T is continuous, when considering the uniform topology on $\mathcal{B}(\mathcal{H})$, are called the *regular points* of T. The points where k_T is not continuous are called the *singular points* of T. The set of all regular points and the set of all singular points of an operator T are denoted by $\rho^r_{SF}(T)$ and $\rho^s_{SF}(T)$, respectively.

The following proposition gives a connection between the two functions defined above.

Proposition 2.1. *k_T is continuous at λ iff $min.ind_T$ is continuous at λ.*

Proof. See [1, proposition 2.6]. ■

The next proposition states the main topological properties of the set of regular points and the set of singular point.

Proposition 2.2. *If T is an operator then:*

(i) $\rho^r_{SF}(T)$ is an open set;
(ii) $\rho^s_{SF}(T)$ has no limit point in $\rho_{SF}(T)$.

Proof. See [1, theorem 2.2]. ■

For a bounded open set D, we will denote by $hull(D)$ the union of D with all the bounded components of the complement. If γ is a simple Jordan loop, we will denote by $ins\gamma$ the inside of γ. In fact, we have

$$hull(D) = \{z \in \mathbf{C};\ \text{there is a simple Jordan loop}\ \gamma \subset D\ \text{such that}\ z \in ins\gamma\}.$$

Corollary 2.3. *If T is an operator and $K \subset hull(\mathcal{P}_\pm(T))$, compact, then there is a finite set of simple, closed Jordan curves $\{\Gamma_j\}_{j=1}^n$ such that $K \subset \cup_{j=1}^n ins\Gamma_j$ and $\Gamma_j \subset \mathcal{P}_\pm(T) \cap \rho_{SF}^r(T)$ for every $1 \le j \le n$. In particular, if there is a component D of $\mathcal{P}_\pm(T)$ such that $K \subset hull(D)$, then we can take $n = 1$.*

Although it looks very innocent, the next proposition is the basis of the whole construction. For an operator T, we will denote the right resolvent by $\rho_r(T)$ and we will denote the left resolvent by $\rho_l(T)$.

Proposition 2.4. *If T is an operator and K is a compact subset of $\rho_r(T) \cap \sigma_p(T)$, then there is a vector x such that $P_{\ker(\lambda - T)}x \neq 0$ for every $\lambda \in K$.*

PROOF. See [1, proposition 1.8]. ■

Next, we will state the properties of the Apostol triangular representation. If L is a subset of $\mathcal{H}$ we will denote by $CSp(L)$ the closed linear span of L. Also, we will use $\sigma_{p0}(T)$ for the set of all isolated eigenvalues of finite (geometric) multiplicity of T (sometimes called normal eigenvalues). Not only are these eigenvalues isolated, but the corresponding spectral subspace in the Riesz decomposition has finite dimension.

Theorem 2.5. *If T is an operator acting on a Hilbert space $\mathcal{H}$, and*

$$\mathcal{H}_r(T) = CSp\{\ker(\lambda - T)\,;\ \lambda \in \rho_{SF}^r(T)\}\mathcal{H}_l(T) = CSp\{\ker(\lambda - T)^*\,;\ \lambda \in \rho_{SF}^r(T)\},$$

then:
(i) $\mathcal{H}_r(T) \perp \mathcal{H}_l(T)$.
If $\mathcal{H}_0(T) = (\mathcal{H}_r(T) \oplus \mathcal{H}_l(T))^\perp$ and T_r, T_0 and T_l are the compressions of T to $\mathcal{H}_r(T)$, $\mathcal{H}_0(T)$ and $\mathcal{H}_l(T)$, respectively, then:
(ii) $\mathcal{H}_r(T)$, $\mathcal{H}_r(T) \oplus \mathcal{H}_0(T) \in LatT$,
and so, with respect to the decomposition $\mathcal{H} = \mathcal{H}_r(T) \oplus \mathcal{H}_0(T) \oplus \mathcal{H}_l(T)$,

$$T = \begin{pmatrix} T_r & * & * \\ 0 & T_0 & * \\ 0 & 0 & T_l \end{pmatrix},$$

such that:
(iii) $\rho_{SF}(T) \subseteq \rho_r(T_r) \cap \rho_l(T_l)$;
(iv) $\rho_{SF}^r(T) \subseteq \rho(T_0)$;
(v) $\rho_{SF}^s(T) \subset \sigma_{p0}(T_0)$;

(vi) $\sigma_{p0}(T) \subseteq \rho(T_r) \cap \rho(T_l) \cap \sigma_{p0}(T_o)$;
(vii) $min.ind_{T_r}(\lambda) = 0$ *for every* $\lambda \in \rho_{SF}(T_r)$.

Proof. See [1, theorem 2.8]. ■

Proposition 2.6. *If* T *is an operator, then* $\rho_{SF}(T) \cap cl(int\sigma_p(T)) \subset \rho_r(T_r) \cap \sigma_p(T_r)$.

Proof. Let $\lambda \in \rho_{SF}(T) \cap cl(int\sigma_p(T))$. As $\lambda \in \rho_{SF}(T)$, by (iii) of Theorem 2.5, we conclude that $\lambda \in \rho_r(T_r)$. Suppose that $\lambda \notin \sigma_p(T_r)$. Then, since $\lambda - T_r$ has closed range and no kernel, $\lambda \in \rho_l(T_r)$ and so $\lambda \in \rho(T_r)$. Hence, there is an $\varepsilon > 0$ such that $B_\varepsilon(\lambda) \subset \rho(T_r) \cap \rho_{SF}(T)$. The set $B_\varepsilon(\lambda) \cap int\sigma_p(T)$ is open, nonempty and it is included in $\rho_{SF}(T)$. Therefore, by (ii) of Proposition 2.2, there is a $\mu \in B_\varepsilon(\lambda) \cap \sigma_p(T) \cap \rho^r_{SF}(T)$. But $\ker(\mu - T_r) = \ker(\mu - T) \neq (0)$, which implies $\mu \in \sigma(T_r)$, and this is a contradiction. ■

As a corollary, we will obtain a generalization of Proposition 2.4 for arbitrary minimal index.

Corollary 2.7. *(i) If* T *is an operator and* K *is a compact subset of* $\rho_{SF}(T) \cap cl(int\sigma_p(T))$*, then there is a vector* x *such that* $P_{\ker(\lambda - T)}x \neq 0$ *for every* $\lambda \in K$.
(ii) If T *is an operator and* K *is a compact subset of* $\rho^r_{SF}(T) \cap \sigma_p(T)$*, then there is a vector* x *and* $0 < m \leq M$ *such that* $m \leq \|P_{\ker(\lambda - T)}x\| \leq M$ *for every* $\lambda \in K$.

Proof. (i) By Proposition 2.6, $K \subset \rho_r(T_r) \cap \sigma_p(T_r)$. Hence, using Proposition 2.4, there is a vector x such that $P_{\ker(\lambda - T_r)}x \neq 0$ for every $\lambda \in K$, and as for $\lambda \in K$, $\ker(\lambda - T_r) = \ker(\lambda - T)$, we obtain the conclusion.
(ii) We actually have $K \subset \rho_{SF}(T) \cap int\sigma_p(T)$, so by (i), there is a vector x such that $P_{\ker(\lambda - T)}x \neq 0$ for every $\lambda \in K$. As $K \subset \rho^r_{SF}(T)$, the function $\lambda \longmapsto \|P_{\ker(\lambda - T)}x\|$ is continuous, it is defined on the compact set K and it never takes the value 0. Hence, it has a maximum and a nonzero minimum. ■

3. Main result

The following proposition is an easy consequence of the continuity of the Fredholm index.

Proposition 3.1. *For an operator* T*, and* K *a compact subset of* $hull(\mathcal{P}_\pm(T))$*, there is an* $\varepsilon > 0$ *such that* $K \subset hull(\mathcal{P}_\pm(S))$ *for every operator* S *with the property that* $||T - S|| < \varepsilon$.

For K, a compact subset of $\mathbf{C}$, and $\varepsilon > 0$ we will denote by

$$K_\varepsilon = \{z \in \mathbf{C} : dist(z, K) \leq \varepsilon\}.$$

Corollary 3.2. *If* T *is an operator and* $K \subset \rho_{SF}(T)$*, compact, then there is an*

$\varepsilon > 0$ such that $K_\varepsilon \subset \rho_{SF}(T)$ and the boundary of K_ε consists entirely of regular points.

Proposition 3.3. *Let T be an operator and let $\{f_\alpha\}$ be a net of analytic functions in a neighborhood of $\sigma(T)$. If $f_\alpha(T)$ is convergent in the strong operator topology to some operator S, then for every K, compact subset of $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$, there is α_0 such that $\{f_\alpha\}_{\alpha \geq \alpha_0}$ is uniformly bounded on K.*

PROOF. By Corollary 3.2, there is $\varepsilon > 0$ such that $K_\varepsilon \subset \rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$ and the boundary of K_ε consists entirely of regular points. Then the boundary of K_ε^* is a subset of $\rho_{SF}^r(T^*) \cap \sigma_p(T^*)$. By (ii) of Proposition 2.7, there is $y \in \mathcal{H}$ and there are $0 < m < M$ such that $m \leq \|P_{\ker(\lambda - T)^*} y\| \leq M$ for every λ in the boundary of K_ε.

Since $f_\alpha(T)y$ converges to Sy, there is α_0 such that $||f_\alpha(T)y|| \leq ||Sy|| + 1$ for every $\alpha \geq \alpha_0$.

Let $y_\lambda = P_{\ker(\lambda - T)^*} y$. Then,

$$\begin{aligned} m^2\,|f_\alpha(\lambda)| &= m^2 |f_\alpha^*(\overline{\lambda})| \leq |f_\alpha^*(\overline{\lambda})| \langle y_\lambda, y_\lambda \rangle = |f_\alpha^*(\overline{\lambda})| \langle y_\lambda, y \rangle = |\langle f_\alpha^*(T^*) y_\lambda, y \rangle| \\ &= |\langle y_\lambda, f_\alpha(T) y \rangle| \leq ||y_\lambda|| ||f_\alpha(T) y|| \leq M(||Sy|| + 1) \end{aligned}$$

for every $\alpha \geq \alpha_0$ and every λ in the boundary of K_ε. An application of the maximum modulus theorem (see [4, theorem 1.4, p. 129]) concludes the proof. ■

The next theorem establishes the uniform convergence claimed in Section 1 above.

Theorem 3.4. *If $f_\alpha(T)$ is convergent in the strong operator topology to $f(T)$, then there is an increasing sequence of indices α_n such that f_{α_n} converges to f uniformly on the compact subsets of $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$. Moreover, if x_m is a sequence of points in $\mathcal{H}$, the subsequence can be chosen such that $f_{\alpha_n}(T)x_m$ converges to $f(T)x_m$ for every $m \geq 1$.*

PROOF. Let K_n be a sequence of compact subsets of $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$, such that $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T) = \cup_n K_n$ and $K_n \subset int K_{n+1}$ for every $n \geq 1$. Let $\{z_n\}$ be a sequence of points in $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$, which has a limit point in every component of $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$. Thus $f_\alpha(z_n)$ converges to $f(z_n)$ for every $n \geq 1$.

Let α_1 such that $\{f_\alpha\}_{\alpha \geq \alpha_1}$ is uniformly bounded on K_1, $|f_\alpha(z_1) - f(z_1)| < \frac{1}{2}$ and $||f_\alpha(T)x_1 - f(T)x_1|| < \frac{1}{2}$ for $\alpha \geq \alpha_1$. Let $\alpha_2 > \alpha_1$ such that $\{f_\alpha\}_{\alpha \geq \alpha_2}$ is uniformly bounded on K_2, $|f_\alpha(z_j) - f(z_j)| < \frac{1}{2^2}$ and $||f_\alpha(T)x_j - f(T)x_j|| < \frac{1}{2^2}$ for $\alpha \geq \alpha_2$ and $j = 1, 2$. Let $\alpha_n > \alpha_{n-1}$ such that $\{f_\alpha\}_{\alpha \geq \alpha_n}$ is uniformly bounded on K_n, $|f_\alpha(z_j) - f(z_j)| < \frac{1}{2^n}$ and $||f_\alpha(T)x_j - f(T)x_j|| < \frac{1}{2^n}$ for $\alpha \geq \alpha_n$ and $j = 1, 2, \ldots, n$.

It can easily be seen that $\{f_{\alpha_n}\}$ is locally bounded on $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$. Therefore, by Montel's theorem (see [4, theorem 2.9, p. 153]), $\{f_{\alpha_n}\}$ is a normal family. Hence, there is g, an analytic function on $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$, such that a subsequence of $\{f_{\alpha_n}\}$ converges to g uniformly on the compact subsets. To simplify the notation, the subsequence will also be denoted by $\{f_{\alpha_n}\}$. In particular, we have that $f_{\alpha_n}(z_j)$ converges to $g(z_j)$ for every $j \geq 1$. Since, by construction, the same sequence converges to $f(z_j)$, we conclude that $f(z_j) = g(z_j)$ for every $j \geq 1$. Thus $f = g$ on a set that has a limit point in every component of $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$. This implies that $f = g$ on $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$.

The proof of the *Moreover* statement from Theorem 3.4 is obvious. ■

Corollary 3.5. *If $f_n(T)$ is convergent in the strong operator topology to $f(T)$, then f_n converges to f uniformly on the compact subsets of $(\rho_{SF}(T) \cap \sigma_c(T)) \setminus \sigma_{p0}(T)$.*

Proof. According to Theorem 3.4, every subsequence of f_n has a subsequence converging to f. It is not difficult to see that this, in fact, implies that f_n converges to f uniformly on the compact subsets of $\rho_{SF}(T) \cap \sigma_c(T)$. ■

Remark 3.6. *Theorem 3.4 and Corollary 3.5 are true in the case of convergence in the weak * topology.*

Remark 3.7. *If $||f_\alpha(T)||$ are bounded, then the result in Theorem 3.4 can be extended to $(\rho_{SF}(T) \cap \sigma(T)) \setminus \sigma_{p0}(T)$.*

Remark 3.8. *Theorem 3.4 and Corollary 3.5 are true, in particular, on the domain where the Fredholm index is negative.*

4. Applications

For an operator T on a Hilbert space, $P(T)$, $P^\infty(T)$ and $\mathcal{A}(T)$ will denote the uniform closure, the weak $*$ closure and the closure in the strong operator topology of the set of all polynomials in T and I, respectively.

Let A be a subset of the complex plane. For a Hilbert Space $\mathcal{H}$ we define:

$$\begin{aligned}
\mathcal{U}_A(\mathcal{H}) &= \{\, T \in \mathcal{B}(\mathcal{H}) \,:\, A \subset \rho(T) \text{ and } (\lambda - T)^{-1} \in P(T) \text{ for every } \lambda \in A \,\},\\
\mathcal{W}_A^*(\mathcal{H}) &= \{\, T \in \mathcal{B}(\mathcal{H}) \,:\, A \subset \rho(T) \text{ and } (\lambda - T)^{-1} \in P^\infty(T) \text{ for every } \lambda \in A \,\},\\
\mathcal{W}_A(\mathcal{H}) &= \{\, T \in \mathcal{B}(\mathcal{H}) \,:\, A \subset \rho(T) \text{and } (\lambda - T)^{-1} \in \mathcal{A}(T) \text{ for every } \lambda \in A \,\}.
\end{aligned}$$

We have $\mathcal{U}_A(\mathcal{H}) \subset \mathcal{W}_A^*(\mathcal{H}) \subset \mathcal{W}_A(\mathcal{H})$. It is completely apparent that all three classes are invariant under similarities and so are their interiors and their closures. Also, $T \in \mathcal{U}_A(\mathcal{H})$ if and only if $T^* \in \mathcal{U}_{A^*}(\mathcal{H})$, where A^* denotes the conjugate set of A. Similar relations hold for the other two classes. It is not difficult to see that if T is in one of the classes (on $\mathcal{H}$) and $\mathcal{M}$ is an invariant subspace of T, then the restriction of T to $\mathcal{M}$ is in the same class (on $\mathcal{M}$).

If A is the complement of the unit circle and T is a diagonal operator with the diagonal dense in the unit circle, then $T \in \mathcal{W}_A^*(\mathcal{H}) \setminus \mathcal{U}_A(\mathcal{H})$. See also [6, example 7.6, p. 304].

For an operator T, we will use $\sigma_{lre}(T)$ to denote the *left and right essential spectrum* of T(that is, the intersection of the left essential spectrum and right essential spectrum). Recall that if C is a component of $\sigma_{lre}(T)$ that is not a component of $\sigma_e(T)$, then some part of the boundary of C is included in the boundary of $\mathcal{P}_{\pm\infty}(T)$.

First, we will characterize the closure of the three classes.

Theorem 4.1. *For an operator T the following are equivalent:*
(a) $T \in cl\mathcal{U}_A(\mathcal{H})$,
(b) $T \in cl\mathcal{W}_A(\mathcal{H})$,
(c) $hull(\mathcal{P}_\pm(T)) \cap A = \emptyset$ *and if C is a component of $\sigma_{lre}(T) \cup \sigma_{p0}(T)$ then C is not included in $intA$.*

PROOF. It is clear that (a) implies (b).

(b) *implies* (c)
Suppose that there is $\lambda \in hull(\mathcal{P}_\pm(T)) \cap A$. Without loss of generality we can assume that $\lambda \in hull(\mathcal{P}_-(T))$ (if not, consider the adjoint). By Proposition 3.1, there is $\varepsilon > 0$ such that $\lambda \in hull(\mathcal{P}_-(S))$ for every $||T - S|| < \varepsilon$. As $T \in cl\mathcal{W}_A(\mathcal{H})$, there is such an S in $\mathcal{W}_A(\mathcal{H})$. Let Γ be a simple Jordan loop such that $\Gamma \subset \mathcal{P}_-(S)$ and $\lambda \in ins\Gamma$.

Since $\lambda \in A$, there is a net of polynomials, p_α, such that $p_\alpha(S)$ converges in the strong operator topology to $(\lambda - S)^{-1}$. By Theorem 3.4, there is a subsequence p_{α_n} such $p_{\alpha_n}(z)$ converges to $(\lambda - z)^{-1}$, uniformly on the compact subsets of $\mathcal{P}_-(S)$. In particular, $p_{\alpha_n}(z)$ converges to $(\lambda - z)^{-1}$ uniformly of Γ, which is a contradiction, because $\lambda \in ins\Gamma$. Thus, $hull(\mathcal{P}_-(T)) \cap A = \emptyset$.

Suppose now that there is C, a component of $\sigma_{lre}(T) \cup \sigma_{p0}(T)$ such that $C \subset intA$.

If C is a component of $\sigma(T)$, then the upper semi-continuity of the spectrum implies that there is $\varepsilon > 0$, with the property that $\sigma(S) \cap intA \neq \emptyset$ for every $||T - S|| < \varepsilon$. Since $T \in cl\mathcal{W}_A(\mathcal{H})$, there is such an S in $\mathcal{W}_A(\mathcal{H})$. Therefore $\sigma(S) \cap intA \neq \emptyset$ for some $S \in \mathcal{W}_A(\mathcal{H})$. This is a contradiction, because $A \subset \rho(S)$.

If C is a component of the essential spectrum then a similar argument, using the upper semi-continuity of the essential spectrum, leads to a contradiction.

If C is not a component of $\sigma(T)$ and is not a component of $\sigma_e(T)$, then C is a component of $\sigma_{lre}(T)$, which is not a component of $\sigma_e(T)$. Thus, some part of the boundary of C is included in the boundary of $\mathcal{P}_{\pm\infty}(T)$. Hence, $intA \cap \mathcal{P}_{\pm\infty}(T) \neq \emptyset$. Since this implies that $hull(\mathcal{P}_\pm(T)) \cap A \neq \emptyset$, we get, again, a contradiction.

Therefore, if C is a component of $\sigma_{lre}(T) \cup \sigma_{p0}(T)$, then C is not included in $intA$.

(c) *implies* (a)
Let T be an operator satisfying the conditions in (c). If C is a component of $\sigma_{lre}(T) \cup \sigma_{p0}(T)$, then there is a point $z_C \in C$ such that either $z_C \notin A$ or z_C is a limit point of $\mathbf{C} \setminus A$.

Let $\varepsilon > 0$. By [3, proposition 2.1 and theorem 2.3], there is an operator S such that $||T - S|| < \frac{\varepsilon}{2}$ and $\sigma(S) = \sigma_0 \cup \sigma_1$, where $\sigma_0 \subset \mathcal{P}_\pm(T)$ and σ_1 is a finite subset of $\{z_C\}_C$. Let $\sigma_1 = \{z_1, \ldots, z_k\}$. There is an invertible operator R such that $RSR^{-1} = B_0 \oplus B_1 \oplus \ldots B_k$, where $\sigma(B_k) = \sigma_0$ and $\sigma(B_j) = \{z_j\}$ for every $1 \leq j \leq k$.

If $z_j \in hull(\mathcal{P}_\pm(T))$ or $z_j \notin clhull(\mathcal{P}_\pm(T)) \cup A$, let $w_j = z_j$. If z_j is in the boundary of $hull(\mathcal{P}_\pm(T))$, let $w_j \in hull(\mathcal{P}_\pm(T))$ such that $|z_j - w_j| < \frac{\varepsilon}{2||R||\,||R^{-1}||}$. If $z_j \notin clhull(\mathcal{P}_\pm(T))$ and $z_j \in A$, then, since z_j is a limit point of $\mathbf{C} \setminus A$, we can find $w_j \notin clhull(\mathcal{P}_\pm(T)) \cup A$ such that $|z_j - w_j| < \frac{\varepsilon}{2||R||\,||R^{-1}||}$.

Let $F = B_0 \oplus (B_1 + (w_1 - z_1)I) \oplus \ldots \oplus (B_k + (w_k - z_k)I)$. We have that $||RSR^{-1} - F|| < \frac{\varepsilon}{2||R||\,||R^{-1}||}$. Then,

$$||T - R^{-1}FR|| \leq ||T - S|| + ||S - R^{-1}FR|| \leq \tfrac{\varepsilon}{2} + ||R^{-1}RSR^{-1}R - R^{-1}FR|| \\ \leq \tfrac{\varepsilon}{2} + ||R||\,||R^{-1}||\,||RSR^{-1} - F|| < \varepsilon.$$

Thus, in order to show that $T \in cl\mathcal{U}_A(\mathcal{H})$, it will be enough to prove that $R^{-1}FR \in \mathcal{U}_A(\mathcal{H})$. Moreover, since $\mathcal{U}_A(\mathcal{H})$ is invariant under similarities, it suffices to show that $F \in \mathcal{U}_A(\mathcal{H})$. It is clear that $A \subset \rho(F)$.

Let $\lambda \in A$. If $w_j \in hull(\mathcal{P}_\pm(T))$, let r_j such that $B_{r_j}(w_j) \subset hull(\mathcal{P}_\pm(T))$, where $B_{r_j}(w_j)$ denotes the open ball of radius r_j centered in w_j. If $w_j \notin clhull(\mathcal{P}_\pm(T))$, let $r_j < \min\{dist(w_j, clhull(\mathcal{P}_\pm(T))), |w_j - \lambda|\}$. We have that $\sigma(F) \subset hull(\mathcal{P}_\pm(T)) \cup \cup_{j=1}^{k} B_{r_j}(w_j)$, which is a finite union of simple, connected sets and does not contain λ. By a corollary of Runge's Theorem (see [4, corollary 1.15, p. 200]), there is a sequence of polynomials, p_n, such that $p_n(z)$ converges to $(\lambda - z)^{-1}$ uniformly on the compact subsets of $hull(\mathcal{P}_\pm(T)) \cup (\cup_{j=1}^{k} B_{r_j}(w_j))$. The continuity of the analytic functional calculus implies that $p_n(F)$ converges uniformly to $(\lambda - F)^{-1}$. Therefore $F \in \mathcal{U}_A(\mathcal{H})$. ■

Corollary 4.2. $cl\mathcal{U}_A(\mathcal{H}) = cl\mathcal{W}_A^*(\mathcal{H}) = cl\mathcal{W}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : hull(\mathcal{P}_\pm(T)) \cap A = \emptyset$ and if C is a component of $\sigma_{lre}(T) \cup \sigma_{p0}(T)$, then C is not included in $intA\}$.

Next we will characterize the interior.

Theorem 4.3. *For an operator T, the following are equivalent:*
(a) $T \in int\mathcal{U}_A(\mathcal{H})$,
(b) $T \in int\mathcal{W}_A(\mathcal{H})$,
(c) $\widehat{\sigma(T)} \cap clA = \emptyset$.

Proof. It is clear that (a) implies (b).

(b) *implies* (c)

Suppose that there is T in $int\mathcal{W}_A(\mathcal{H})$ such that $\widehat{\sigma(T)} \cap clA \neq \emptyset$. Since $\sigma(T) \cap A = \emptyset$, either there is a sequence $\{\lambda_n\}$ in A convergent to some point in the boundary of $\sigma(T)$, or there is $\lambda \in A$ such that $\lambda \in \widehat{\sigma(T)} \setminus \sigma(T)$.

In the first case, by [3, proposition 2.1], for every $\varepsilon > 0$, there is an operator S_ε such that $||T - S_\varepsilon|| < \varepsilon$ and $\sigma(S_\varepsilon) \supset \{z : dist(z, \sigma(T)) < \delta_\varepsilon\}$ for some $\delta_\varepsilon > 0$. This implies that there is N such that $\lambda_n \in \sigma(S_\varepsilon)$, and so $S_\varepsilon \notin \mathcal{W}_A(\mathcal{H})$. Therefore, $T \notin int\mathcal{W}_A(\mathcal{H})$.

To deal with the second case we will use a technique we learned from [8]. Let $\varepsilon > 0$ such that $||T - S|| < \varepsilon$ implies that S belongs to $int\mathcal{W}_A(\mathcal{H})$. By [3, proposition 2.1], there is an operator S with the following properties: $||T - S|| < \varepsilon$, $\sigma(S)$ is a perfect set, $\sigma_{lre}(S)$ is the closure of a bounded open set whose boundary consists of finitely many pairwise disjoint smooth simple Jordan loops, $\sigma(T) \subset \sigma(S)$, $\sigma_{p0}(S)$ is a finite subset of $\sigma_{p0}(T)$, and $\rho_{SF}(S)$ has a finite number of components. Hence, $int\mathcal{W}_A(\mathcal{H})$ contains an operator with all these properties.

Let $D_1, \dots, D_k$ be the components of $\rho_{SF}(S)$ where the index is positive, let $D_{k+1}, \dots, D_r$ be the components with negative index, and let n_j be the index on D_j for $1 \le j \le r$. For $1 \le j \le k$, let B_j be the n_jth inflation of the adjoint of the Bergman operator on D_j. For $k+1 \le j \le r$, let B_j be the n_jth inflation of the Bergman operator on D_j. Let B_0 be the infinite inflation of the Bergman operator on $int\sigma_{lre}(S)$ direct sum with its adjoint. Let B_{r+1} be the Riesz summand of S corresponding to $\sigma_{p0}(S)$. Finally, let $B = \oplus_{j=0}^{r+1} B_j$.

By the Similarity Orbit Theorem (reduced form, see [2, theorem 9.1, p. 5]), S is in the closure of the similarity orbit of B. Let $\delta > 0$ such that $||S - R|| < \delta$ implies that r belongs to $\mathcal{W}_A(\mathcal{H})$. Thus, there is an operator similar to B in $\mathcal{W}_A(\mathcal{H})$. Therefore B itself is in $\mathcal{W}_A$ of the corresponding space. We have that

$$\lambda \in \widehat{\sigma(T)} \subset \widehat{\sigma(S)} = \widehat{\sigma(B)}.$$

It is not very difficult to see that there is an invariant subspace of B, $\mathcal{M}$, such that $\sigma(B_{\mathcal{M}}) = \widehat{\sigma(B)}$, where $B_{\mathcal{M}}$ denotes the restriction of B to $\mathcal{M}$. This is a contradiction, since $B_{\mathcal{M}} \in \mathcal{W}_A(\mathcal{M})$ and so $A \subset \rho(B_{\mathcal{M}})$.

(c) *implies* (a)

Let T be an operator such that $\widehat{\sigma(T)} \cap clA = \emptyset$, and let V be an open set with simply connected components such that $\widehat{\sigma(T)} \subset V$ and $V \cap clA = \emptyset$. The upper semi-continuity of the spectrum implies that there is an $\varepsilon > 0$ such that whenever $||T - S|| < \varepsilon$, we have $\sigma(S) \subset V$. Since V has simply connected components, in fact $\widehat{\sigma(S)} \subset V$. It is clear now that every such S is in $\mathcal{U}_A(\mathcal{H})$. Therefore T is in $int\mathcal{U}_A(\mathcal{H})$. ■

Corollary 4.4. *$int\mathcal{U}_A(\mathcal{H}) = int\mathcal{W}_A^*(\mathcal{H}) = int\mathcal{W}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \widehat{\sigma(T)} \cap clA = \emptyset\}$.*

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