SMOOTH UNIVALENT UNIVERSAL FUNCTIONS

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Abstract

Let Ω be a Jordan region in $\mathbb{C}$ and let $f \in H(\Omega)$ be a universal function (in various senses). In this paper, we prove that it is possible for f to have simultaneously the following properties: (i) f is univalent, (ii) each derivative $f^{(\ell)}$ $(\ell \geq 0)$ extends continuously on the closure of Ω.

1. Introduction

Let $O \subset \mathbb{C}$ be an open set in the complex plane. We denote by $H(O)$ the space of all functions that are holomorphic in O endowed with the topology of uniform convergence on compacta. We are interested in functions $f \in H(O)$, which somehow perform approximations. Depending on the way these approximations are performed, we introduce and study different classes of such functions, which we call universal functions. Some of these classes contain only 'bad' functions and some other classes may contain functions with good properties, e.g. functions that are smooth on the boundary or univalent functions.

To be more precise we give two examples. However, some standard notation is first needed. $\mathbb{N} = \{1, 2, ...\}$ is the set of natural numbers, while $\mathbb{Q}$ is the set of rational numbers. If $A \subset \mathbb{C}$ then we denote by $\partial A, \overline{A}$ and A^o its boundary, its closure and its interior in $\mathbb{C}$, respectively. The open disk of radius $r > 0$ and center at $\alpha \in \mathbb{C}$ is denoted, as usual, by $D(\alpha, r)$; in particular, D will stand for the open unit disk $D(0, 1)$. Finally, if f is a function holomorphic in a neighborhood of a point ζ, then we denote by $S_N(f, \zeta) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}(z - \zeta)^n$ the value at z of the Nth partial sum of the Taylor development of f around ζ. A function $f \in H(D)$ is called a universal Taylor series if, for every compact set $K \subset D^c$ with K^c connected and

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for every function $g : K \longrightarrow \mathbb{C}$ continuous on K and holomorphic in K^o, there exists a subsequence $(S_{\lambda_n}(f,0))$ of $(S_n(f,0))$ that converges to g uniformly on K. The class $U(D,0)$, which contains all universal Taylor series, is G_δ and dense in $H(D)$ (see [15]). Moreover, the class $U(D,0)$ is disjoint from the class of Nevanlinna (see [14]). Thus, the class $U(D,0)$ contains neither univalent functions nor functions that are smooth on the boundary. This universality is related to the phenomenon of overconvergence.

Now if we consider the larger class $U_1(D,0)$ of functions $f \in H(D)$, which make the same approximations as before but on compact sets $K \subset D^c$, with K^c connected such that $K \cap \partial D = \emptyset$, then we obtain a class (see [1; 8]) that contains univalent functions (see [17]) and functions that are smooth on the boundary (see [12]).

The two classes mentioned above, $U(D,0)$, $U_1(D,0)$, share some similarities and it was not known whether they were really different classes. The results we have previously described showed that these classes do not coincide (see [12; 13; 14]). In fact, they have essential differences as one can see from the results of the present paper also.

Schneider, see [16], constructed univalent functions that are universal in several senses simultaneously. But he did not deal with the possibility of having these functions smooth up to the boundary. In this paper, we will study certain classes of universal functions and we will prove that they contain functions that are simultaneously smooth on the boundary and univalent.

More specifically, in Section 2 we will prove that there exist smooth univalent functions $f \in H(D)$ with the following property: for every compact set $L \subset D$ with L^c connected, and for every function $g : L \longrightarrow \mathbb{C}$ continuous on L and holomorphic in L^o, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that $f^{(\lambda_n)} \xrightarrow{n\to+\infty} g$ uniformly on L. The functions that have the aforementioned property are called universal under derivatives (since its derivatives make the approximations), and we denote by $U_{der}(D)$ the class of all these functions (see [2; 4; 5; 11]). Our result is in fact more general, since we work on any Jordan domain Ω (and not only on D) and since we obtain a residuality result (see Theorem 2.9). At this point, we would like to mention that the class $U_{der}(D)$ of universal functions under derivatives can be interpreted in terms of the notion of hypercyclicity, which has been an interest of operator theorists in recent years. Let us recall this well known notion: if X is a topological vector space, a linear operator $T : X \to X$ is called hypercyclic if there is a vector $x \in X$ so that its orbit $\{x, Tx, ...\}$ is dense in X. Furthermore, the vector x is called the hypercyclic vector for T. For an excellent introduction to this subject, we refer the reader to the survey article [5]. Therefore, the universal functions belonging to the class $U_{der}(D)$ can be viewed as hypercyclic vectors for the differentiation operator.

In Section 3, we prove analogous results for a class of universal functions with respect to overconvergence denoted by $\widetilde{U}_1(\Omega)$ (see [12]), where Ω is again a Jordan domain. The class $\widetilde{U}_1(\Omega)$ contains all functions $f \in H(\Omega)$ such that for every compact set $K \subset \overline{\Omega}^c$ with K^c connected, and for every function $g : K \longrightarrow \mathbb{C}$ continuous on K and holomorphic in K^o there exists a sequence

$\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that:

$$\sup_{\zeta\in\Omega}\sup_{z\in K}|S_{\lambda_n}(f,\zeta)(z)-g(z)|\longrightarrow 0 \quad \text{as} \quad n\to+\infty.$$

Our results in Sections 2 and 3 strengthen partially the main result of the first and third authors in a recent paper [3, theorem 2.8], where the functions under consideration were just *continuous*—not smooth—on the boundary (the word 'partially' has been used because in [3] the approximation is made with the *same* approximating sequence for both kinds of universality).

Finally, in Section 4 we study another class of universal functions with respect to overconvergence denoted by $U_1(\mathbb{C}\setminus\overline{\Omega}\,,\,\zeta)$ (see [1; 2; 8; 9; 10]), which contains all functions holomorphic in $\mathbb{C}\setminus\overline{\Omega}$ (Ω always denotes a Jordan domain and now $\zeta\in\mathbb{C}\setminus\overline{\Omega}$), such that for every compact set $K\subset\Omega$ with K^c connected and for every function $g:K\longrightarrow\mathbb{C}$ continuous on K and holomorphic in K^o, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that:

$$\sup_{z\in K}|S_{\lambda_n}(f,\zeta)(z)-g(z)|\xrightarrow{n\to+\infty}0.$$

We prove that this class contains functions univalent and smooth on $(\mathbb{C}\setminus\Omega)\cup\{\infty\}$ (see Theorem 4.5).

By the above results, the different universalities can be divided to two kinds. One kind yields universal functions with wild properties, as that of $U(D,0)$ or universal functions under translations and dilations [5]; the second kind contains universal functions with nicer properties, as, for example, univalence and smoothness up to the boundary. Some previous results assured in some cases the existence of such functions, which were univalent or separately smooth. Our results guarantee the existence of such universal functions that are simultaneously univalent and smooth. This stronger result follows by an application of Baire's category theorem to conveniently chosen function spaces without use of any heavy complex analytic argument. For the role of Baire's theorem in various branches of analysis, we refer readers to [5; 6; 7].

2. Universal functions under derivatives

Throughout this section Ω will be a Jordan domain, that is, the bounded component of the complement of a Jordan curve; a Jordan curve is a topological image in $\mathbb{C}$ of the unit circle.

Definition 2.1. *A function $f\in H(\Omega)$ belongs to the class $U_{der}(\Omega)$ if, for every compact set $L\subset\Omega$ with L^c connected and for every function $g:L\longrightarrow\mathbb{C}$ continuous on L and holomorphic in L^o, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that:*

$$\sup_{z\in L}|f^{(\lambda_n)}(z)-g(z)|\xrightarrow{n\to+\infty}0.$$

For the above definition, we refer readers to [11] (see also [2; 4; 5]), and the class $U_{der}(\Omega)$ was proved to be G_δ and dense in $H(\Omega)$.

We denote by $A^\infty(\Omega)$ the space of all functions f that are holomorphic in Ω, such that $f^{(\ell)}$, $\ell = 0, 1, 2, \ldots$ is continuous on $\overline{\Omega}$. We consider for this space the topology induced by the seminorms

$$\|f\|_{\ \ell} = \sup_{z \in \overline{\Omega}} |f^{(\ell)}(z)| \ , \quad (\ell = 0, 1, \ldots).$$

Moreover, we say that a function $f \in A^\infty(\Omega)$ belongs to the space $X^\infty(\Omega)$ if, for every $\varepsilon > 0$ and for every $N \in \mathbb{N}$, there exists a polynomial p such that

$$\sup_{z \in \overline{\Omega}} |f^{(\ell)}(z) - p^{(\ell)}(z)| < \varepsilon \quad (\ell = 0, 1, \ldots, N).$$

Equivalently, $X^\infty(\Omega)$ is the closure of the set of polynomials in $A^\infty(\Omega)$ endowed with its natural topology. $X^\infty(\Omega)$ is a closed subspace of $A^\infty(\Omega)$, thus endowed with the relative topology it is a Frechét space and Baire's theorem applies. Under adequate conditions—for instance if $\partial\Omega$ is rectifiable (see [12])—one obtains that $X^\infty(\Omega) = A^\infty(\Omega)$ (so this holds for $\Omega = D$). It seems to be unknown whether the last equality holds for a general Jordan domain.
For a fixed p positive natural number by $A^p(\Omega)$, $p > 0$, we denote the space of all functions $f \in H(\Omega)$ such that $f^{(\ell)}$, $\ell = 0, 1, \ldots, p$ extends continuously on $\overline{\Omega}$ endowed with the topology induced by the seminorms:

$$\sup_{z \in \overline{\Omega}} |f^{(\ell)}(z)| \ , \ \ell = 0, 1, \ldots, p.$$

Again, we consider a space $X^p(\Omega)$ that contains all functions $f \in A^p(\Omega)$ that can be approximated by polynomials (the topology is again the relative topology from the space $A^p(\Omega)$).

Definition 2.2. *A holomorphic function $f \in A^\infty(\Omega)$ belongs to the class $U^\infty_{der}(\Omega)$ if for every function $\varphi \in X^\infty(\Omega)$ there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that for every $\ell = 0, 1, 2, \ldots$ the following holds:*

$$\sup_{z \in \overline{\Omega}} |f^{(\lambda_n + \ell)}(z) - \varphi^{(\ell)}(z)| \xrightarrow{n \to +\infty} 0.$$

In other words, $f \in U^\infty_{der}(\Omega)$ whenever the closure in $A^\infty(\Omega)$ of its sequence of derivatives contains $X^\infty(\Omega)$, or equivalently, contains the polynomials.

Proposition 2.3. *Let Ω be a Jordan domain and let $f \in U^\infty_{der}(\Omega)$. Then we have the following:*
(i) *For every compact set $L \subset \overline{\Omega}$ with L^c connected and for every function $g : L \longrightarrow \mathbb{C}$ continuous on L and holomorphic in L^o, there exists a sequence*

$\{\lambda_n\}_{n\in\mathbb{N}}$ *of natural numbers such that:*

$$\sup_{z\in L}|f^{(\lambda_n)}(z)-g(z)|\to 0 \quad as\ n\to+\infty.$$

In particular, this holds for all compact sets $L\subset\Omega$ *with connected complement. It follows that* $U_{der}^{\infty}(\Omega)\subset U_{der}(\Omega)$.
(ii) *For every simply connected open set* $S\subset\Omega$, *the set* $\{f^{(n)}\ ;\ n=0,1,2,\dots\}$ *is dense in* $H(S)$ *with the topology of uniform convergence on compacta.*
(iii) *For every open set* $S\subset\Omega$ *such that* $\overline{S}^c$ *is connected, the set* $\{f^{(n)}\ ;\ n=0,1,2,\dots\}$ *is dense in* $A(S)$, *the space of holomorphic functions in* S *and continuous on* S *with the topology of uniform convergence on* $\overline{S}$.
(iv) *Let* $S\subset\Omega$ *be an open set and let* $p\in\mathbb{N}$. *Then the set* $\{f^{(n)}\ ;\ n=0,1,2,\dots\}$ *is dense in* $X^p(S)$ *with the topology of the latter space.*

PROOF. (i) Let $f\in U_{der}^{\infty}(\Omega)$. In addition, let $L\subset\overline{\Omega}$ be a compact set with connected complement. We consider a continuous function $g:L\longrightarrow\mathbb{C}$ that is holomorphic in L^o. Applying Mergelyan's theorem (see [16]) we can find a sequence of polynomials p_n such that:

$$\sup_{z\in L}|g(z)-p_n(z)|\xrightarrow{n\to+\infty}0. \tag{1}$$

Since $p_n\in X^{\infty}(\Omega)$ and $f\in U_{der}^{\infty}(\Omega)$, for every $n\in\mathbb{N}$ we can find a natural number λ_n such that:

$$\sup_{z\in\overline{\Omega}}|f^{(\lambda_n)}(z)-p_n(z)|<\frac{1}{n}. \tag{2}$$

Relations (1) and (2) imply that

$$\sup_{z\in L}|f^{(\lambda_n)}(z)-g(z)|\xrightarrow{n\to+\infty}0,$$

as required. The proofs of (ii), (iii) and (iv) are similar and they are omitted. ■

Our aim is to prove that the class $U_{der}^{\infty}(\Omega)$ is non-empty and that it contains univalent functions. This will follow from the result of Theorem 2.9 below. To begin with, we need the following simple lemmas.

Lemma 2.4. *Let* φ *be a polynomial. Then there exists a sequence of polynomials* $\{p_n\}_{n\in\mathbb{N}}$ *such that:* $p_n^{(n)}(z)=\varphi(z)$ *and* $p_n\to 0$ *compactly on* $\mathbb{C}$.

Lemma 2.4 is well known and its simple proof is omitted; see [4].

Lemma 2.5. *Let* Ω *be a Jordan region and let* $f\in H(\Omega)$ *be a univalent function. In addition, let* $f_n\in H(\Omega)$ $n=0,1,2,\dots$ *be a sequence of functions such that*

$f_n \longrightarrow f$ compactly on Ω. Then, for every compact set $L \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, f_n is one to one on L.

The proof of Lemma 2.5 is a well known application of Rouche's Theorem and is omitted. In fact, this lemma holds for every open set Ω.

We now consider one last space, which we denote by $X_B^\infty(\Omega)$ and which contains all functions $f \in A^\infty(\Omega)$ that in the topology of $A^\infty(\Omega)$ can be approximated by polynomials that are univalent on a neighborhood of $\overline{\Omega}$ depending on f. In other words, $X_B^\infty(\Omega)$ is the closure in $A^\infty(\Omega)$ of the polynomials that are univalent on a neighborhood of $\overline{\Omega}$. We notice that such polynomials always exist, as, for example, $p(z) = z$.
For every $j, N, s \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N}$ we set

$$\widetilde{O}(\Omega, f_j, N, s, n) =$$

$$\{f \in A^\infty(\Omega) : \sup_{z\in\overline{\Omega}} |f^{(n+\ell)}(z) - f_j^{(\ell)}(z)| < \frac{1}{s}\ ,\ \ell = 0, 1, \ldots, N\},$$

where $\{f_j : j = 1, 2, \ldots\}$ is an enumeration of the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.

Lemma 2.6. *The following equality holds:*

$$U_{der}^\infty(\Omega) \cap X_B^\infty(\Omega) = \bigcap_{j=1}^\infty \bigcap_{N=1}^\infty \bigcap_{s=1}^\infty \bigcup_{n=0}^\infty \big[\widetilde{O}(\Omega, f_j, N, s, n) \cap X_B^\infty(\Omega)\big].$$

Proof. The inclusion '$\subset$' is obvious. As for the inverse inclusion, let

$$f \in \bigcap_{j=1}^\infty \bigcap_{N=1}^\infty \bigcap_{s=1}^\infty \bigcup_{n=0}^\infty \big[\widetilde{O}(\Omega, f_j, N, s, n) \cap X_B^\infty(\Omega)\big]$$

and in addition let $\varphi \in X^\infty(\Omega)$. Since $X^\infty(\Omega)$ is the closure of the set of polynomials in $A^\infty(\Omega)$ endowed with its natural topology, it follows that there exists a sequence of polynomials p_N, $N = 1, 2, \ldots$ such that

$$\sup_{z\in\overline{\Omega}} |\varphi^{(\ell)}(z) - p_N^{(\ell)}(z)| \xrightarrow{N\to+\infty} 0 \quad \text{for all } \ell = 0, 1, 2, \ldots.$$

Without loss of generality, we may assume that the coefficients of the polynomials p_N belong to the set $\mathbb{Q} + i\mathbb{Q}$. Thus, for every $N \in \mathbb{N}$ there exists $\lambda_N \in \mathbb{N}$ with

$$\sup_{z\in\overline{\Omega}} |f^{(\lambda_N+\ell)}(z) - p_N^{(\ell)}(z)| < \frac{1}{N} \quad \text{for all } \ \ell = 0, 1, \ldots, N.$$

Therefore,

$$\sup_{z\in\overline{\Omega}} |f^{(\lambda_N+\ell)}(z) - \varphi^{(\ell)}(z)| \xrightarrow{N\to+\infty} 0.$$

This implies that $f \in U^{\infty}_{der}(\Omega) \cap X^{\infty}_{B}(\Omega)$. ■

Lemma 2.7. *The set* $\widetilde{O}(\Omega, f_j, N, s, n) \cap X^{\infty}_{B}(\Omega)$ *is open in* $X^{\infty}_{B}(\Omega)$.

PROOF. The assertion follows because the subset $\widetilde{O}(\Omega, f_j, N, s, n)$ is open in $A^{\infty}(\Omega)$, which in turn holds because such a subset is a finite intersection of open subsets in $A^{\infty}(\Omega)$, namely, of the subsets $(S^n)^{-1}(\{f \in A^{\infty}(\Omega) : \|f^{(n)} - f_j\|_{\ell} < 1/s\})$ $(\ell = 0, 1, ..., N)$. Here, S is the derivation mapping $S : f \to f'$, which is clearly a continuous operator on $A^{\infty}(\Omega)$. ■

Lemma 2.8. *The set* $\bigcup\limits_{n=0}^{\infty}\Big[\widetilde{O}(\Omega, f_j, N, s, n) \cap X^{\infty}_{B}(\Omega)\Big]$ *is dense in* $X^{\infty}_{B}(\Omega)$.

PROOF. Let $g \in X^{\infty}_{B}(\Omega)$, $\varepsilon > 0$ and $N_0 \in \mathbb{N}$.
We will find $f \in \bigcup\limits_{n=1}^{\infty}\Big[\widetilde{O}(\Omega, f_j, N, s, n) \cap X^{\infty}_{B}(\Omega)\Big]$, such that

$$\sup_{z\in\overline{\Omega}} |f^{(\ell)}(z) - g^{(\ell)}(z)| < \varepsilon \qquad \text{for all} \quad \ell = 0, 1, \ldots, N_0.$$

Since $g \in X^{\infty}_{B}(\Omega)$, there exists a polynomial p univalent on an open neighborhood G of $\overline{\Omega}$ such that

$$\sup_{z\in\overline{\Omega}} |g^{(\ell)}(z) - p^{(\ell)}(z)| < \frac{\varepsilon}{2} \qquad \text{for all} \quad \ell = 0, 1, \ldots, N_0.$$

Using Lemma 2.4 we can find a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that

$$p_n^{(n)}(z) = f_j(z) \qquad \text{for all} \quad z \in \mathbb{C} \quad \text{and} \quad p_n \to 0 \quad \text{compactly on} \quad \mathbb{C}.$$

Obviously,

$$p + p_n \xrightarrow{n\to+\infty} p \quad \text{compactly on } \mathbb{C}.$$

Hence, if L is a compact set such that $\overline{\Omega} \subset L^o \subset L \subset G$, in view of Lemma 2.5, the polynomial $p + p_n$ is one-to-one on L for n large enough. Thus, there exists $n_0 \in \mathbb{N}$, $n_0 > \text{degree}(p)$ such that

$$\sup_{z\in\overline{\Omega}} |p_{n_0}^{(\ell)}(z)| < \frac{\varepsilon}{2} \qquad \ell = 0, 1, \ldots, N_0$$

and $p + p_{n_0}$ is one-to-one on L.
We set

$$f = p + p_{n_0}.$$

Then $f^{(n_0)}(z) = p^{(n_0)}(z) + p_{n_0}^{(n_0)}(z) = f_j(z)$ for all $z \in \mathbb{C}$ and $f \in X_B^\infty(\Omega)$. Thus, $f \in \left[\widetilde{O}(\Omega, f_j, N, s, n_0) \cap X_B^\infty(\Omega)\right]$. Moreover,

$$\sup_{z\in\overline{\Omega}} |g^{(\ell)}(z) - f^{(\ell)}(z)| < \frac{\varepsilon}{2} + \sup_{z\in\overline{\Omega}} |p_{n_0}^{(\ell)}| < \varepsilon \quad \text{for} \quad \ell = 0, 1, \ldots, N_0,$$

as required. ■

Theorem 2.9. *The class $U_{der}^\infty(\Omega) \cap X_B^\infty(\Omega)$ is G_δ and dense in $X_B^\infty(\Omega)$.*

Proof. Lemmas 2.6, 2.7 and 2.8 imply that the class $U_{der}^\infty(\Omega) \cap X_B^\infty(\Omega)$ is a denumerable intersection of sets that are open and dense in $X_B^\infty(\Omega)$. The result follows from Baire's Theorem. ■

Remark 1. The above result yields that the class $U_{der}^\infty(\Omega)$ is non empty and that it contains functions that are both smooth and univalent. Since $U_{der}^\infty(\Omega) \subset U_{der}(\Omega)$, the same holds for the original class of universal functions under derivatives. Therefore we have obtained, as a consequence, part of [3, theorem 2.8].

Remark 2. With the same method it is also possible to prove that $U_{der}^\infty(\Omega) \cap X^\infty(\Omega)$ is G_δ and dense in $X^\infty(\Omega)$.

3. Universal functions with respect to overconvergence

Throughout this section, Ω will also stand for a Jordan domain.

Definition 3.1. *A function $f \in H(\Omega)$ belongs to the class $\widetilde{U}_1(\Omega)$ if, for every compact set $K \subset \overline{\Omega}^c$ with K^c connected and for every function $g : K \longrightarrow \mathbb{C}$ continuous on K and holomorphic in K^o, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that:*

$$\sup_{\zeta\in\Omega} \sup_{z\in K} |S_{\lambda_n}(f,\zeta)(z) - g(z)| \longrightarrow 0 \quad \textit{as} \quad n \to +\infty.$$

The above definition was introduced in [12], and if Ω is a Jordan domain with rectifiable boundary then $\widetilde{U}_1(\Omega)$ is G_δ and dense in $A^\infty(\Omega)$.

Recall that $X_B^\infty(\Omega)$ denotes the closure in $A^\infty(\Omega)$ of the polynomials that are univalent in a neighborhood of $\overline{\Omega}$. That is, a function $f \in A^\infty(\Omega)$ belongs to $X_B^\infty(\Omega)$, if for every $\varepsilon > 0$ and for every $N \in \mathbb{N}$ there exists a polynomial p that is univalent on a neighborhood of $\overline{\Omega}$ such that:

$$\sup_{z\in\overline{\Omega}} |f^{(\ell)}(z) - p^{(\ell)}(z)| < \varepsilon \ \text{ for } \ \ell = 0, 1, \ldots, N.$$

The set $X_B^\infty(\Omega)$ is a closed subset of $A^\infty(\Omega)$. Thus, endowed with the relative topology from $A^\infty(\Omega)$ it is complete and Baire's Theorem applies.

We fix a sequence $\{K_m\}_{m\in\mathbb{N}}$ of compact sets in $\overline{\Omega}^c$ with connected complement such that for every compact set $K \subset \overline{\Omega}^c$ with connected complement there exists

$m \in \mathbb{N}$ such that $K \subset K_m$ (see [12, lemma 2]). Again, f_j $(j = 1, 2, \dots)$ will stand for an enumeration of the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.
For $m, j, s \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N}$ we consider the set:

$$E(\Omega, K_m, f_j, s, n) = \{f \in H(\Omega) : \sup_{\zeta \in \Omega} \sup_{z \in K_m} |S_n(f, \zeta)(z) - f_j(z)| < \frac{1}{s}\}.$$

Lemma 3.2. *The following equality holds.*

$$\widetilde{U}_1(\Omega) \cap X_B^\infty(\Omega) = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} \big[E(\Omega, K_m, f_j, s, n) \cap X_B^\infty(\Omega)\big].$$

PROOF. See [12, lemma 3]. ■

Lemma 3.3. *The set $E(\Omega, K_m, f_j, s, n) \cap X_B^\infty(\Omega)$ is open in $X_B^\infty(\Omega)$.*

PROOF. Let $f \in E(\Omega, K_m, f_j, s, n) \cap X_B^\infty(\Omega)$. Let $M = \sup_{\zeta \in \Omega} \sup_{z \in K_m} |z - \zeta|$.
We set

$$a = \frac{\frac{1}{s} - \sup_{\zeta \in \Omega} \sup_{z \in K_m} |S_n(f, \zeta)(z) - f_j(z)|}{\sum_{\ell=0}^{n} \frac{M^\ell}{\ell!}}.$$

Then $a > 0$ and the open set

$$V = \{g \in X_B^\infty(\Omega) : \sup_{\zeta \in \Omega} |g^{(\ell)}(\zeta) - f^{(\ell)}(\zeta)| < a \quad \text{for} \quad \ell = 0, 1, \dots, n\}$$

contains f and is contained in $E(\Omega, K_m, f_j, s, n) \cap X_B^\infty(\Omega)$. Indeed, if $g \in V$ then we have

$$\sup_{\zeta \in \Omega} \sup_{z \in K_m} |S_n(g, \zeta)(z) - f_j(z)|$$

$$\leq \sup_{\zeta \in \Omega} \sup_{z \in K_m} |S_n(g, \zeta)(z) - S_n(f, \zeta)(z)| + \sup_{\zeta \in \Omega} \sup_{z \in K_m} |S_n(f, \zeta)(z) - f_j(z)|$$

$$= \sup_{\zeta \in \Omega} \sup_{z \in K_m} \Big|\sum_{\ell=0}^{n} \frac{g^{(\ell)}(\zeta) - f^{(\ell)}(\zeta)}{\ell!} (z - \zeta)^\ell\Big| + \sup_{\zeta \in \Omega} \sup_{z \in K_m} |S_n(f, \zeta)(z) - f_j(z)|$$

$$< a \sum_{\ell=0}^{n} \frac{M^\ell}{\ell!} + \sup_{\zeta \in \Omega} \sup_{z \in K_m} |S_n(f, \zeta)(z) - f_j(z)| = \frac{1}{s}.$$

This completes the proof. ■

Lemma 3.4. *The set $\bigcup_{n=0}^{\infty} \big[E(\Omega, K_m, f_j, s, n) \cap X_B^\infty(\Omega)\big]$ is dense in $X_B^\infty(\Omega)$.*

Proof. Let $g \in X_B^\infty(\Omega)$, $N \in \mathbb{N}$ and let $\varepsilon > 0$. Then there exists a polynomial p, univalent on a neighborhood G of $\overline{\Omega}$ such that:

$$\sup_{z\in\overline{\Omega}} |g^{(\ell)}(z) - p^{(\ell)}(z)| < \frac{\varepsilon}{2} \quad \text{for} \quad \ell = 0, 1, \dots, N. \tag{3}$$

Since $\overline{\Omega} \cap K_m = \emptyset$, there exist two simply connected, open and disjoint sets A, B such that $\overline{\Omega} \subset A$ and $K_m \subset B$ (see for instance [2] and [5]). Without loss of generality we assume that $A \subset G$. Let L be a compact set such that $\overline{\Omega} \subset L^o \subset L \subset A$. Runge's Theorem implies that there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that:

$$p_n(z) \xrightarrow{n\to+\infty} p(z) \qquad \text{compactly on A}$$

and

$$p_n(z) \xrightarrow{n\to+\infty} f_j(z) \qquad \text{compactly on B .}$$

Thus, in view of Weierstrass's Theorem and Lemma 2.5, there exists a polynomial $\tilde{p}$ from the above mentioned sequence of polynomials such that:

- $\tilde{p}$ is one-to-one on L,
- $\sup_{z\in L} |\tilde{p}^{(\ell)}(z) - p^{(\ell)}(z)| < \frac{\varepsilon}{2}$ for $\ell = 0, 1, \dots, N$

and

- $\sup_{z\in K_m} |\tilde{p}(z) - f_j(z)| < \frac{1}{s}$.

We set $f(z) = \tilde{p}(z)$. Then, obviously, $f \in X_B^\infty(\Omega)$ and (3) implies that

$$\sup_{z\in\overline{\Omega}} |g^{(\ell)}(z) - f^{(\ell)}(z)| < \varepsilon \quad \text{for} \quad \ell = 0, 1, \dots, N.$$

Moreover, for $n > \text{degree}(\tilde{p})$ we obtain

$$\sup_{\zeta\in\Omega} \sup_{z\in K_m} |S_n(f,\zeta)(z) - f_j(z)| = \sup_{z\in K_m} |f(z) - f_j(z)| < \frac{1}{s}.$$

This completes the proof. ∎

Theorem 3.5. *The set $\widetilde{U}_1(\Omega) \cap X_B^\infty(\Omega)$ is G_δ and dense in $X_B^\infty(\Omega)$.*

Proof. Lemmas 3.2, 3.3 and 3.4 show that the set $\widetilde{U}_1(\Omega) \cap X_B^\infty(\Omega)$ is a denumerable intersection of sets that are open and dense in $X_B^\infty(\Omega)$. In view of Baire's Theorem the result follows. ∎

Remark 3. Theorem 3.5 implies the existence of smooth and univalent elements of $\widetilde{U}_1(\Omega)$. This result is generic in the space $X_B^\infty(\Omega)$.
Let $\mathcal{F} = \{f \in A^\infty(\Omega) : f \text{ is constant or } f \text{ is univalent in } \Omega\}$. Then $X_B^\infty(\Omega) \subset \mathcal{F}$ by Hurwitz's theorem. Note that a constant function cannot be universal. In the particular case where Ω is the open unit disk, we have equality: $X_B^\infty(\Omega) = \mathcal{F}$ (it has been used that each constant $\alpha \in \mathbb{C}$ is the limit of the sequence $(\alpha + (z/n))$,

which consists of functions that are univalent in D). In the general case we do not know whether the equality holds.

4. Universal functions on doubly connected domains

Let $\Omega \subset \mathbb{C}$ be a Jordan domain and let $\zeta \in \overline{\Omega}^c$. We denote by $A_0^\infty(\mathbb{C} \setminus \overline{\Omega})$ the space of all functions $f \in H(\mathbb{C} \setminus \overline{\Omega})$ that satisfy the following:
(i) $f^{(\ell)}$ extends continuously on $\partial\Omega$ $\ell = 0, 1, \ldots$.
(ii) $\lim_{z\to\infty} f(z) \in \mathbb{C}$.
For this space we consider the topology induced by the seminorms

$$\sup_{z\in\mathbb{C}\setminus\Omega} |f^{(\ell)}(z)| \quad \text{for} \quad \ell = 0, 1, \ldots.$$

Definition 4.1. *A function $f \in H(\mathbb{C} \setminus \overline{\Omega})$ belongs to the class $U_1(\mathbb{C} \setminus \overline{\Omega},\ \zeta)$ if, for every compact set $K \subset \Omega$ with K^c connected and for every function $g : K \longrightarrow \mathbb{C}$ continuous on K and holomorphic in K^o, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that:*

$$\sup_{z\in K} |S_{\lambda_n}(f,\zeta)(z) - g(z)| \longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

The above class of functions was proved to be G_δ and dense in $H(\mathbb{C} \setminus \overline{\Omega})$ (see [2]). We denote by Y the set

$$Y = \{q \in A_0^\infty(\mathbb{C} \setminus \overline{\Omega}) : q \ \text{ rational function with at most one pole in } \Omega,\\ q \text{ univalent on an open neighboorhood of } \mathbb{C} \setminus \Omega\}.$$

And we set

$$Y^\infty(\mathbb{C} \setminus \overline{\Omega}) = \overline{Y}.$$

The closure of the set above is with respect to the topology of $A_0^\infty(\mathbb{C} \setminus \overline{\Omega})$. Thus, $Y^\infty(\mathbb{C}\setminus\overline{\Omega})$ is a closed subset of $A_0^\infty(\mathbb{C}\setminus\overline{\Omega})$. Therefore $Y^\infty(\mathbb{C}\setminus\overline{\Omega})$ endowed with the topology of $A_0^\infty(\mathbb{C}\setminus\overline{\Omega})$ is a complete space and Baire's Theorem is at our disposal.

First of all, let us fix a sequence $\{K_m\}_{m\in\mathbb{N}}$ of compact sets in Ω with connected complement, such that for every compact set $K \subset \Omega$ with connected complement there exists $m \in \mathbb{N}$ such that $K \subset K_m$, see [16, theorem 13.3, p. 267] and, as before, let f_j , $j = 1, 2, \ldots$ be an enumeration of the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. For $m, j, s \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N}$ we consider the set:

$$E(\mathbb{C} \setminus \overline{\Omega}, K_m, f_j, s, n) = \{f \in H(\mathbb{C} \setminus \overline{\Omega}) : \sup_{z\in K_m} |S_n(f,\zeta)(z) - f_j(z)| < \frac{1}{s}\}.$$

Lemma 4.2. *For $m, j, s \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N}$ the following equality holds:*

$$U_1(\mathbb{C} \setminus \overline{\Omega}, \zeta) \cap Y^\infty(\mathbb{C} \setminus \overline{\Omega}) = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} \big[E(\mathbb{C} \setminus \overline{\Omega}, K_m, f_j, s, n) \cap Y^\infty(\mathbb{C} \setminus \overline{\Omega})\big].$$

Proof. See [2, proposition 3.6]. ■

Lemma 4.3. *The set* $E(\mathbb{C}\setminus\overline{\Omega}, K_m, f_j, s, n)\cap Y^\infty(\mathbb{C}\setminus\overline{\Omega})$ *is open in* $Y^\infty(\mathbb{C}\setminus\overline{\Omega})$.

Proof. Let $g\in E(\mathbb{C}\setminus\overline{\Omega}, K_m, f_j, s, n)\cap Y^\infty(\mathbb{C}\setminus\overline{\Omega})$. Let $M=\sup\limits_{z\in K_m}|z-\zeta|$ and let $a=\frac{1}{s}-\sup_{z\in K_m}|S_n(g,\zeta)(z)-f_j(z)|$. We set

$$V=\{f\in Y^\infty(\mathbb{C}\setminus\overline{\Omega}):|f^{(\ell)}(\zeta)-g^{(\ell)}(\zeta)|<\frac{\ell!}{M^\ell(n+1)}a\ ,\ \ell=0,1,\ldots,n\}.$$

Then V is an open neighborhood of g, which is contained in $E(\mathbb{C}\setminus\overline{\Omega}, K_m, f_j, s, n)\cap Y^\infty(\mathbb{C}\setminus\overline{\Omega})$. Indeed, for $f\in V$ and with the use of Cauchy estimates we have the following:

$$\sup_{z\in K_m}|S_n(f,\zeta)(z)-f_j(z)|\leq\sup_{z\in K_m}|S_n(f,\zeta)(z)-S_n(g,\zeta)(z)|$$

$$+\sup_{z\in K_m}|S_n(g,\zeta)(z)-f_j(z)|\leq a+\sup_{z\in K_m}|S_n(g,\zeta)(z)-f_j(z)|\frac{1}{s}.$$

■

Lemma 4.4. *For every* $m,j,s\in\mathbb{N}\setminus\{0\}$ *the set* $\bigcup\limits_{n=0}^{\infty}\big[E(\mathbb{C}\setminus\overline{\Omega}, K_m, f_j, s, n)\cap Y^\infty(\mathbb{C}\setminus\overline{\Omega})\big]$ *is dense in* $Y^\infty(\mathbb{C}\setminus\overline{\Omega})$.

Proof. Let $g\in Y^\infty(\mathbb{C}\setminus\overline{\Omega})$, $N\in\mathbb{N}$ and let $\varepsilon>0$. According to the definition of $Y^\infty(\mathbb{C}\setminus\overline{\Omega})$, there exists a rational function $q\in A_0^\infty(\mathbb{C}\setminus\overline{\Omega})$ univalent on an open neighborhood of $\mathbb{C}\setminus\Omega$ with, at most, one pole in Ω such that:

$$\sup_{z\in\Omega^c}|g^{(\ell)}(z)-q^{(\ell)}(z)|<\frac{\varepsilon}{2}\quad\text{for}\quad\ell=0,1,\ldots,N.\tag{4}$$

Let $a\in\Omega$ such that $|\zeta-a|>\sup\limits_{z\in K_m}|\zeta-z|$. Then there exists a Jordan curve γ in Ω such that K_m, a and the possible pole of q is in its interior. Let F be the unbounded component of $\mathbb{C}\setminus\gamma$. We consider the mapping $\varphi:\mathbb{C}\cup\{\infty\}\longrightarrow\mathbb{C}\cup\{\infty\}$ given by $\varphi(z)=\frac{1}{z-a}$. We denote by G the image of $\mathbb{C}\setminus\overline{\Omega}$ under the above mapping. We also denote by $\widetilde{F}$ the image of F and by $\widetilde{K}_m$ the image of K_m. Finally, we consider a simply connected, open set $\widetilde{E}\supset\widetilde{K}_m$, such that $\widetilde{E}\cap\widetilde{F}=\emptyset$. Runge's Theorem implies that there exists a sequence of polynomials $\{p_\sigma\}_{\sigma\in\mathbb{N}}$ such that:

$$p_\sigma(w)\xrightarrow{\sigma\to+\infty}f_j(\frac{1}{w}+a)\quad\text{compactly on}\quad\widetilde{E},\tag{5}$$

$$p_\sigma(w)\xrightarrow{\sigma\to+\infty}q(\frac{1}{w}+a)\quad\text{compactly on}\quad\widetilde{F}.\tag{6}$$

We note that the function $q(\frac{1}{w}+a)$ is univalent on an open neighborhood of $\overline{G}$ and, in view of Lemma 2.5, the same holds eventually for all p_σ. Relation (5) implies that there exists $\sigma_1 \in \mathbb{N}$ such that:

$$\sup_{w\in \widetilde{K}_m} |p_\sigma(w) - f_j(\frac{1}{w}+a)| < \frac{1}{2s}\ , \qquad \sigma > \sigma_1.$$

That is,

$$\sup_{z\in K_m} |p_\sigma(\frac{1}{z-a}) - f_j(z)| < \frac{1}{2s}\ , \qquad \sigma > \sigma_1. \tag{7}$$

Relation (6) implies that

$$p_\sigma(\frac{1}{z-a}) \xrightarrow{\sigma\to+\infty} q(z) \qquad \text{compactly on } \ F.$$

Thus, there exists (in view also of Weierstrass's Theorem) $\sigma_2 \in \mathbb{N}$ such that

$$\sup_{z\in \mathbb{C}\setminus\Omega} |\big[p_\sigma(\frac{1}{z-a})\big]^{(\ell)} - q^{(\ell)}(z)| < \frac{\varepsilon}{2} \quad \text{for} \quad \ell = 0,1,\ldots,N \quad \text{and} \quad \sigma > \sigma_2, \tag{8}$$

and $p_\sigma(\dfrac{1}{z-a})$ $(\sigma > \sigma_2)$ is univalent on an open neighborhood of $\mathbb{C}\setminus\Omega$ (see relation (6), and note that the function $\dfrac{1}{z-a}$ is univalent on $\mathbb{C}\setminus\{a\}$). Let $\sigma_0 = \max\{\sigma_1,\sigma_2\}$. We set $f(z) = p_{\sigma_0}(\dfrac{1}{z-a})$. Then, obviously, $f \in Y^\infty(\mathbb{C}\setminus\overline{\Omega})$. Relations (4) and (8) imply that

$$\sup_{z\in \mathbb{C}\setminus\Omega} |g^{(\ell)}(z) - f^{(\ell)}(z)| < \varepsilon \quad \text{for} \quad \ell = 0,1,\ldots,N.$$

Moreover, f is a rational function with at most one pole at a, and since a was chosen so that $|\zeta - a| > \sup_{z\in K_m} |\zeta - z|$, the Taylor expansion of f with center at ζ converges to f in the disk $D(\zeta, |\zeta-a|) \supset K_m$. Hence, there exists $n_o \in \mathbb{N}$ such that

$$\sup_{z\in K_m} |S_{n_0}(f,\zeta)(z) - f_j(z)| < \frac{1}{2s}. \tag{9}$$

By relations (7) and (9) we have

$$\sup_{z\in K_m} |S_{n_0}(f,\zeta)(z) - f_j(z)| < \frac{1}{s},$$

which implies that $f \in E(\mathbb{C}\setminus\overline{\Omega}, K_m, f_j, s, n_0)$. Thus, $f \in \bigcup_{n=0}^{\infty}\Big[E(\mathbb{C}\setminus\overline{\Omega}, K_m, f_j, s, n) \cap Y^\infty(\mathbb{C}\setminus\overline{\Omega})\Big]$ and the result follows. ■

Theorem 4.5. *The set* $U_1(\mathbb{C} \setminus \overline{\Omega},\ \zeta) \cap Y^\infty(\mathbb{C} \setminus \overline{\Omega})$ *is* G_δ *and dense in* $Y^\infty(\mathbb{C} \setminus \overline{\Omega})$.

Proof. Lemmas 5.2, 5.3 and 5.4 imply that the set $U_1(\mathbb{C} \setminus \overline{\Omega},\ \zeta) \cap Y^\infty(\mathbb{C} \setminus \overline{\Omega})$ is a denumerable intersection of sets that are open and dense in $Y^\infty(\mathbb{C} \setminus \overline{\Omega})$. The result follows from Baire's Theorem. ■

Remark 4. We note that a similar situation to that of the last remark in Section 3 holds for the space $Y^\infty(\mathbb{C} \setminus \overline{\Omega})$. Namely, if Ω is the open unit disk, then

$$Y^\infty(\mathbb{C} \setminus \overline{\Omega}) = \{f \in A_0^\infty(\mathbb{C} \setminus \overline{\Omega}) : f \text{ is constant or univalent in } \mathbb{C} \setminus \overline{\Omega}\}.$$

References

[1] C.K. Chui and M.N. Parnes, Approximation by overconvergence of a power series, *Journal of Mathematical Analysis and Applications* **36** (1971), 693–6.

[2] G. Costakis, Some remarks on universal functions and Taylor series, *Mathematical Proceedings of the Cambridge Philosophical Society* **128** (2000), 157–75.

[3] G. Costakis and V. Vlachou, A generic result concerning univalent universal functions, *Archiv der Mathematik* **82** (2004), 344–51.

[4] R.M. Gethner and J.H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, *Proceedings of the American Mathematical Society* **100** (1987), 281–8.

[5] K.G. Grosse-Erdmann, Holomorphe Monster und universelle Funktionen, *Mitteilungen aus dem Mathematischen Seminar Giessen* **176** (1987), 1–84.

[6] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, *Bulletin of the American Mathematical Society* **36** (1999), 345–81.

[7] J-P. Kahane, Baire's Category Theorem and Trigonometric Series, *Journal d'Analyse Mathématique* **80** (2000), 143–82.

[8] W. Luh, Approximation analytischer Funktionen durch überkonvergente Potenzreihen und deren Matrix-Transformierten, *Mitteilungen aus dem Mathematischen Seminar Giessen* **88** (1970), 1–56.

[9] W. Luh, Über die Anwendung von Übersummierbarkeit zur Approximation regulären Funktionen. Topics in analysis, *Lecture Notes in Mathematics*, vol. **419**. Springer, Berlin, 1974, pp 260–7.

[10] W. Luh, Über den Satz von Mergelyan, *Journal of Approximation Theory* **16** (1976), 194–8.

[11] G.R. MacLane, Sequences of derivatives and normal families, *Journal d'Analyse Mathématique* **2** (1952–3), 72–87.

[12] A. Melas and V. Nestoridis, On various types of universal Taylor series, *Complex Variables Theory and Application* **44** (2001), 245–58.

[13] A. Melas and V. Nestoridis, Universality of Taylor series as a generic property of holomorphic functions, *Advances in Mathematics* **157** (2001), 138–76.

[14] A. Melas, V. Nestoridis and I. Papadoperakis, Growth of coefficients of universal Taylor series and comparison of two classes of functions, *Journal d'Analyse Mathématique* **73** (1997), 187–202.

[15] V. Nestoridis, Universal Taylor series, *Annales de l'Institute Fourier*, Grenoble, **46** (1996), 1293–306.

[16] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York–Toronto–London, 1966.

[17] I. Schneider, Schlichte Funktionen mit universellen Approximationseigenschaften, *Mitteilungen aus dem Mathematischen Seminar Giessen* **230** (1997), 1–72.