GROUPS WITH CONJUGATE-PERMUTABLE CONDITIONS

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Abstract

For a group G, a subgroup H of G is said to be conjugate permutable if $HH^x = H^xH$ for any $x \in G$. This concept was introduced by Foguel. In this note, we call a subgroup H self-conjugate permutable in G if $HH^x = H^xH$ implies $H^x = H$. This is the dual of the concept of conjugate permutable subgroups. A C1-group is a group all of whose cyclic subgroups are self-conjugate permutable; and a C2-group is a group all of whose cyclic subgroups are conjugate permutable or self-conjugate permutable. Some properties of conjugate permutable subgroups and self-conjugate permutable subgroups for a any group are obtained, and the structure of finite C1 and C2-groups are described.

1. Introduction

In [3], Foguel introduced the following concept, which was further studied in [4] and [5].

Definition 1.1. [3] Let G be a group. A subgroup H of G is called conjugate permutable in G, denoted by $H <_{C-P} G$, if $HH^x = H^xH$ for all $x \in G$.

In the same paper, Foguel showed that for a finite group any conjugate permutable subgroup is subnormal [3, corollary 1.1]. This is an interesting result. In [5], the finite groups all of whose subgroups are conjugate permutable subgroups (ECP-groups) was investigated. In particular, the finite ECP-groups are nilpotent [5, theorem 3.5] and all finite ECP-p-groups ($p \geq 5$) are regular [5, theorem 3.12].

In this note, we introduce the following dual of conjugate permutable subgroups:

Definition 1.2. Let G be a group and H a subgroup of G. We call H self-conjugate permutable in G, if $HH^x = H^xH$ implies that $H^x = H$.

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Obviously, a subgroup N of G is normal if and only if N is both conjugate permutable and self-conjugate permutable in G. Also, it is easy to see that for a finite group G, all of its maximal subgroups and Hall subgroups are self conjugate permutable. In this paper we give some new results on conjugate permutable subgroups and consider the following conditions:
C1: every cyclic subgroup H of G is self-conjugate permutable (C1-group);
C2: every cyclic subgroup of G is either conjugate permutable or self-conjugate permutable (C2-group).

2. Conjugate permutable subgroups

Lemma 2.1. *Let G be any group and let H be a conjugate permutable subgroup of G. Then HH^x is also a conjugate permutable subgroup for all $x \in G$.*

Proof. The result, which is lemma 1.1 in [3], is obvious. ■

Theorem 2.2. *Let G be a group satisfying the maximal condition and the minimal condition on subgroups. Then every conjugate permutable subgroup of G is subnormal in G.*

Proof. Let H be a conjugate permutable subgroup of G. If H is normal in G, then nothing needs to be shown. Therefore assume that H is not normal in G. Then there exists $x_1 \in G$ such that $H^{x_1} \neq H$. Write $K_1 = HH^{x_1}$. By Lemma 2.1, K_1 is conjugate permutable in G and $H < K_1$. If H is non-normal in K_1, then there exists $x_2 \in K_1$ such that $H^{x_2} \neq H$. Let $K_2 = HH^{x_2}$. Then K_2 is a conjugate permutable subgroup of G and $K_2 < K_1$. Thus, we get $K_1 > K_2 > H$. If $H < HH^y$ always implies that H is non-normal in HH^y, then repeating the above argument, we can find an infinite chain of subgroups:

$$G > K_1 > K_2 > \ldots > K_r > \ldots > H.$$

This is contrary to the hypothesis that G satisfies the minimal condition on subgroups. Hence, without loss of generality, we can require $H \lhd K_1$. Because K_1 is also conjugate permutable in G, applying the above conclusion to K_1, we know that if K_1 is non-normal in G, then there is $x_2 \in G$ such that $K_1 < K_1K_1^{x_2}$ and $K_1 \lhd K_1K_1^{x_2}$. Write $K_2 = K_1K_1^{x_2}$. Since G satisfies the maximal condition on subgroups, we have a finite chain of subgroups:

$$H = K_0 \lhd K_1 \lhd K_2 \lhd \ldots \lhd K_s \lhd G,$$

such that $K_1 = HH^{x_1}$, $K_i = K_{i-1}K_{i-1}^{x_i}$ and $K_{x_i} < K_{i+1}$ for all i. This shows that H is subnormal in G. ■

Corollary 2.3. *Let G be as in Theorem 2.2. Assume that H is a conjugate permutable subgroup of G. Then the following statements are true:*

(1) *If H is solvable, then H^G, the normal closure of H in G, is solvable;*
(2) *If H is a p-group (p a prime), then H^G is also a p-group;*

(3) *If H is a Sylow p-subgroup of G, then H is normal in G.*

PROOF. The case when H is normal in G is trivial. Suppose that H is not normal. Then from the proof of Theorem 2.2 there exists a finite chain of subgroups of G: $H = K_0 \lhd K_1 \lhd \ldots \lhd K_r \lhd G$, where $K_i = K_{i-1}K_{i-1}^{x_i}$ $(i > 0)$ for some $x_i \in G$, $K_{i-1} < K_i$, $i = 1, 2, \ldots, r$, and all K_i are conjugate permutable in G by Lemma 2.1. If H is solvable, then $K_1/H \cong H^{x_1}/H \cap H^{x_1}$ is solvable, and hence K_1 is solvable. By the induction, we see that K_r is solvable and so is H^G. Thus (1) holds. If H is a p-group, then H^{x_1} is also a p-group. For any $g \in K_1 = HH^{x_1}$, we have $g = ab$ for some $a \in H$ and $b \in H^{x_1}$. Then ab is a p-element (modulo H), and since H is a p-group, it follows that ab is a p-element and hence $g = ab$ is a p-element. That is, K_1 is a p-group. By induction, K_r is a p-group and (2) holds. If H is a Sylow p-subgroup, then H is a maximal p-subgroup. But K_1 is a p-group that contains H properly. This is a contradiction, which completes the proof. ■

Similarly, we have:

Theorem 2.4. *Let G be any group and let H be a subgroup of G of finite index. Assume that H is conjugate permutable in G. Then H is subnormal in G and all conclusions of Corollary 2.3 hold.*

Corollary 2.5 ([3, corollary 1.1]). *Let G be a finite group. If $H <_{C-P} G$, then H is subnormal in G.*

Corollary 2.6. *Let G be a finite group and let H be a nilpotent subgroup. If H is conjugate permutable in G, then $H \leq F(G)$, the Fitting subgroup of G.*

PROOF. This follows from Corollary 2.5. ■

Theorem 2.7. *Let G be an ECP-group satisfying the minimal condition on subgroups. Then G is locally nilpotent.*

PROOF. For any $H < G$, if H is not normal in G, then from the proof of Theorem 2.2, we know that there is $x \in G$ such that $H \lhd HH^x$ and $H < HH^x$. So G satisfies the normalizer condition. By [6, 12.2.2], G is locally nilpotent. ■

3. The structures of finite C1- and C2-groups

Lemma 3.1. *Both C1- and C2-groups are closed under taking subgroups.*

PROOF. Assume that G is a C1-group and H a subgroup of G. For any cyclic subgroup A of H and $x \in H$, if AA^x is a subgroup of H, then $x \in N_G(A)$. Thus $x \in H \cap N_G(A) = N_H(A)$, as desired. The proof of C2-groups is similar. ■

The following lemma is an essential fact.

Lemma 3.2. *Let G be any group and let H be a subgroup of G. If H is subnormal of defect 2 in a group G (meaning $H \lhd K \lhd G$), then H permutes with its conjugates.*

Lemma 3.3. *Let G be a nilpotent group and let H be a self-conjugate permutable subgroup of G. Then H is normal in G.*

Proof. If $N_G(H) < G$, then there exists a subgroup K such that $N_G(H) < K$ and $N_G(H)$ is normal in K. By Lemma 3.2, H is conjugate permutable. Hence $H \trianglelefteq K$, which is a contradiction. ■

A Hamiltonian group is a finite non-abelian group all of whose subgroups are normal. From Lemma 3.3, we obtain a characterization of a Hamiltonian group.

Corollary 3.4. *Let G be a finite, non-abelian, nilpotent group. Then G is a C2-group if and only if G is a Hamiltonian group.*

Theorem 3.5. *Let G be a finite C1-group. Then the following conclusions hold:*

(1) *G is super-solvable;*
(2) *Every Sylow subgroup of G of odd order is abelian, and a Sylow 2-subgroup of G is either abelian or a Hamiltonian group;*
(3) *G' is abelian, hence G is meta-abelian.*

Proof. Assume that (1) is not true and let G be a counter-example of minimal order. By Lemma 3.1, the condition is inherited by subgroups. So G is a minimal non-super-solvable group, a non-super-solvable group all of whose proper subgroups are super-solvable. By Doerk's theorem [1], G has a normal Sylow p-subgroup P, such that $P/\Phi(P)$ is a chief factor of G and $\Phi(P) \leq Z(P)$. Let A be a cyclic subgroup of P, such that $A \nsubseteq \Phi(P)$. By Lemma 3.3, A is normal in P. Since P is a normal subgroup of G, apply Lemma 3.2 to see that A is conjugate permutable in G. On the other hand, by hypothesis that A is self-conjugate permutable in G, so $A \lhd G$. Consequently, $P/\Phi(P)$ is cyclic of order p, and hence G is super-solvable, which is a contradiction. Thus (1) holds.

As the condition is inherited by subgroups, (2) follows from Corollary 3.4.

By conclusion (1), G is super-solvable, so G' is nilpotent [2, VI, 9.1]. Then conclusion (2) shows that G' is either abelian or Hamiltonian. We thus know that every subgroup of G' is normal in G'. It follows by Lemma 3.2 that any subgroup of G', particularly every cyclic subgroup A, is conjugate permutable in G. Since G is a C1-group, we have $A \lhd G$. Suppose that G' is Hamiltonian. Then G' contains Q_8. Let C be a cyclic subgroup of order 4 of Q_8, the quaternion group of order 8. Then C is normal in G. In particular, $G/C_G(C)$ is abelian. Hence $G' \leq C_G(C)$. But then, Q_8 centralizes C and hence Q_8 would be abelian, which is absurd. Hence we can conclude that G' is abelian. The proof of the theorem is now complete. ■

Theorem 3.6. *Let G be a finite C2-group and let $F(G)$ denote the Fitting subgroup of G. Then the following hold:*

(1) *$G/F(G)$ is super-solvable and all its Sylow subgroups are either abelian or Hamiltonian;*
(2) *$F(G)$ is a CCP-group (i.e., each cyclic subgroup of $F(G)$ is conjugate permutable).*

PROOF. By Lemma 3.1, the condition of the theorem is inherited by subgroups. We first show that G is solvable. We argue by induction on the order of G. If G is 2-nilpotent, let H be the normal 2-complement; then H is solvable by induction, and hence G is solvable. Thus, we may assume that $G/O_2(G)$ is not 2-nilpotent. Then we can choose a non-2-nilpotent subgroup $H/O_2(G)$ with the smallest possible order. Then every proper subgroup of $H/O_2(G)$ is 2-nilpotent. By a theorem of Itô, see [2, IV, 5.4], $H/O_2(G)$ is a minimal, non-nilpotent group with a normal Sylow 2-subgroup. Take $K \leq H$ minimal satisfying $H = O_2(G)K$. Write $N = O_2(G) \cap K$. Then $N \leq \Phi(K)$ and $K/N \cong H/O_2(G)$. So K/N is a minimal, non-2-nilpotent group. Thus we have $K = PQ$, where P is the normal Sylow 2-subgroup and Q is a Sylow q-subgroup, q an odd prime. Let A be any cyclic subgroup of P such that $A \not\subseteq N$. If A is conjugate permutable in G, Corollary 2.6 indicates that $A \leq O_2(G)$. Consequently, $A \leq N$, which is a contradiction. We thus deduce that A is not conjugate permutable for all such A. By hypothesis, we have that A is self-conjugate permutable in G. It follows from Lemma 3.1 that A is self-conjugate permutable in P. By Lemma 3.3, A is normal in P. Because $P \lhd K$, by Lemma 3.2 we have that A is conjugate permutable in K. So $A \lhd K$. Thus, we know that all such A are normal in K. By the structure of a Schmidt group [6, Theorem 9.19], K/N is of order $2q^n$ and hence K/N is 2-nilpotent, which is a contradiction. The solvability of G is now proved.

We now claim that $G/F(G)$ is super-solvable. Let U/N be any chief factor of G such that $F(G) \leq N$. Because we have shown that G is solvable, U/N is an elementary abelian p-group for some prime p. Assume that $|U/N| = p^n, n \geq 2$. For any fixed $x \in G$, let $H = U\langle x\rangle$. Choose a subgroup K of H such that $K = NK$ and K is of the smallest possible order. Let $N_0 = N \cap K$. Then $N_0 \leq \Phi(K)$ and $K/N_0 \cong H/N$. Thus U/N is a homomorphic image of K. Let P be the inverse image of U/N in K. Then P is normal in K and $P/N_0 \cong U/N$. So P/N_0 is a normal p-subgroup of K/N_0. Notice that $N_0 \leq \Phi(K)$ and apply [6, theorem 3.5], we have that P is nilpotent. Let A be any cyclic subgroup of P such that $A \not\subseteq N_0$. If A is conjugate permutable in G, by Lemma 2.6, $A \leq F(G)$ and hence $A \leq N_0$, which is a contradiction. Now by hypothesis, A is self-conjugate permutable in G, particularly in P. By Lemma 3.3, A is normal in P. Because $P \lhd K$, Lemma 3.2 indicates that A is normal in K. That is, every subgroup of P/N_0 is normal in K/N_0. It follows that every subgroup of U/N is normal in H/N. That is, every element x of G normalizes every subgroup of U/N. Consequently, U/N is cyclic, which is a contradiction. Thus, we can conclude that $F(G)$ is super-solvable.

Let P be a Sylow subgroup of G. For any cyclic subgroup C of P that is not contained in $P \cap F(G)$. Then C is not subnormal in G. Apply Theorem 1.6 to see that C is not conjugate permutable in G. Thus, C is self-conjugate permutable in G and hence in P. By Lemma 3.3, C is normal in P. Consequently every cyclic subgroup of $PF(G)/F(G)$ is normal in $PF(G)/F(G)$, so $PF(G)/F(G)$ is either abelian or a Hamiltonian group. Conclusion (1) is therefore proved.

We now show conclusion (2). Write $F = F(G)$. If some cyclic subgroup C of F is not conjugate permutable in F, then C is self-conjugate permutable in G by hypothesis, and hence in F. Thus, from Lemma 3.3 we have $N_F(C) = F$. Consequently, C is normal in F; in particular C is conjugate permutable in F. This is a contradiction. The proof is now complete. ■

Remark. We give two examples or C1-groups and C2-groups without proofs.

(1) The finite groups with all Sylow subgroups cyclic are C1-groups.
(2) The finite non-abelian groups with all proper subgroups abelian are C2-groups.

Theorem 3.7. *Let G be a finite group and let p be the smallest prime divisor of $|G|$. Assume that every cyclic subgroup of G of order p is self-conjugate permutable, and if $p = 2$, in addition, each cyclic subgroup of G of order 4 is also self-conjugate permutable. Then G is p-nilpotent.*

Proof. It is clear that the condition is inherited by subgroups. If G is a counter-example of minimal order, then G is a minimal non-p-nilpotent group. By Itô's theorem [2, IV, 5.4], $G = PQ$ with normal Sylow p-subgroup P, the exponent of P is p if p is odd, and at most 4 if $p = 2$. If $p > 2$, let A be any cyclic subgroup of P of order p. If $p = 2$, let A be any cyclic subgroup of P of order 2 or 4. By hypothesis, A is self-conjugate permutable in G, and hence in P. So A is normal in P by Lemma 3.3. Because P is normal in G, it follows by Lemma 3.2 that A is conjugate permutable in G. Therefore A is normal in G. Consequently, P is cyclic and thus G is p-nilpotent, which is a contradiction. ■

We notice that in both simple groups A_5 and $PSL(2,7)$, every subgroup of order ≥ 3 is self-conjugate permutable. This information should be valuable for characterizing some finite simple groups by means of self-conjugate permutable subgroups. In particular we have the following:

Question. What can be said about the finite simple groups whose cyclic subgroups of order ≥ 3 are self-conjugate permutable?

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