# AN APPLICATION OF A FIXED-POINT THEOREM TO BEST APPROXIMATION FOR GENERALIZED AFFINE MAPPING

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Abstract

The aim of the present paper is to obtain a fixed-point theorem for generalized affine mapping, and a class of relatively nonexpansive commutative mappings that is used to find the best approximation result. In the process, related results due to Jungck and Sessa, and many others, are generalized and improved.

## 1. Introduction

During the last four decades several interesting and valuable studies relating to the application of fixed-point theorems have been presented in the field of best approximation theory. An excellent example can be seen in [10].

Historically, Meinardus [5] was the first to employ a fixed-point theorem to establish the existence of an invariant approximation. Later, Brosowski [1] obtained a celebrated result and generalized the Meinardus's result. Since then, a number of results have been proved following in the direction of Brosowski [1], (see [3; 6; 7; 8; 9; 11; 12]). Jungck and Sessa [4] further weakened the hypothesis of Sahab, Khan and Sessa [6] by replacing the weak and strong topology for relatively nonexpansive commutative maps.

Recently, the concept of affine with respect to a point, which is a generalization of an affine mapping, was introduced by Vijayaraju and Marudai [12].

The purpose of the present paper is to use the concept of generalized affine mapping and the result of Hadzic [2] in order to find a fixed-point theorem to a class of relatively nonexpansive commutative mappings for a number of mappings, which are then used to find the best approximation result. In the process, related results due to Jungck and Sessa [4], and hence to Brosowski [1], Hicks and Humpheries [3], Sahab, Khan and Sessa [6] and Singh [7] are also generalized and improved.

## 2. Preliminaries

To prove our results, we recall the following definitions:

**Definition 2.1.** [4]. Let $X$ be a normed space and let $C$ be a non-empty subset

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of $X$. Let $x_0 \in X$. An element $y \in C$ is called a best approximant to $x_0 \in X$, if:

$$\|x_0 - y\| = d(x_0, C) = \inf\{\|x_0 - z\| : z \in C\}.$$

Let $D$ be the set of best $C-$approximants to $x_0$, and so

$$D = \{y \in C : \|x_0 - y\| = d(x_0, C)\}.$$

**Definition 2.2.** [4]. Let $X$ be a normed space. A set $C$ in $X$ is said to be convex, if $\lambda x + (1 - \lambda)y \in C$, whenever $x, y \in C$ and $0 \leq \lambda \leq 1$.

A set $C$ in $X$ is said to be star-shaped, if there exists at least one point $q \in C$ such that the line segment $[x, q]$ joining $x$ to $q$ is contained in $C$ for all $x \in C$ (that is, $\lambda x + (1 - \lambda)q \in C$, for all $x \in C$ and $0 < \lambda < 1$). In this case, $q$ is called the star-center of $C$.

Each convex set is star-shaped with respect to each of its points, but not conversely.

**Definition 2.3.** [12]. Let $C$ be a convex subset of normed linear space $X$. Then self-mapping $T$ of $C$ is said to be affine if:

$$T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y),$$

for all $x, y \in C$ and $\lambda \in (0, 1)$.

We give the definition providing the notion of affine with respect to a point, which is a generalization of an affine mapping, introduced by Vijayaraju and Marudai [12].

**Definition 2.4.** [12]. Let $C$ be a nonempty, convex subset of normed linear space $X$, and let $q \in C$. A self-mapping $T$ of $C$ is said to be affine with respect to $q$ if:

$$T(\lambda x + (1 - \lambda)q) = \lambda T(x) + (1 - \lambda)T(q),$$

for all $x \in C$ and $\lambda \in (0, 1)$.

The following example shows that an affine mapping with respect to a point need not be affine.

**Example 2.1.** [12]. Let $X = \mathbb{R}$ and let $C = [0, 1]$. Define $T$ on $C$ by

$$Tx = \begin{cases} 1, & \text{if} \quad x \in [0, 1), \\ 0, & \text{if} \quad x = 1. \end{cases}$$

Then we have

$$T(\lambda x + (1-\lambda)1/2) = \begin{cases} 1, & \text{if } \ x \in [0,1) \text{ and } \lambda \in (0,1), \\ 0, & \text{if } \ x = 1 \text{ and } \lambda = 1. \end{cases}$$

If $x \in [0,1)$ and $\lambda \in (0,1)$, then $T(x) = 1 = T(1/2)$ and therefore

$$T(\lambda x + (1-\lambda)1/2) = 1 = \lambda T(x) + (1-\lambda)T(1/2).$$

If $x = 1$ and $\lambda = 1$, then

$$T(\lambda x + (1-\lambda)1/2) = 0 = \lambda T(x) + (1-\lambda)T(1/2).$$

Therefore $T$ is affine with respect to $1/2$. If $x = 1$ and $\lambda = 1/2$, then

$$T(\lambda x + (1-\lambda)1/2) = T(3/4) = 1 \neq 1/2 = \lambda T(1) + (1-\lambda)T(1/2).$$

Hence, $T$ is not affine.

**Definition 2.5.** [6]. A map $T : C \to C$ is said to be $I-$contraction, if there exists a self-map $I$ on $C$ and a real number $k \in (0,1)$ such that

$$\|Tx - Ty\| \leq k\|Ix - Iy\|,$$

for all $x, y \in C$.
If, in the above inequality, $k = 1$, then $T$ is called $I-$nonexpansive.

**Definition 2.6.** [4]. A map $T : C \to X$ ($C \subseteq X$) is said to be demiclosed if and only if, whenever $\{x_n\}$ is a sequence in $C$ converging weakly to $x \in C$ and $\{Tx_n\}$ converges strongly to $y \in X$, then $Tx = y$.

**Definition 2.7.** A pair of self-mappings $(I, T)$ of a Banach space $X$ is said to be commutative on $C$ ($C \subset X$), if $ITx = TIx$ for all $x \in C$.

Throughout this paper, $F(T)$(resp.$F(I)$) denotes the set of fixed point of mapping $T$(resp.$I$).

We also use the following result, due to Hadzic [2]:

**Theorem 2.1.** *[2] Let $S, T : X \to X$ be continuous mappings and let $\Im$ be a family of self-mappings $\mathcal{A} : X \to X$ such that:*
(i) *$A(X) \subseteq S(X) \cap T(X)$, for all $\mathcal{A} \in \Im$;*
(ii) *$\mathcal{A}$ commutes with $S$ and $T$, for each $\mathcal{A} \in \Im$;*
(iii) *$d(\mathcal{A}x, \mathcal{B}y) \leq q \ d(Sx, Ty)$,*
*for any $x, y \in X$ and for all $\mathcal{A}, \mathcal{B} \in \Im$ where $0 \leq q < 1$. Then $S$, $T$ and $\mathcal{A}$ have a unique, common fixed point in $X$ for all $\mathcal{A} \in \Im$.*

## 3. Main Results

Throughout this section, $X$ denotes a Banach space and $C$ is a weakly compact subset of $X$. We have the following common fixed-point result for such a space:

**Theorem 3.1.** *Let $T$ and $S$ be self-mappings of $C$ and let $\Im$ be a family of self-mappings $\mathcal{A}: C \to C$ such that $\mathcal{A}(X) \subseteq S(X) \cap T(X)$; $\mathcal{A}$ commutes with $S$ and $T$. Suppose $C$ is $p-$star-shaped with $p \in F(T) \cap F(S)$; $T$ and $S$ are affine with respect to $p$ and continuous in the weak topology. If $T, S$ and $\mathcal{A}, \mathcal{B} \in \Im$ satisfy*

$$\|\mathcal{A}x - \mathcal{B}y\| \leq \|Sx - Ty\|, \tag{3.1}$$

*for all $x \neq y \in C$, then $F(T) \cap F(S) \cap F(\mathcal{A}) \neq \phi$, for all $\mathcal{A} \in \Im$, provided $T - \mathcal{A}$ and $S - \mathcal{A}$ are demiclosed.*

Proof. Choose a sequence $k_n \in (0, 1)$ with $\{k_n\} \to 1$ as $n \to \infty$. Define for each $n \geq 1$, for all $\mathcal{A} \in \Im$ and for all $x \in C$, a mapping $\mathcal{A}_n$ by

$$\mathcal{A}_n(x) = k_n \mathcal{A}x + (1 - k_n)p.$$

Since $C$ is $p-$star-shaped with $p \in F(T)$, $T$ is affine with respect to $p$ and $\mathcal{A}(C) \subset T(C)$, it follows that

$$\mathcal{A}_n(x) = k_n \mathcal{A}x + (1 - k_n)p = k_n \mathcal{A}x + (1 - k_n)T(p) \in T(C).$$

Hence, $\mathcal{A}_n(C) \subset T(C)$ for each $n$. Since $T$ is affine with respect to $p$ and $\mathcal{A}$ and $T$ are commutative, we have

$$\begin{aligned} \mathcal{A}_n Tx &= k_n \mathcal{A}Tx + (1 - k_n)p \\ &= k_n T\mathcal{A}x + (1 - k_n)Tp \\ &= T(k_n \mathcal{A}x + (1 - k_n)p) \\ &= T\mathcal{A}_n x, \end{aligned}$$

for each $x \in C$. Thus $\mathcal{A}_n$ and $T$ are commutative for each $n$ and $\mathcal{A}_n(C) \subseteq C = T(C)$. Similarly, we can prove $\mathcal{A}_n$ and $S$ are commutative for each $n$ and $\mathcal{A}_n(C) \subseteq C = S(C)$. Therefore $\mathcal{A}_n(C) \subseteq T(C) \cap S(C)$. Similarly, we can define $\mathcal{B}_n$ for $\mathcal{B} \in \Im$.

Moreover,

$$\|\mathcal{A}_n x - \mathcal{B}_n y\| = k_n \|\mathcal{A}x - \mathcal{B}y\| \leq k_n \|Sx - Ty\|,$$

for all $x, y \in C$. Furthermore, $C$ is complete since the weak topology is Hausdorff and $C$ is weakly compact. Therefore, maps $\mathcal{A} \in \Im$ and $T, S$ satisfy all the conditions of Theorem 2.1, which guarantees that $F(T) \cap F(S) \cap F(\mathcal{A}) = \{x_n\}$ for each $n$ and for all $\mathcal{A} \in \Im$. Since $C$ is weakly compact, there exists a sequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to^w z \in C$ weakly as $m \to \infty$. By the weak continuity of $T$, we have $z \in F(T)$. Also,

$$(T - \mathcal{A})x_m = (1 - (k_m)^{-1})(x_m - p).$$

This implies that $(T-\mathcal{A})x_m \to 0$ strongly as $m \to \infty$, since $\{x_m\}$ is bounded and $k_m \to 1$ as $m \to \infty$. Now, the demiclosedness of $T-\mathcal{A}$ guarantees that $0 = (T-\mathcal{A})z$, that is, $Tz = \mathcal{A}z$. Similarly, we can show $\mathcal{A}z = Sz$, when $S-\mathcal{A}$ is demiclosed. Hence

$$F(T) \cap F(S) \cap F(\mathcal{A}) \neq \phi,$$

for all $\mathcal{A} \in \Im$. This completes the proof. ■

*3.1. Application to best approximation*

**Theorem 3.2.** *Let $T, S$ be self-mappings of $X$ and let $\Im$ be a family of self-mappings $\mathcal{A} : C \to C$. Let $A(\partial C) \subseteq C$ and $x_0 \in F(T) \cap F(S) \cap F(\mathcal{A})$. Suppose $T, S$ are affine with respect to p and continuous in the weak topology on $D$, and $T(D) = D = S(D)$. If $D$ is nonempty, weakly compact and p-star-shaped with $p \in F(S) \cap F(T)$. If $\mathcal{A} \in \Im$ commuting with $T$ and $S$ on $D$ and satisfy for all $x \in D \cup \{x_0\}$,*

$$\|\mathcal{A}x - \mathcal{B}y\| \leq \|Sx - Ty\|,$$

*then $D \cap F(T) \cap F(S) \cap F(\mathcal{A}) \neq \phi$, for all $\mathcal{A} \in \Im$, provided $T - \mathcal{A}$ and $S - \mathcal{A}$ are demiclosed.*

PROOF. Let $y \in D$. Then $y \in \partial C$ and so $\mathcal{A}y \in C$, because $\mathcal{A}(\partial C) \subseteq C$. Now, since $Tx_0 = Sx_0 = x_0 = \mathcal{A}x_0$, we have

$$\|\mathcal{A}y - x_0\| = \|\mathcal{A}y - \mathcal{B}x_0\| \leq \|Sy - Tx_0\| = \|Sy - x_0\| = dist(x_0, C).$$

This shows that $\mathcal{A}y \in D$. Consequently, $T(D) = S(D) = D = \mathcal{A}(D)$ for all $\mathcal{A} \in \Im$. Now, Theorem 3.1 guarantees that

$$D \cap F(T) \cap F(S) \cap F(\mathcal{A}) \neq \phi,$$

for all $\mathcal{A} \in \Im$. This completes the proof. ■

*Remark 3.3.* In the light of the comment given by Vijayaraju and Marudai [12] that an affine mapping with respect to a point need not be affine, and using a result of Hadzic [2] to a class of relatively nonexpansive mappings for a number of mappings, our results generalize the results of Jungck and Sessa [4]. Hence, related results due to Brosowski [1], Hicks and Humpheries [3], Sahab, Khan and Sessa [6] and Singh [7] are also generalized and improved.

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