KOLIHA–DRAZIN INVERTIBLES FORM A REGULARITY

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Abstract

In this paper we show that the set of all Koliha–Drazin invertible elements in a complex unital Banach algebra forms a regularity as defined by Kordula and Müller, and we explore the properties of the set as a regularity. We also use this result to simplify the proof that the set of all Drazin invertible elements of a complex unital Banach algebra forms a regularity.

1. Regularities and axiomatic spectral theory

According to Kordula and Müller [5] and Müller [6, p. 50], a *regularity* in a complex unital Banach algebra $\mathcal{A}$ is defined as a set $\mathcal{R} \subset \mathcal{A}$, such that

(i) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in \mathcal{R} \iff a^n \in \mathcal{R}$;

(ii) if a, b are relatively prime elements of $\mathcal{A}$, then $ab \in \mathcal{R} \iff a \in \mathcal{R}$ and $b \in \mathcal{R}$ (a, b are *relatively prime* if there exist $x, y \in \mathcal{A}$ such that $\{a, b, x, y\}$ is a commuting set and $ax + by = 1$).

The concept of regularity is at the heart of the Kordula–Müller axiomatic spectral theory, where the spectrum of $a \in \mathcal{A}$ relative to the regularity $\mathcal{R}$ is defined by:

$$\sigma_{\mathcal{R}}(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \mathcal{R}\}.$$

If $\mathcal{R}$ is the set $\mathcal{A}^{\mathsf{inv}}$ of all invertibles in $\mathcal{A}$, we obtain the ordinary spectrum $\sigma(a)$ of a. The crucial property of the spectrum relative to a regularity is that it obeys a restricted spectral mapping theorem [6, theorem I.6.7],

$$\sigma_{\mathcal{R}}(f(a)) = f(\sigma_{\mathcal{R}}(a))$$

for every function f analytic in a neighbourhood of $\sigma(a)$ non-constant on each component of its domain.

An element a of $\mathcal{A}$ is called *Koliha–Drazin invertible* (or KD-*invertible*) [4, definition 4.1] if there exists $b \in \mathcal{A}$ such that:

$$ab = ba, \quad bab = b, \quad aba = a + w, w \in \mathcal{A}^{\mathsf{qnil}}, \tag{1.1}$$

where $\mathcal{A}^{\mathsf{qnil}}$ is the set of all quasinilpotent elements of $\mathcal{A}$. The KD-inverse of a is unique if it exists, and is denoted by a^{D}. The element $a^{\pi} = 1 - aa^{\mathsf{D}}$ is idempotent,

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and is called the *spectral idempotent of* a; both a^{D} and a^{π} double commute with a. An element $a \in \mathcal{A}$ is KD-invertible if and only if 0 is not an accumulation point of $\sigma(a)$ [4, theorem 4.2]. For further properties of the KD-inverse see [4; 7; 8]. The set of all KD-invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{\mathsf{KD}}$. We say that an element a is *Drazin invertible* if and only if it is KD-invertible with $w = aa^{\pi}$ nilpotent; the set of all Drazin invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{\mathsf{D}}$.

It is known from [1, theorem 2.3] that the set of all Drazin invertible elements forms a regularity. The purpose of this note is to prove that the KD-invertibles in a complex unital Banach algebra also form a regularity.

Lemma 1.1. *Let $\mathcal{A}$ be a complex unital Banach algebra and let $a, b \in \mathcal{A}$ be relatively prime. If $ab \in \mathcal{A}^{\mathsf{KD}}$, then $a \in \mathcal{A}^{\mathsf{KD}}$.*

PROOF. There exist $x, y \in \mathcal{A}$ such that a, b, x, y all commute and $ax + by = 1$. Let $p = (ab)^{\pi}$ be the spectral idempotent of ab.

First we show that $u = b(ab)^{\mathsf{D}}$ is the KD-inverse of $a(1-p)$. The double commutativity of the Drazin inverse and of the spectral idempotent with the element ensures that $a, b, x, y, (ab)^{\mathsf{D}}, p, u$ all commute. Hence, u commutes with $a(1-p)$. We note that $u(1-p) = u$ as $(ab)^{\mathsf{D}}p = 0$. Further, $ua = au = ab(ab)^{\mathsf{D}} = 1-p$, and

$$ua(1-p)u = (1-p)^2u = u,$$
$$a(1-p) - a(1-p)ua(1-p) = a(1-p) - a(1-p)^3 = 0 \in \mathcal{A}^{\mathsf{qnil}}.$$

Thus, $(a(1-p))^{\mathsf{D}} = u$. In fact, we have $a(1-p)$ is Drazin invertible as $(a(1-p))^{\pi} = 1 - ua(1-p) = p$ and $a(1-p)(a(1-p))^{\pi} = a(1-p)p = 0$ is nilpotent.

Next we show that $axp \in \mathcal{A}^{\mathsf{KD}}$ and $v = xp(axp)^{\mathsf{D}}$ is the KD-inverse of ap. Since $abp \in \mathcal{A}^{\mathsf{qnil}}$, $axp - (axp)^2 = ax(1-ax)p = axbyp = (abp)xy \in \mathcal{A}^{\mathsf{qnil}}$ (xy commutes with abp). By the spectral mapping theorem for the ordinary spectrum,

$$f(\sigma(axp)) = \sigma(f(axp)) = \sigma(axp - (axp)^2) = \{0\},$$

with $f(\lambda) = \lambda - \lambda^2$. Consequently, $\sigma(axp) \subset \{0, 1\}$; so $axp, 1 - axp \in \mathcal{A}^{\mathsf{KD}}$, and $(axp)^{\pi} + (1 - axp)^{\pi} = 1$. Then, v commutes with ap, while:

$$\begin{aligned} v(ap)v &= xp(axp)^{\mathsf{D}}apxp(axp)^{\mathsf{D}} = xp(axp)^{\mathsf{D}} = v, \\ ap - apvap &= ap(1 - axp(axp)^{\mathsf{D}}) = ap(axp)^{\pi} = ap(1 - (1 - axp)^{\pi}) \\ &= ap(1 - axp)(1 - axp)^{\mathsf{D}} = ap(1 - ax)(1 - axp)^{\mathsf{D}} \\ &= apby(1 - axp)^{\mathsf{D}} \in \mathcal{A}^{\mathsf{qnil}}, \end{aligned}$$

as $apb = abp$ is quasinilpotent and commutes with $y(1 - axp)^{\mathsf{D}}$. Hence $(ap)^{\mathsf{D}} = v$.

We have proved that that $a(1-p)$ and ap are KD-invertible. Since $a(1-p)ap = 0 = apa(1-p)$, by [4, theorem 5.7], $a = a(1-p) + ap \in \mathcal{A}^{\mathsf{KD}}$. ■

It is worth noting the explicit form of a^{D} in the preceding lemma ($p = (ab)^\pi$):

$$a^{\mathsf{D}} = b(ab)^{\mathsf{D}} + xp(axp)^{\mathsf{D}} \quad \text{and} \quad aa^{\mathsf{D}} = ab(ab)^{\mathsf{D}} + axp(axp)^{\mathsf{D}}.$$

Theorem 1.2. *The set $\mathcal{A}^{\mathsf{KD}}$ of all* KD-*invertible elements in a complex unital Banach algebra $\mathcal{A}$ forms a regularity.*

Proof. (i) Let $a \in \mathcal{A}$. The spectral mapping theorem for the ordinary spectrum says that $\sigma(a^n) = \{\lambda^n : \lambda \in \sigma(a)\}$. This shows that 0 is not an accumulation spectral point of a if and only if it is not an accumulation spectral point of a^n. Thus $a \in \mathcal{A}^{\mathsf{KD}} \iff a^n \in \mathcal{A}^{\mathsf{KD}}$.

(ii) If a, b commute and are KD-invertible, then ab is also KD-invertible by [4, theorem 5.5]. By Lemma 1.1, if a, b are relatively prime and $ab \in \mathcal{A}^{\mathsf{KD}}$, then $a, b \in \mathcal{A}^{\mathsf{KD}}$. ■

From the preceding theorem we can recover [1, theorem 2.3]. See also [6, theorem III.22.10].

Corollary 1.3. *The set $\mathcal{A}^{\mathsf{D}}$ of all Drazin invertible elements of a complex unital Banach algebra $\mathcal{A}$ forms a regularity.*

Proof. (i) It is well known that $a \in \mathcal{A}^{\mathsf{D}} \iff a^n \in \mathcal{A}^{\mathsf{D}}$ for any $n \in \mathbb{N}$.

(ii) If $a, b \in \mathcal{A}$ are commuting Drazin invertible elements, then ab is Drazin invertible with $(ab)^{\mathsf{D}} = a^{\mathsf{D}}b^{\mathsf{D}}$ (see Drazin [2]).

Suppose that $ax + by = 1$, with a, b, x, y commuting and $ab \in \mathcal{A}^{\mathsf{D}}$. Write $p = (ab)^\pi$. From Lemma 1.1, $a, b \in \mathcal{A}^{\mathsf{KD}}$ and $a^{\mathsf{D}} = b(ab)^{\mathsf{D}} + xp(axp)^{\mathsf{D}}$.

Assume first that $abp = 0$. In this case $(axp)^2 = axp - abpxy = axp$, and $(axp)^{\mathsf{D}} = axp$. Then:

$$a^\pi = 1 - aa^{\mathsf{D}} = 1 - ab(ab)^{\mathsf{D}} - (axp)^2 = p - a^2x^2p = (1 - a^2x^2)p = (1 + ax)byp,$$

and $aa^\pi = (1 + ax)abpy = 0$. Thus $a \in \mathcal{A}^{\mathsf{D}}$. By symmetry, $bb^\pi = 0$ and $b \in \mathcal{A}^{\mathsf{D}}$.

For the general case, assume that $(abp)^n = a^n b^n p = 0$ for $n \in \mathbb{N}$. There exist elements $x_n, y_n \in \mathcal{A}$ such that $a^n x_n + b^n y_n = 1$ (expand $(ax + by)^{2n-1}$). By the preceding part of the proof applied to a^n and b^n in place of a, b, we get that $a^n a^\pi = (aa^\pi)^n = 0$. Thus $a \in \mathcal{A}^{\mathsf{D}}$. By symmetry, also $b \in \mathcal{A}^{\mathsf{D}}$ and $(bb^\pi)^n = 0$. ■

We define the KD-spectrum and D-spectrum of an element $a \in \mathcal{A}$ by:

$$\sigma_{\mathsf{KD}}(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \mathcal{A}^{\mathsf{KD}}\}, \quad \sigma_{\mathsf{D}}(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \mathcal{A}^{\mathsf{D}}\}.$$

The spectral mapping theorem for $\sigma_{\mathsf{D}}(a)$ is proved in [1]; for $\sigma_{\mathsf{KD}}(a)$ it follows from our Theorem 1.2 and from [6, theorem I.6.7]:

Theorem 1.4. *Let $\mathcal{A}$ be a complex unital Banach algebra and let $a \in \mathcal{A}$. If f is* *any function holomorphic in an open neighbourhood of the ordinary spectrum* $\sigma(a)$ *of* a *and non-constant on any component of* $\sigma(a)$, *then* $f(\sigma_{\mathsf{KD}}(a)) = \sigma_{\mathsf{KD}}(f(a))$.

We explore some of the properties of $\sigma_{\mathsf{KD}}(a)$ and $\sigma_{\mathsf{D}}(a)$. By $\mathsf{acc}\,\sigma(a)$ and $\mathsf{iso}\,\sigma(a)$ we denote the set of all accumulation and isolated points of $\sigma(a)$, respectively, and write $\Pi(a)$ for the set of all poles of the resolvent $(\lambda 1 - a)^{-1}$.

Proposition 1.5. *Let* $\mathcal{A}$ *be a complex unital Banach algebra and let* $a \in \mathcal{A}$. *Then*:
(i) $\sigma_{\mathsf{KD}}(a) = \mathsf{acc}\,\sigma(a)$, $\sigma_{\mathsf{D}}(a) = \mathsf{acc}\,\sigma(a) \cup (\mathsf{iso}\,\sigma(a) \setminus \Pi(a))$.
(ii) $\sigma_{\mathsf{KD}}(a) \subset \sigma_{\mathsf{D}}(a) \subset \sigma(a)$ *with both* $\sigma_{\mathsf{KD}}(a)$ *and* $\sigma_{\mathsf{D}}(a)$ *closed* (*possibly empty*).
(iii) $\sigma_{\mathsf{KD}}(a) = \emptyset$ *if and only if* $\sigma(a)$ *is a finite set.*
(iv) $\sigma_{\mathsf{D}}(a) = \emptyset$ *if and only if* $\sigma(a)$ *consists of a finite number of points none of which is a pole of the resolvent of* a.

Proof. (i) This follows from the definition when we recall that $a \in \mathcal{A}^{\mathsf{KD}}$ if and only if $0 \notin \mathsf{acc}\,\sigma(a)$, and $a \in \mathcal{A}^{\mathsf{D}}$ if and only if $a \in \mathcal{A}^{\mathsf{KD}}$ and $0 \in \Pi(a)$.

(ii) The inclusion is clear. The closure is proved when we observe that $\mathsf{acc}\,\sigma(a)$ is closed and the limit points of $\mathsf{iso}\,\sigma(a) \setminus \Pi(a)$ belong to $\mathsf{acc}\,\sigma(a)$.

(iii) If $\sigma(a)$ is finite, its every point is isolated and therefore not in $\sigma_{\mathsf{KD}}(a)$. Conversely, if $\sigma(a)$ is infinite, then being compact it has an accumulation point μ, and $\mu \in \sigma_{\mathsf{KD}}(a)$.

(iv) This follows from (iii) and (i). ■

We proceed to consider some of the set properties of regularities defined in [6, p. 54]. First, we give an example that demonstrates that neither the KD- nor the D-invertibles possess property (P1) defined by Müller in [6, p. 51] in the category of complex unital Banach algebras.

Example 1.6. A regularity $\mathcal{R}$ is said to have property (P1) if

$$ab \in \mathcal{R} \iff a \in \mathcal{R} \text{ and } b \in \mathcal{R} \text{ whenever } a, b \in \mathcal{A} \text{ commute.} \tag{P1}$$

For a counterexample, let $\mathcal{A}$ be the space ℓ^∞ with pointwise addition and multiplication. Set $a = (\frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots)$, $b = (0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots)$. Then $ab = 0 \in \mathcal{A}^{\mathsf{D}} \subset \mathcal{A}^{\mathsf{KD}}$, but neither a nor b is in $\mathcal{A}^{\mathsf{KD}}$ or $\mathcal{A}^{\mathsf{D}}$ as $0 \in \mathsf{acc}\,\sigma(a)$.

In fact, $\mathcal{A}^{\mathsf{KD}}$ satisfies property (P1) only in a very special case.

Theorem 1.7. *Let* $\mathcal{A}$ *be a complex unital Banach algebra. Then the following conditions are equivalent*:
(i) $\mathcal{A}^{\mathsf{KD}}$ *has property* (P1);
(ii) $\mathcal{A}^{\mathsf{KD}} = \mathcal{A}$;
(iii) $\sigma_{\mathsf{KD}}(a) = \emptyset$ *for all* $a \in \mathcal{A}$;
(iv) *each element of* $\mathcal{A}$ *has finite spectrum*;
(v) *the quotient algebra* $\mathcal{A}/\mathsf{rad}(\mathcal{A})$, *where* $\mathsf{rad}(\mathcal{A})$ *is the Jacobson radical of* $\mathcal{A}$, *is finite dimensional.*

PROOF. First we prove the implications (i)→(ii)→(iii)→(iv)→(i). Suppose that $\mathcal{A}^{\mathsf{KD}}$ satisfies (P1). As 0 belongs to $\mathcal{A}^{\mathsf{KD}}$ and commutes with all elements, for any $a \in \mathcal{A}$ the product $0a$ is in $\mathcal{A}^{\mathsf{KD}}$; so $a \in \mathcal{A}^{\mathsf{KD}}$ for all $a \in \mathcal{A}$, and $\mathcal{A}^{\mathsf{KD}} = \mathcal{A}$. Condition (ii) clearly implies (iii). Condition (iv) follows from (iii) in view of Proposition 1.5. If each element of a Banach algebra has finite spectrum, then all elements of the spectrum are isolated, and each element of the algebra is KD-invertible. Then, trivially, $\mathcal{A}^{\mathsf{KD}}$ satisfies (P1). Hence (iv) implies (i).

Conditions (iv) and (v) are equivalent in view of the Kaplansky finite spectrum lemma [3, lemma 7] and of [6, theorem I.1.41], which ensures the equality of $\sigma(a)$ and $\sigma(a + \mathrm{r}ad(\mathcal{A}))$. ■

Finally, the spectral continuity properties (P2), (P3), (P4) defined in [6, p. 54] hold for $\sigma_{\mathsf{KD}}(a)$ and $\sigma_{\mathsf{D}}(a)$ only in this special case.

Theorem 1.8. *For a complex unital Banach algebra $\mathcal{A}$ the set of* KD*-invertibles satisfies properties* (P2), (P3), (P4) *if and only if $\mathcal{A}^{\mathsf{KD}} = \mathcal{A}$.*

PROOF. If $\mathcal{A}^{\mathsf{KD}}$ is the entire algebra then these properties are trivially satisfied.

Conversely, suppose that $\mathcal{A}^{\mathsf{KD}}$ is not the entire algebra. Then there exists $a \in \mathcal{A} \setminus \mathcal{A}^{\mathsf{KD}}$, that is, $0 \in \mathsf{acc}\,\sigma(a)$. Let $a_n = n^{-1}a$. Then $0 \in \mathsf{acc}\,\sigma(a_n)$ and $a_n \to 0$ as $n \to \infty$ but $0 \notin \mathsf{acc}\,\sigma(0)$, so property (P3) does not hold. Hence, neither do properties (P2) or (P4). ■

REFERENCES

[1] M. Berkani and M. Sarih, An Atkinson-type theorem for B-Fredholm operators, *Studia Mathematica* **148** (3) (2001), 251–7.

[2] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, *American Mathematical Monthly* **65** (1958), 506–14.

[3] I. Kaplansky, Ring isomorphism of Banach algebras, *Canadian Journal of Mathematics* **6** (1954), 374–81

[4] J.J. Koliha, A generalized Drazin inverse, *Glasgow Mathematical Journal* **38** (3) (1996), 367–81.

[5] V. Kordula and V. Müller, On the axiomatic theory of spectrum, *Studia Mathematica* **119** (2) (1996), 109–28.

[6] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Operator Theory: Advances and Applications 139, Birkhäuser Verlag, Basel, 2003.

[7] V. Rakočević, Koliha–Drazin invertible operators and commuting Riesz perturbations, *Acta Scientiarum Mathematicarum (Szeged)* **68** (2002), 953–63.

[8] V. Rakočević, Koliha–Drazin inverse and commuting perturbations, *Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento* **73** (3–4) (2004), 101–25.