THE HYPERCIRCLE AND J.L. SYNGE

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Abstract

The contribution of the late Professor John Lighton Synge to the formulation and development of the hypercircle method in numerical analysis is described. This paper represents an attempt to render a more balanced appraisal, since Synge's contribution appears to have been somewhat undervalued by some recent reviewers. The paper begins with a brief historical introduction, followed by an outline of the hypercircle method and its more recent applications. Some comparison with its successor, the finite element method, is also made.

1. General Outline

The hypercircle method is a precursor of the finite element method, which is used extensively nowadays in the numerical analysis of engineering and applied science problems. The principal purpose of this article is to describe the major contribution of the late Professor J.L. Synge (1897–1995) to the construction of the hypercircle method, since, in the opinion of this author, his contribution appears to have been undervalued by some recent reviewers. In addition, the influence of the hypercircle is followed in a variety of fields of application, and hopefully a more balanced appraisal of this technique can be attained. We commence with a brief account of the method and its historical foundations.

In a review paper on the finite element method (so named, apparently, by R.W. Clough in 1960), J.T. Oden [32] traces the history of approximation of a function on a mesh of triangles or on different sub-structures to a variety of earlier authors in the nineteenth and twentieth centuries. More immediately, however, the hypercircle appears to have its origin in the work of Kelvin, Rayleigh and of Ritz [38] in 1909; Trefftz [52] in 1926; and probably Courant [10] in 1943. In 1947 Prager and Synge [37] published a paper entitled 'Approximations in elasticity based on the concept of function space'. Here, the hypercircle was introduced as a numerical method for obtaining upper and lower bounds on functionals associated with the solutions of boundary value problems in linear elasticity theory. Basic development of the method took place in the ensuing decade—with extension to a wider class of problems at a later time. The original intention was to attempt to use the intuition of Euclidean geometry to aid the visualisation of a state of stress by substituting for six components of stress a single vector in function space. Then the hypercircle appears as the locus of possible positions of a point corresponding to the unknown

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solution of the elastic problem. And the minimum principles of elastic equilibrium appear analogous to the minimum distance from a point to a plane.

The course of development of Synge's work in the decade to 1957 shows his preference for a geometrical approach to problem solving—in contrast to the purely analytic approach of contemporaries such as Diaz and Greenberg [11] and Payne and Weinberger [34].

In the hypercircle method, some basic properties of function space are used, and the context of application is that of linear, elliptic, partial differential equations. We envisage the solution of a typical problem, such as a vector in Hilbert space whose end point lies at the intersection of two function subspaces, each of which corresponds to functions satisfying relaxed forms of the original problem. Solutions of these relaxed problems are easily generated, whereas the solution of the original problem is unknown but is supposed unique. If the metric of the function space is positive-definite the two relaxed subspaces are orthogonal, and thus an easy calculation—using the analogy with Euclidean geometry–can locate the solution on a hypercircle, which is the intersection of a hypersphere with a hyperplane.

Upper and lower bounds on the norm of the solution vector then follow by use of an analogue of Pythagoras's theorem. These bounds depend on the radius of the hypercircle and can be brought closer together by minimising this radius. By defining auxiliary test functions in each of the relaxed subspaces, finite dimensional subspaces are generated and can be used in the minimising process. This results in two distinct sets of linear algebraic equations that must be solved, and this generates the bulk of the numerical work.

In the case that the problem requires an indefinite metric in the function space—as in the application to vibration problems—the hypersphere is replaced by the analogue of a hyperbola, and the bounds become much less definite. But the situation can be recovered by using non-orthogonal subspaces, and then bounds on the solution require knowledge of the eigenvalue spectrum (see Weinberger [55]).

2. General features of the hypercircle method in the case of linear, elliptic, partial differential equations and a positive definite metric—the case of most applications

(a) A key feature of all approximative techniques, such as the finite element and hypercircle methods, is the division of the physical domain of interest into a number of subdomains, which resemble a mesh. These subdomains may be of any shape, but it is conventional to start with triangles. In these subdomains one specifies locally defined approximating functions with linear polynomials to commence with. In the hypercircle method, these are functions of two co-ordinates for plane problems. Synge calls these test functions, and they are used in linear combinations with arbitrary weights to define functions satisfying each of the relaxed problems mentioned above. This defines the radius of the hypercircle, and minimising this radius generates two separate sets of linear simultaneous equations, which must be solved for the weights.

An important paper of Synge's [47] from 1952 describes in detail, for plane problems, a class of continuous linear test functions called pyramid functions. These functions lead to banded matrices in the linear equations for the weights that are reasonably easy to solve. The pyramid functions were used in a number of applications as described in the 1957 monograph, in which [49, p. 209] an original convergence property of these functions was studied. Here also is found the maximal angle result for convergence of the pyramid functions to a continuous function; that is, that the maximum angle of a mesh triangle must be bounded away from pi. These results have been used and elaborated on by many subsequent researchers.

(b) The method generally yields bounds in the mean square sense on the solution, but, in classical elasticity problems, if a pointwise bound is desired, then Green type functions are made available for this purpose. However, two extra sets of simultaneous equations must be solved—making four sets in all— and this could be regarded as too much labour to undertake for one pointwise result.

(c) The method is available for irregular physical domains—not just the circles and squares of classical analytical applications.

(d) Exact upper and lower bounds are made available by the method for even a coarse mesh, which would lead to small numbers of linear equations to be solved for the weights. These bounds cannot be expected to be close, but they can be brought closer by refining the mesh. Of course, this is at the expense of solving larger sets of equations. The emphasis on obtaining upper and lower bounds on the solution was an essential and early feature of the method.

(e) Variational principles lie at the heart of the method. They are significant in that they can suggest both the choice of an inner product in function space and the form to be taken by each of the relaxed problems. In the hypercircle, while the elastic applications involved minimising the potential energy for the upper bound (Ritz [38]) and minimising the complementary energy for the lower bound (Trefftz [52]), another variational principle was needed for the problems in electrostatics and elastostatics that involve Laplace's and Poisson's equations. This was provided by McConnell [28] in 1951.

(f) The continuity conditions to be satisfied by the conforming test functions in each of the relaxed problems are carefully set out. While the test functions available were adequate for both upper and lower bounding processes in the case of Laplace's equation, and for the displacement boundary conditions for the elastic problems, it was not easy to find conforming test functions for the biharmonic problems involving traction boundary conditions. A further comment on resolution of this difficulty will be made in the final section below.

(g) As regards boundary conditions, many problems considered were set in domains that could be completely covered by the straight-edged test functions used without any mismatch at the boundary, so that boundary conditions were satisfied exactly. However, in Synge [49, p. 305], a method for handling curvilinear boundaries was described, with a later application to unidirectional viscous flow in a semicircular channel. A criticism of the method made in Oden's 1991 review [33, p. 6] refers to his opinion that 'Synge's treatment of boundary conditions was clearly not in the spirit of finite elements...' It is difficult to understand the meaning of this statement.

Synge always gave due importance to boundary conditions in the formulation of the relaxed problems. Perhaps the small variety of boundary conditions referred to in the hypercircle work is alluded to. Here, however, one has to note the small numbers of researchers working in the hypercircle project, and the constraint of continuity of the test functions that was always observed by them. In contrast, not all of those working in the finite element field observed this considerable restriction—which allowed considerable rein to those neglecting it.

3. Applications of the hypercircle method

The 1957 monograph [49] contains a number of applications of the hypercircle method, all but one using the limited calculating power of the desk calculators of the time. The exceptional case was the study of the torsional rigidity of a hollow square—for which upper and lower bounds were found that differed by 0.12 percent of the mean value. For this, 432 linear equations were solved for the lower bound, and 431 equations for the upper bound, using a computer (see Synge and Cahill [50]).

Hart [21] also employed a computer when he applied the method to the biharmonic equation in the plane, using piecewise quadratic patch functions defined in mesh octagons. He established upper and lower bounds on the strain energy in a thin square elastic plate under uniform transverse load and with all edges clamped. These bounds were brought together to within 1.5 percent of their mean by solving two sets of 91 equations. Naturally, these equations were much more complicated than those in the torsion problem. A convergence result for these patch functions was also supplied.

Synge published nine papers on the hypercircle method [41; 42; 43; 44; 45; 46; 47; 48; 50] between 1947 and 1957, and the summarising monograph [49] appeared in 1957. Others extended the method in various ways; a representative sample of their papers—listed from earlier to later times—follows. For example, McMahon [29] in 1953 used the hypercircle method with tetrahedral test functions in three dimensions to find a lower bound for the electrostatic capacity of a cube. Swift and Higgins [51] used it for waveguides; Maple [27], Prager [35; 36], Greenberg and Truell [20] and Edelman [16] all used it for elastic problems. Perhaps most significantly, Ackroyd [1] found this geometrical method very appropriate in an extension to the Boltzmann equation, where it was used to fix upper and lower bounds on the neutron density in nuclear reactors—a crucial variable indeed. Dillon and O'Brien [13] made stress calculations; Shubinsky and Brown [40] considered the hypercircle method's relation to energy theorems; Collins [9] extended it to consideration of unilateral problems; Duggan [15] discussed its relation to the Euler–Hamilton variational theory; Arthurs [3; 4], Arthurs and Hart [5] and Hart [23] used hypercircle estimates for nonlinear problems; Do and Bailey [14] also used the hypercircle method in nonlinear problems; Nelson and Goodman [30; 31] employed it in the analysis of multistorey frame buildings; and Auchmuty [6], in 1992, obtained a posteriori estimates for linear equations.

For a more detailed review of the hypercircle and applications up to 1969, see the article by Hart [22].

4. Final Comments

Nearly 60 years after the publication of the Prager–Synge paper of 1947, and nearly 50 years after Synge's 1957 monograph [49] appeared, thousands of papers and many books on a variety of methods for approximating the solution of a much wider range of problems have been written. These involve both linear and nonlinear equations in both solid and fluid continua. While the hypercircle method was based on the assumption of a variational principle, many later researchers not only used this assumption, but also treated different problems where no functional existed—as in the case of the diffusion equation—using the Galerkin or weighted residual methods. Also, while the systematic trial functions used by Synge were linear, and those used by Hart were quadratic, trial functions and piecewise polynomials used by others were often of higher degree.

There appear to be two streams of these papers involving the finite element method. In the mathematical stream, beginning from the 1960s, much more elaborate functional analysis is now used, in which a weak solution in a Sobolev space is sought for a wider variety of variational principles. However, the final aim is usually the same as that considered in Synge's 1957 monograph, namely to bound the error of the finite element solution in terms of some power of the mesh length. In the engineering stream, with the paper usually acknowledged as being the first appearing in 1956 (Turner *et al.* [53]), followed by major applications in the case of elastic solids by Argyris in 1964 [2] and others, and an early paper on the convergence of the method in 1968 (see Johnson and McLay [25]), many engineers have boldly set forth to solve practical problems using a wide variety of shapes for the mesh elements. In some cases the continuity conditions assuring conformity of trial functions have been neglected—but convergence of the method with refinement of the mesh is assessed by the patch test. See the extensive reviews by Oden [32; 33] and Zienkiewicz [56; 57]. Ciarlet [8] has many references to earlier work, including one to a paper by Friedrichs [18] that contains a proof of convergence in the case of the Neumann and Dirichlet problems.

In Prager and Synge's 1947 paper [37] and Synge's papers and monograph [41; 42; 43; 44; 45; 46; 47; 48; 49; 50] from 1947 to 1957, the utility of using not just the extremum principle of minimum potential energy, but also the dual principle of minimum complementary energy for obtaining upper and lower bounds for elastic continua, is clearly set out. Later, in 1965, Fraeijs de Veubeke [17] set out these ideas for the engineers (Zienkiewicz, [56, p. 250; 57, p. 11]). Also, a notable contribution of Zienkiewicz and Fraeijs de Veubeke [58]—using Southwell's analogy between the bending problem and the in-plane stretching problem for elastic plates—was to indicate a way around the difficulty, already noted, of finding conforming test functions in the traction case of boundary conditions. More recently, alternative non-geometric methods were described in books by Arthurs [3] and Sewell [39] on complementary variational principles and maximum and minimum principles, respectively.

Synge planned his expository monograph [49] as a leisurely description of the hypercircle method for mathematical physicists and engineers. To this end he used only elementary concepts of function space, fearing that his prospective readers might be repelled by a more axiomatic treatment (see preface of [49]).

However, although his 1957 monograph received good reviews at the time (see J.B. Diaz [12], H.J. Greenberg [19] and W. Wasow [54]), many engineers may have found even this mild use of function space unpalatable; while, when the mathematicians began to take interest in the 1960s and 1970s, they may have thought that the functional approach was not used at sufficient depth. As a result, it appears that Synge's work on the hypercircle is somewhat undervalued and overly criticised in some review articles. For example, two leading reviewing authors—in 1996 Zienkiewicz [57] and in 1994 Babuska [7]—omit mention of this work completely. Also, to quote Oden again in 1991 (see [33, p. 6]),

> Synge described his 'method of the hypercircle' in which he also spoke of piecewise linear approximation on triangular meshes, but not in a rich variational setting and not in a way in which approximations were built by either partitioning a domain into triangles or assembling triangles to approximate a domain...

This present reviewer believes, however, that the alleged poverty of the variational setting is probably due to the time at which it was formulated. And the statement about not partitioning a domain into triangles is plainly incorrect—as a casual perusal of Synge's monograph [49] will rapidly confirm.

However, discerning authors such as Huebner *et al.* [24, p. 9] regard Synge's work as a 'key example of finite element ideas'; and an earlier Oden [32, p.4] recognises the hypercircle as being 'in the true spirit of the finite element'. More positively, in 1996 Krizek and Neittaanmaki [26, p. 1] state that 'the first monograph on the FEM is probably that of Synge in 1957. There we can already find the original proof of the approximation properties of continuous and piecewise linear functions over a triangulated plane domain'. And later in the same book [26, p. 75], we find the maximum angle condition for linear triangular elements acknowledged to Synge, on page 211 of his 1957 monograph [49]. It is also relevant to note in Krizek and Neittaanmaki [26, p. 62] the use of the hypercircle in current mathematical analysis. However, it must be said that the development of the hypercircle method suffered considerably from the scarcity of computers and software in the 1940s and 1950s. The desk machines of the time were capable only of the four basic arithmetical operations.

To summarise, it is clear from the list of authors above that many, including engineers, found the hypercircle a useful concept, and some, for example Swift and Higgins [51] and Ackroyd [1] were very appreciative. Indeed, the latter author (see [1, p. 848]) believes that Synge's geometrical approach for the Boltzmann equation in the nuclear application gives information that appears to be lacking if using other methods of solution. And some basic results that have been carried over into the finite element method are due to Synge—although often unattributed. A recent, brief survey of the literature also indicates that, in addition to being a seminal influence, the hypercircle is still in use as a tool of analysis by a number of specialists.

In all of his work, and particularly in this topic, one can perceive John Synge's strong geometrical approach to mathematics. Indeed, he affirmed to this reviewer that he could not have invented the pyramid functions without it. Finally, I cannot do better than to quote from the last sentences of Greenberg's review [19] of Synge's 1957 monograph:

> In his introduction, the author refers to other treatments of the problem of bounding solutions of boundary problems which proceed without diagrams or geometrical ideas. These he recommends to readers who 'prefer to take their analysis neat'. This reviewer, however, heartily recommends to all readers the stimulating mixture which Professor Synge has concocted for us.

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