ON THE RIESZ IDEMPOTENT FOR A CLASS OF OPERATORS

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Abstract

A result from Stampfli says that, for hyponormal operators on Hilbert space, the Riesz idempotent at an isolated point of a spectrum is self-adjoint, with its range given by null space of both the operator and its adjoint. In this note, we extend this result to operators that are algebraically class $H(q)$. The arguments involve local spectral theory and extensions of a-Browder's theorem.

1. Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on infinite dimensional separable Hilbert space H. For $A \in B(H)$, let $H_0(A)$ denote the quasi-nilpotent part

$$H_0(A) = \{x \in H : \lim_{n\to\infty} ||A^n x||^{\frac{1}{n}} = 0\}$$

of the operator A, and let $H(q)$ denote the class of $A \in B(H)$ for which $H_0(A-\lambda I) = (A-\lambda I)^{-q}(0)$ for all complex numbers λ and some integer $q \geq 1$. The class $H(q)$ is large; it contains, amongst others, the classes consisting of generalized scalar, hyponormal, p-hyponormal $(0 < p < 1)$ and M-hyponormal operators on a Hilbert space (see [3; 6; 11]. An operator of class $H(q)$ is said to be algebraically class $H(q)$, $a - H(q)$, if there exists a non constant polynomial p for which $p(A)$ is a class $H(q)$.

Let $A \in B(H)$ and let $\lambda \in \sigma(A)$ be an isolated point of $\sigma(A)$. If there exists a closed disk D_λ centered at λ that satisfies $D_\lambda \cap \sigma(A) = \{\lambda\}$. The operator

$$E = \frac{1}{2\pi i}\int_{\partial D_\lambda} (\lambda - A)^{-1} d\lambda$$

is called the Riesz idempotent with respect to λ, which has properties that $E^2 = E$, $EA = AE$, $ker(A-\lambda) \subset EH$ and $\sigma(A|_{EH}) = \{\lambda\}$. In [15], Stampfli proved that if A is hyponormal and $\lambda \in \sigma(A)$ is isolated, then the Riesz idempotent E with respect to λ is self-adjoint and satisfies

$$EH = ker(A-\lambda) = ker(A-\lambda)^*.$$

In [5], M. Cho and K. Tanahashi have extended Stampfli's result to p-hyponormal

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and *log*-hyponormal operators. Recently, A. Uchiyama and K. Tanahashi [18] extended the above result to the case of a class A operator. In [17], Tanahashi, Uchiyama and Cho proved that the above result holds for a (p, k)-quasi-hyponormal operator. In this paper, Uchiyama and Tanahashi's result [18] is extended to algebraically class $H(q)$. We prove that if A is a class $a - H(q)$ and λ is a non-zero isolated eigenvalue of $\sigma(A)$, then $EH = ker(A - \lambda) = ker(A - \lambda)^*$, where E is the Riesz idempotent with respect to λ. In this case, E is self-adjoint, i.e, it is an orthogonal projection.

Browder's theorem and a-Browder's theorem are related to an important property that has a leading role on local spectral theory: the single valued extension property, see the recent monograph by Laursen and Neumann [11]. The study of operators satisfying Browder's theorem is of significant interest, and is currently being undertaken by a number of mathematicians around the world. In order to generalise some recent results in the literature, we prove that a-Browder's theorem holds for a large class of operators containing the classes of normal, hyponormal, p-hyponormal and M-hyponormal operators. An operator $A \in B(H)$ is said to be p-hyponormal if $(A^*A)^p \geq (AA^*)^p$, where $p > 0$. This definition is due to Aluthge [1], and many authors have studied interesting properties of p-hyponormal operators by using the Aluthge transform (see [1; 12]). A is M-hyponormal if there exists a positive number M such that:

$$(A - \lambda I)(A - \lambda I)^* \leq M^2(A - \lambda I)^*(A - \lambda I) \text{for all} \, \lambda \in \mathbb{C}.$$

The class of M-hyponormal operators, which properly contains the class of hyponormal operators, has been studied over the recent past.

2. Main results

We begin by recalling the definition of the the single valued extension property (SVEP), which will be used for the sequel.

Definition 2.1. *We say that $A \in B(H)$ has the single valued extension property (SVEP) if, for every open set U of $\mathbb{C}$, the only analytic function $f : U \to H$ that satisfies the equation $(A - \lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$ on U.*

Lemma 2.1. *[2] Let A be class $H(q)$. If $0 \neq \lambda \in \mathbb{C}$, $x \in H$ and $Ax = \lambda x$, then $A^*x = \overline{\lambda}x$.*

Lemma 2.2. *[8] Let $A \in B(H)$ be algebraically class $H(q)$. Then A is isoloid.*

Lemma 2.3. *[8] Let $A \in B(H)$ be algebraically class $H(q)$. Then A has SVEP.*

Theorem 2.1. *Let A be a class $H(q)$ operator and let λ be a non-zero isolated point of $\sigma(A)$. Let D_λ denote the closed disk that centered λ, such that $D_\lambda \cap \sigma(A) = \{\lambda\}$.*

Then the Riesz idempotent

$$E = \frac{1}{2\pi i}\int_{\partial D_\lambda}(\lambda - A)^{-1}d\lambda$$

satisfies

$$EH = ker(A-\lambda) = ker(A-\lambda)^*.$$

In particular, E is self-adjoint.

PROOF. If $A \in H(q)$, then λ is an eigenvalue of A and $EH = ker(A-\lambda)$ by Lemma 2.2. Since $ker(A-\lambda) \subset ker(A-\lambda)^*$ by Lemma 2.1, it suffices to show that $ker(A-\lambda)^* \subset ker(A-\lambda)$. Since $ker(A-\lambda)$ is a reducing subspace of A by Lemma 2.1, and the restriction of a class $H(q)$ to its reducing subspaces is also a class $H(q)$, A can therefore be written as follows:

$$A = \lambda \oplus A_1 \text{ on } H = Ker(A-\lambda) \oplus (Ker(A-\lambda))^\perp,$$

where A_1 is $a - H(q)$ with $ker(A_1 - \lambda) = \{0\}$. Since

$$\lambda \in \sigma(A) = \{\lambda\} \cup \sigma(A_1)$$

is isolated, only two cases occur: one is that $\lambda \notin \sigma(A_1)$ and the other is that λ is an isolated point of $\sigma(A_1)$, and this contradicts the fact that $ker(A_1-\lambda) = \{0\}$. Since A_1 is invertible as an operator on $(Ker(A-\lambda))^\perp$, then $ker(A-\lambda) = ker(A-\lambda)^*$.

Next, we show that E is self-adjoint. Since

$$EH = ker(A-\lambda) = ker(A-\lambda)^*,$$

we have

$$((z-A)^*)^{-1}E = \overline{(z-\lambda)^{-1}}E.$$

Therefore,

$$E^*E = -\frac{1}{2\pi i}\int_{\partial D_\lambda}((z-A)^*)^{-1}Ed\overline{z} = -\frac{1}{2\pi i}\int_{\partial D_\lambda}\overline{(z-A)^{-1}}Ed\overline{z} =$$

$$\overline{(\frac{1}{2\pi i}\int_{\partial D_\lambda}(z-A)^{-1}dz)}E = E.$$

This completes the proof. ■

Corollary 2.1. *Let $A \in B(H)$ be M-hyponormal and let λ be a non-zero isolated point of $\sigma(A)$. Let D_λ denote the closed disk that centered λ, such that $D_\lambda \cap \sigma(A) = \{\lambda\}$. Then the Riesz idempotent*

$$E = \frac{1}{2\pi i}\int_{\partial D_\lambda}(z-A)^{-1}dz$$

satisfies

$$EH = ker(A-\lambda) = ker(A-\lambda)^*$$

In particular, E is self-adjoint.

It is known that the SVEP is stable under the functional calculus, i.e. if $A \in B(H)$ has the SVEP, then so does $f(A)$ for each analytic function $f(z)$ that is analytic on some open neighborhood of $\sigma(A)$. The following lemma is immediate:

Lemma 2.4. *Let $A \in B(H)$ be algebraically a class $H(q)$ operator. Then $f(A)$ has the SVEP for each analytic function $f(z)$ that is analytic on some open neighborhood of $\sigma(A)$.*

If $A \in B(H)$, we shall write $N(A)$ and $R(T)$ for the null space and the range of A, respectively. Also, let $\alpha(A) := dimN(A)$, $\beta(A) := dimN(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of A, respectively.

An operator $A \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space, and if its range has finite co-dimension. The index of a Fredholm operator is given by:

$$I(A) := \alpha(A) - \beta(A).$$

A is called Weyl if it is of index zero, and Browder if it is Fredholm of finite ascent and descent, equivalently ([10, theorem 7.9.3]) if A is Fredholm and $A-\lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in \mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by [9; 10]:

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : \ A-\lambda \text{ is not Fredholm}\},$$

$$\sigma_w(A) = \{\lambda \in \mathbb{C} : \ A-\lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(A) = \{\lambda \in \mathbb{C} : \ A-\lambda \text{ is not Browder}\},$$

respectively. Evidently,

$$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) = \sigma_e(A) \cup acc\sigma(A),$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$. If we write $isoK = K \setminus accK$, then we let

$$\pi_{00}(A) := \{\lambda \in iso\,\sigma A : \ 0 < \alpha(A-\lambda) < \infty\},$$

$$p_{00}(A) := \sigma(A) \setminus \sigma_b(A).$$

By definition,

$$\sigma_{ea}(A) = \cap \{\sigma_a(A+K) : \ K \in K(H)\},$$

and

$$\sigma_{ab}(A) = \cap \{\sigma_a(A+K) : TK = KT \,\text{and} K \in K(H)\}.$$

We say that Browder's theorem holds for A if

$$\sigma(A) \setminus \sigma_w(A) = p_{00}(A).$$

Theorem 2.2. *[7]. Let $A \in B(H)$ be algebraically class $H(q)$. Then $f(A)$ satisfies Browder's theorem for each analytic function $f(z)$ that is analytic on some open neighborhood of $\sigma(A)$.*

The essential approximate point spectrum $\sigma_{ea}(A)$ is defined by

$$\sigma_{ea}(A) = \cap\{\sigma_a(A+K) : K \text{ is a compact operator }\},$$

where $\sigma_a(A)$ is the approximate point spectrum of A. We consider the set

$$\Phi_+^-(H) = \{A \in B(H) : A \text{ is left semi-Fredholm and } \operatorname{ind} A \leq 0\}.$$

V. Rakočević [13] proved that

$$\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+^-(H)\},$$

and the inclusion $\sigma_{ea}(f(A)) \subset f(\sigma_{ea}(A))$ holds for all functions $f(z)$ that are analytic on some open neighborhood of $\sigma(A)$ with no restriction on A. The next theorem shows the spectral mapping theorem on the essential approximate point spectrum of algebraically class $H(q)$ operators.

Lemma 2.5. *Let $A \in B(H)$ and let $\lambda \in \mathbb{C}$. If $A - \lambda$ is semi-Fredholm and it has finite ascent, then ind$(A-\lambda) \leq 0$.*

PROOF. If $A - \lambda$ has finite descent, then ind $(A-\lambda) = 0$ by [16, theorem V 6.2]. If $A - \lambda$ does not have finite descent, then

$$n \operatorname{ind}\ (A-\lambda) = \dim N(A-\lambda)^n - \dim R((A-\lambda)^n)^\perp \to -\infty.$$

Hence, ind$(A-\lambda) < 0$. ■

Corollary 2.2. *Let $A \in B(H)$ be algebraically class $H(q)$. If $A - \lambda$ is semi-Fredholm for some $\lambda \in \mathbb{C}$, then ind$(A-\lambda) \leq 0$.*

Theorem 2.3. *Let $A \in B(H)$ be algebraically class $H(q)$. Then*

$$\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for every functions $f(z)$ that is analytic on some open neighborhood G of $\sigma(A)$.

Proof. It suffices to show that $f(\sigma_{ea}(A)) \subseteq \sigma_{ea}(f(A))$. We may assume that f is nonconstant. Let $\lambda \notin \sigma_{ea}(f(A))$ and let

$$f(z) - \lambda = g(z)\Pi_{j=1}^{n}(z - \lambda_j),$$

where $\lambda_j \in G$ and $g(z) \neq 0$ for all $z \in G$. Then,

$$f(A) - \lambda = g(T)\Pi_{j=1}^{n}(A - \lambda_j).$$

Since $\lambda \notin \sigma_{ea}(f(A))$ and all operators on the right side of the above equality commute, each $(A-\lambda_j)$ is left semi-Fredholm and ind $(A-\lambda_j) \leq 0$ by the previous corollary. Thus, $\lambda_j \notin \sigma_{ea}(A)$ and $\lambda \notin f(\sigma_{ea}(A))$. ∎

We say that a-Browder's theorem holds for A if $\sigma_{ea}(A) = \sigma_{ab}(A)$. It is well known that

$$a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

In general, [4] Weyl's theorem does not hold for operators having the SVEP only, but a-Browder's theorem does hold for operators having the SVEP only, as we will show in Theorem 2.4.

Theorem 2.4. *Let $A \in B(H)$. If A has the SVEP, then a-Browder's theorem holds for A.*

Proof. It is well known that $\sigma_{ea}(A) \subseteq \sigma_{ab}(A)$. Conversely, assume that $\lambda \in \sigma_a(A) \setminus \sigma_{ea}(A)$. Then $A - \lambda \in \Phi_+^-(H)$ and $A - \lambda$ is not bounded below. Since A has the SEVP and $A - \lambda \in \Phi_+^-(H)$, [3, theorem 2.6] implies that $A - \lambda$ has finite acsent. Hence [14, theorem 2.1] would imply that $\lambda \in \sigma_a(A) \setminus \sigma_{ab}(A)$. This implies that a-Browder theorem holds for A. ∎

Corollary 2.3. *Let $A \in B(H)$ be algebraically class $H(q)$. Then a-Browder's theorem holds for $f(A)$ for each analytic function $f(z)$ that is analytic on some open neighborhood of $\sigma(A)$*

Proof. By applying Theorem 2.3 we get

$$\sigma_{ab}(f(A)) = f(\sigma_{ab}(A)) = f(\sigma_{ea}(A)) = \sigma_{ea}(f(A)).$$

Therefore a-Browder's theorem holds for $f(A)$. ∎

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