THE WEYL CALCULUS: FINITE DIMENSIONAL ASPECTS

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Abstract

The Weyl calculus for a pair $\mathbf{A} = (A_1, A_2)$ of self-adjoint $(n \times n)$-matrices, due to H. Weyl, associates a matrix $W_{\mathbf{A}}(f)$ to each smooth function f defined on $\mathbb{R}^2$ in a linear but typically not multiplicative way. Letting $c_{\mathbf{A}}(\lambda) := \det((A_1 - \lambda_1 I)^2 + (A_2 - \lambda_2 I)^2)$ for $\lambda \in \mathbb{R}^2$ denote the joint characteristic polynomial of the pair $\mathbf{A}$, it is known, for $n \leq 3$, that $A_1A_2 = A_2A_1$ if and only if $W_{\mathbf{A}}(c_{\mathbf{A}}) = 0$. It is an open question whether this is still true for $n \geq 4$. Our aim here is to pursue two new approaches: the role of the canonical order structure for self-adjoint matrices; and topological invariants arising from continuity properties of the non-linear map $(\mathbf{A}, f) \mapsto W_{\mathbf{A}}(f)$.

1. Introduction

For a pair $\mathbf{A} = (A_1, A_2)$ of self-adjoint operators in a Hilbert space, the *Weyl functional calculus* $W_{\mathbf{A}} : f \mapsto W_{\mathbf{A}}(f)$ for $\mathbf{A}$ is a means of constructing bounded operators $W_{\mathbf{A}}(f)$ for suitable smooth functions f defined on $\mathbb{R}^2$. It was introduced by H. Weyl [22] in his treatment of quantum mechanics and is based on Fourier theory, namely

$$W_{\mathbf{A}}(f) := \tfrac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\xi_1 A_1 + \xi_2 A_2)} \widehat{f}(\xi)\, d\xi; \tag{1.1}$$

see also [1; 2; 4; 8; 14; 20]. In recent years new techniques arising from matrix-valued Fourier p-multiplier theory, Clifford analysis and joint spectral theory have proved valuable in making available a finer analysis of the Weyl calculus, especially in finite dimensions, see [3; 5; 6; 7; 8; 9; 10; 11; 12; 13; 16; 17; 18; 19]. Certain analytic and geometric properties associated with the Weyl calculus $W_{\mathbf{A}}$, and its support, often force A_1 and A_2 to commute. For a summary of such properties we refer to [3, theorem 1] and [19, theorems 1 and 2], for example.

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In [3] a more algebraic condition was introduced. Define the *joint characteristic polynomial* of a pair $\mathbf{A} = (A_1, A_2)$ of self-adjoint $(n \times n)$-matrices A_1 and A_2 by

$$c_{\mathbf{A}}(\lambda) := \det[(A_1 - \lambda_1 I)^2 + (A_2 - \lambda_2 I)^2], \qquad \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2. \tag{1.2}$$

If A_1 and A_2 commute, then $W_{\mathbf{A}}(c_{\mathbf{A}}) = 0$, [3, proposition 1]. This extends the situation for a single self-adjoint matrix A, in which case an analogously defined Weyl calculus coincides with the usual Borel functional calculus as given by the spectral theorem (see [1, lemma 3.1] and properties (2a)–(2f) below). By the Cayley–Hamilton theorem, $c_A(A) = 0$ where $c_A(\lambda) = \det(A - \lambda I)^2$, and consequently $W_A(c_A) = c_A(A) = 0$. The converse statement was established for $n = 2$ and $n = 3$ in [3, theorem 2]. The computational approach used in [3] (for $n = 3$, `Maple` was essential) is not feasible for $n \geq 4$; therefore other ideas will be needed to prove, or disprove, the following:

Conjecture. *Let* $n \in \mathbb{N}$. *Then* $W_{\mathbf{A}}(c_{\mathbf{A}}) = 0$ *if and only if* $A_1 A_2 = A_2 A_1$.

The purpose of this note is to pursue two approaches that seem not to have been considered to date. Consider the real linear space M_n^{sa} of all self-adjoint $(n \times n)$-matrices over $\mathbb{C}$ with its canonical partial order. Note that $c_{\mathbf{A}}$ always has real coefficients and satisfies $c_{\mathbf{A}} \geq 0$ pointwise on $\mathbb{R}^2$, see [3, proposition 2], and that $W_{\mathbf{A}}$ has the property that $W_{\mathbf{A}}(f) \in M_n^{sa}$ whenever f is real-valued (see (2f) below). So, what order properties can be expected of $W_{\mathbf{A}}(c_{\mathbf{A}})$? This aspect, useful in [18] for extending certain commutativity results due to M. Uchiyama related to the Weyl calculus, [21], is considered in Section 3 in relation to the above Conjecture; some interesting inequalities for $c_{\mathbf{A}}$ will be established there. The second approach is to treat $\mathbf{A}$ as a variable in $[M_n^{sa}]^2 := M_n^{sa} \times M_n^{sa}$ and hence, to consider topological aspects of the non-linear map $W : (\mathbf{A}, f) \mapsto W_{\mathbf{A}}(f)$. Indeed, since $c_{\mathbf{A}} \in \mathbb{P}_{2n}$ (the real linear space of all 2-variable polynomials of degree at most $2n$), we may view W as a map from $[M_n^{sa}]^2 \times \mathbb{P}_{2n}$ into M_n^{sa}. The idea is to find a topological invariant (if it exists!) that distinguishes the sets

$$\mathcal{C}_n := \{\mathbf{A} \in [M_n^{sa}]^2 : A_1 A_2 = A_2 A_1\} \tag{1.3}$$

and

$$\mathcal{Z}_n := \{\mathbf{A} \in [M_n^{sa}]^2 : W_{\mathbf{A}}(c_{\mathbf{A}}) = 0\}, \tag{1.4}$$

for $n \geq 4$. As remarked above, we always have $\mathcal{C}_n \subseteq \mathcal{Z}_n$.

2. Preliminaries

Throughout this paper, $n \geq 1$ is an arbitrary integer; M_n denotes the $(n \times n)$-matrices over $\mathbb{C}$ equipped with some matrix norm $\|\cdot\|$ (arbitrary, but fixed); and $\mathbf{A} = (A_1, A_2) \in [M_n^{sa}]^2$. The M_n-valued distribution $f \mapsto W_{\mathbf{A}}(f)$, which is defined via (1.1) on the *Schwartz space* $\mathcal{S}(\mathbb{R}^2)$, has compact support [1, lemma 2.3] and is of finite order [1, lemma 3.8]. Accordingly, it has a unique extension to $C^\infty(\mathbb{R}^2)$, again denoted by $W_{\mathbf{A}}$. Moreover, $W_{\mathbf{A}}$ has the following fundamental properties; see [1], for example.

If $U \in M_n$ is unitary and $U\mathbf{A}U^{-1}$ denotes the pair (UA_1U^{-1}, UA_2U^{-1}), then

$$UW_{\mathbf{A}}(f)U^{-1} = W_{U\mathbf{A}U^{-1}}(f), \qquad f \in C^\infty(\mathbb{R}^2). \tag{2a}$$

If $p : \mathbb{R}^2 \to \mathbb{C}$ is a polynomial of λ_j alone, $j \in \{1, 2\}$, then

$$W_{\mathbf{A}}(p) = p(A_j). \tag{2b}$$

If $M : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation given by $(M\lambda)_j := d_j + \sum_{k=1}^{2} m_{jk}\lambda_k$, for $j \in \{1, 2\}$, and p is any 2-variable polynomial, then

$$W_{M\mathbf{A}}(p) = W_{\mathbf{A}}(Mp), \tag{2c}$$

where $(M\mathbf{A})_j := d_j I + \sum_{k=1}^{2} m_{jk}A_k$ for $j \in \{1, 2\}$, and Mp denotes the function $\lambda \mapsto p(M\lambda)$ for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

If $p(\lambda) = (a_1\lambda_1 + a_2\lambda_2)^m$ for some $m \in \mathbb{N}$ and $a_1, a_2 \in \mathbb{R}$, then

$$W_{\mathbf{A}}(p) = (a_1A_1 + a_2A_2)^m. \tag{2d}$$

If $p(\lambda) = \lambda_1^{k_1}\lambda_2^{k_2}$ and $k = k_1 + k_2$, then

$$W_{\mathbf{A}}(p) = \frac{k_1!\, k_2!}{k!} \sum_{\pi} A_{\pi(1)}A_{\pi(2)} \cdots A_{\pi(k)}, \tag{2e}$$

where the sum is taken over all maps $\pi : \{1, 2, \ldots, k\} \to \{1, 2\}$ that assume the value $j \in \{1, 2\}$ precisely k_j times.

$$W_{\mathbf{A}}(f) \in M_n^{sa} \;\text{ whenever }\; f \in C^\infty(\mathbb{R}^2) \text{ is } \mathbb{R}\text{-valued}. \tag{2f}$$

The following useful properties, needed later, follow directly from properties (2a)–(2f).

Lemma 2.1. *Let* $n \in \mathbb{N}$ *and let* $\mathbf{A} \in [M_n^{sa}]^2$.

(i) *For each* $\alpha \in \mathbb{R} \setminus \{0\}$ *we have, with* $\alpha\mathbf{A} := (\alpha A_1, \alpha A_2)$*, that*

$$c_{\alpha\mathbf{A}}(\lambda) = \alpha^{2n} c_{\mathbf{A}}\left(\alpha^{-1}\lambda_1, \alpha^{-1}\lambda_2\right) \qquad (\lambda \in \mathbb{R}^2),$$

and hence,

$$W_{\alpha\mathbf{A}}\left(c_{\alpha\mathbf{A}}\right) = \alpha^{2n} W_{\mathbf{A}}\left(c_{\mathbf{A}}\right).$$

(ii) *Let* $\widetilde{\mathbf{A}}$ *denote* $(A_1, -A_2)$ *or* $(-A_1, A_2)$*. Then*

$$W_{\alpha\widetilde{\mathbf{A}}}\left(c_{\alpha\widetilde{\mathbf{A}}}\right) = \alpha^{2n}\, W_{\mathbf{A}}(c_{\mathbf{A}}) \qquad (\alpha \in \mathbb{R}).$$

(iii) *For* $\boldsymbol{\varepsilon} \in \mathbb{R}^2$ *and with* $\mathbf{A} + \boldsymbol{\varepsilon} := (A_1 + \varepsilon_1 I,\ A_2 + \varepsilon_2 I)$*, we have*

$$W_{\mathbf{A}+\boldsymbol{\varepsilon}}\left(c_{\mathbf{A}+\boldsymbol{\varepsilon}}\right) = W_{\mathbf{A}}(c_{\mathbf{A}}).$$

(iv) *For each unitary $U \in M_n$ we have $c_{U\mathbf{A}U^{-1}} = c_{\mathbf{A}}$ and*

$$W_{U\mathbf{A}U^{-1}}(c_{U\mathbf{A}U^{-1}}) = UW_{\mathbf{A}}(c_{\mathbf{A}})U^{-1}.$$

3. Order Properties and Examples

We begin with some useful inequalities regarding the joint characteristic polynomial. Given $\mathbf{A} \in [M_n^{sa}]^2$, define the polynomials

$$q_{\mathbf{A}}^{[j]} : \lambda \mapsto \det\left((A_j - \lambda_j I)^2\right), \qquad \lambda \in \mathbb{R}^2,$$

for $j = 1, 2$. By the arithmetic–geometric mean inequality we have

$$q_{\mathbf{A}}^{[1]} + q_{\mathbf{A}}^{[2]} \geq 2\left(q_{\mathbf{A}}^{[1]} \cdot q_{\mathbf{A}}^{[2]}\right)^{1/2}$$

pointwise on $\mathbb{R}^2$. A connection to $c_{\mathbf{A}}$ is via the following result:

Proposition 3.1. *Let $\mathbf{A} \in [M_n^{sa}]^2$. Pointwise on $\mathbb{R}^2$ we have*

$$c_{\mathbf{A}} \geq 2^n \left(q_{\mathbf{A}}^{[1]} \cdot q_{\mathbf{A}}^{[2]}\right)^{1/2} \geq 0 \tag{3.1}$$

and

$$c_{\mathbf{A}} \geq q_{\mathbf{A}}^{[1]} + q_{\mathbf{A}}^{[2]} \geq 0. \tag{3.2}$$

Proof. For $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$ and positive definite matrices $B_1, B_2 \in M_n^{sa}$, it is known that

$$\det(\alpha_1 B_1 + \alpha_2 B_2) \geq [\det(B_1)]^{\alpha_1}[\det(B_2)]^{\alpha_2},$$

[15, theorem 33.3.1]. For points $\lambda \in \mathbb{R}^2$ with $\lambda_j \notin \sigma(A_j)$ for $j = 1, 2$ (the spectrum of A_j), set $B_j = (A_j - \lambda_j I)^2$ to conclude that

$$c_{\mathbf{A}}(\lambda) = 2^n \det\left(\tfrac{1}{2} B_1 + \tfrac{1}{2} B_2\right) \geq 2^n \left(q_{\mathbf{A}}^{[1]}(\lambda) q_{\mathbf{A}}^{[2]}(\lambda)\right)^{1/2}. \tag{3.3}$$

Since $\sigma(A_1) \times \sigma(A_2) \subseteq \mathbb{R}^2$ is finite, we conclude by continuity that (3.3) holds for all $\lambda \in \mathbb{R}^2$.

For positive definite matrices $B_1, B_2 \in M_n^{sa}$ it is also known that

$$\det(B_1 + B_2) \geq \det(B_1) + \det(B_2),$$

[15, p. 150]. For $\lambda \in \mathbb{R}^2$ with $\lambda_j \notin \sigma(A_j)$ apply this to the particular matrices B_1, B_2 given in the previous paragraph to deduce, in a similar fashion, that (3.2) holds. ■

By the Cayley–Hamilton theorem for single matrices, property (2b) and linearity of $f \mapsto W_{\mathbf{A}}(f)$, we can conclude that:

$$W_{\mathbf{A}}\left(q_{\mathbf{A}}^{[1]} + q_{\mathbf{A}}^{[2]}\right) = W_{\mathbf{A}}\left(q_{\mathbf{A}}^{[1]}\right) + W_{\mathbf{A}}\left(q_{\mathbf{A}}^{[2]}\right) = 0.$$

In view of (3.2) or the fact that $c_{\mathbf{A}} \geq 0$, it is then natural to ask whether $W_{\mathbf{A}}(c_{\mathbf{A}}) \geq 0$ always holds? A related question is whether the map $\mathbf{A} \mapsto W_{\mathbf{A}}(c_{\mathbf{A}})$ is monotone (where $\mathbf{A} \geq \mathbf{0}$ means that $A_j \geq 0$ for $j = 1, 2$)? For $n \leq 2$ it is known that always $W_{\mathbf{A}}(c_{\mathbf{A}}) \geq 0$, [3, section 3].

Example 3.2. Consider the non-commuting matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which satisfy the following identities:

$$A_1^2 = A_2^2 = I,$$

$$A_1 A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 A_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$A_2 A_1 A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 A_2 A_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$A_1 A_2 A_1 A_2 = A_2 A_1 A_2 A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Direct calculation gives

$$\begin{aligned} c_{\mathbf{A}}(\lambda) = {} & \lambda_1^6 + \lambda_2^6 - 2\lambda_1^5 + 2\lambda_1^4 - 2\lambda_2^4 + 8\lambda_2^3 + 4\lambda_1^2 - 4\lambda_2^2 - 8\lambda_1 + 8 \\ & + 3\lambda_1^4\lambda_2^2 + 3\lambda_1^2\lambda_2^4 - 2\lambda_1^3\lambda_2^2 - 12\lambda_1^3\lambda_2 - 6\lambda_1\lambda_2^3 + 4\lambda_1^2\lambda_2^2 \\ & + 8\lambda_1^2\lambda_2 + 4\lambda_1\lambda_2^2 - 24\lambda_1\lambda_2. \end{aligned}$$

Using the properties of the Weyl calculus listed in Section 2 and the above identities for $\mathbf{A} = (A_1, A_2)$, we find that

$$W_{\mathbf{A}}(c_{\mathbf{A}}) = \frac{2}{15} \begin{pmatrix} 26 & 0 & 80 \\ 0 & -105 & 0 \\ 80 & 0 & 162 \end{pmatrix},$$

which is not positive definite.

The pair $\mathbf{A}$ is non-positive; however, for a suitable $\boldsymbol{\varepsilon} \in \mathbb{R}^2$, the pair $\mathbf{D} := \mathbf{A} + \boldsymbol{\varepsilon}$ is positive definite. According to Lemma 2.1(iii) we see that

$$W_{\mathbf{D}}(c_{\mathbf{D}}) = W_{\mathbf{A}}(c_{\mathbf{A}}) \ngeq 0.$$

This shows that the map $\mathbf{A} \mapsto W_{\mathbf{A}}(c_{\mathbf{A}})$ is not monotone.

It may be interesting to note that there also exist non-commuting (3×3)-pairs $\mathbf{A}$ for which $W_{\mathbf{A}}(c_{\mathbf{A}})$ *is* positive semi-definite. Indeed, let $S_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $S_2 = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ be the classical *spin matrices*. For any (1×1)-pair of matrices (u, v), with $u, v \in \mathbb{R}$, consider the direct sum matrices $A_1 = S_1 \oplus (u)$ and $A_2 = S_2 \oplus (v)$. By Lemma 3.3 below we have $c_{\mathbf{A}}(\lambda) = c_{\mathbf{S}}(\lambda)c_{(u,v)}(\lambda)$, that is,

$$c_{\mathbf{A}}(\lambda) = \left(\lambda_1^4 + \lambda_2^4 + 2\lambda_1^2\lambda_2^2 + 4\right) \cdot \left((u - \lambda_1)^2 + (v - \lambda_2)^2\right).$$

According to (3.5) below we have:

$$W_{\mathbf{A}}(c_{\mathbf{A}}) = W_{\mathbf{S}}(c_{\mathbf{A}}) \oplus W_{(u,v)}(c_{\mathbf{A}})$$

and, since the (1×1)-pair (u, v) commutes, that $W_{(u,v)}(c_{\mathbf{A}}) = (c_{\mathbf{A}}(u, v)) = (0)$. Expanding the above product for $c_{\mathbf{A}}(\lambda)$ and using the identities $S_1^2 = S_2^2 = I$ and $S_1S_2 = -S_2S_1$, we get

$$W_{\mathbf{S}}(c_{\mathbf{A}}) = \frac{1}{15}\left((168 + 100(u^2 + v^2))I - 168uS_1 - 168vS_2\right).$$

From this and $W_{\mathbf{A}}(c_{\mathbf{A}}) = W_{\mathbf{S}}(c_{\mathbf{A}}) \oplus (0)$ it can be checked that $W_{\mathbf{A}}(c_{\mathbf{A}})$ is positive semi-definite.

Example 3.2 suggests (unfortunately) that the order in $[M_n^{sa}]^2$ is not a useful property for our problem. Yet, Example 3.4 below gives some positive evidence for the Conjecture. First, we require a simple fact, already used in Example 3.2.

Lemma 3.3. *Let $n = k + m$ for some $k, m \in \mathbb{N}$, and let $\mathbf{E} = (E_1, E_2)$ and $\mathbf{F} = (F_1, F_2)$ be elements of $[M_k^{sa}]^2$ and $[M_m^{sa}]^2$, respectively. Define $A_1 := E_1 \oplus F_1$ and $A_2 := E_2 \oplus F_2$ and set $\mathbf{A} = (A_1, A_2) \in [M_n^{sa}]^2$. Then,*

$$c_{\mathbf{A}}(\lambda) = c_{\mathbf{E}}(\lambda)c_{\mathbf{F}}(\lambda), \qquad \lambda \in \mathbb{R}^2, \tag{3.4}$$

and

$$W_{\mathbf{A}}(c_{\mathbf{A}}) = W_{\mathbf{E}}(c_{\mathbf{E}}c_{\mathbf{F}}) \oplus W_{\mathbf{F}}(c_{\mathbf{E}}c_{\mathbf{F}}). \tag{3.5}$$

Proof. Formula (3.4) follows from the fact that the determinant of a direct sum of two matrices is the product of the determinants of the component matrices. The identity (3.5) follows from (3.4) and the formula $W_{\mathbf{A}}(f) = W_{\mathbf{E}}(f) \oplus W_{\mathbf{F}}(f)$, which is evident from (1.1); see the proof of [6, lemma 1(ii)(a)], for example. ■

Example 3.4. Let S_1 and S_2 be the spin matrices as given in Example 3.2. Set $\mathbf{E} = \mathbf{F} = (S_1, S_2)$ and define the non-commuting pair $\mathbf{A} = \mathbf{E} \oplus \mathbf{F} \in [M_4^{sa}]^2$. By Lemma 3.3 we have $W_{\mathbf{A}}(c_{\mathbf{A}}) = W_{\mathbf{E}}(c_{\mathbf{E}}^2) \oplus W_{\mathbf{E}}(c_{\mathbf{E}}^2)$. Direct calculation gives $c_{\mathbf{E}}(\lambda) = \lambda_1^4 + \lambda_2^4 + 2\lambda_1^2\lambda_2^2 + 4$ and hence,

$$c_{\mathbf{A}}(\lambda) = c_{\mathbf{E}}^2(\lambda) = \lambda_1^8 + \lambda_2^8 + 4\lambda_1^6\lambda_2^2 + 4\lambda_1^2\lambda_2^6 + 6\lambda_1^4\lambda_2^4 + 8\lambda_1^4 + 8\lambda_2^4 + 16\lambda_1^2\lambda_2^2 + 16.$$

To find $W_{\mathbf{E}}\left(c_{\mathbf{E}}^2\right)$ involves computing, in particular, $W_{\mathbf{E}}(\lambda_1^4\lambda_2^4)$; via (2e) this alone involves 70 terms! Persevering, it turns out that $W_{\mathbf{E}}\left(c_{\mathbf{E}}^2\right) = 40\frac{68}{105}I$ and, hence, that $W_{\mathbf{A}}(c_{\mathbf{A}}) = 40\frac{68}{105}(I \oplus I) \neq 0$, as expected.

4. Continuity of Various Maps

For each $k \in \mathbb{N}$, the norm of a polynomial from $\mathbb{P}_k$, the real linear space of all 2-variable polynomials of degree at most k, is taken to be the sum of the moduli of its coefficients. For $\mathbf{A} \in [M_n^{sa}]^2$ we define $|||\mathbf{A}||| = \|A_1\| + \|A_2\|$, where $\|\cdot\|$ is the given matrix norm in M_n. The zero and the identity matrix in M_n are denoted by 0_n and I_n, respectively.

Lemma 4.1. *For each $n \in \mathbb{N}$, the non-linear map $\Psi_n : [M_n^{sa}]^2 \to \mathbb{P}_{2n}$ defined by $\Psi_n : \mathbf{A} \mapsto c_{\mathbf{A}}$ is continuous.*

PROOF. Let $M_{n,2}^{sa}[\lambda]$ denote the real vector space of all 2-variable polynomials, with coefficients from M_n^{sa}, that have the form

$$p(\lambda_1, \lambda_2) = S_{0,0} + S_{1,0}\lambda_1 + S_{2,0}\lambda_1^2 + S_{0,1}\lambda_2 + S_{0,2}\lambda_2^2$$

with $S_{j,k} \in M_n^{sa}$, and equipped with the norm

$$\left| p(\lambda) \right| := \sum_{j,k} \|S_{j,k}\| \,.$$

Define $G : [M_n^{sa}]^2 \to M_{n,2}^{sa}[\lambda]$ by $\mathbf{A} \mapsto \sum_{j=1}^2 (A_j - \lambda_j I)^2$. For $\mathbf{A}, \mathbf{B} \in [M_n^{sa}]^2$ we have:

$$G(\mathbf{A}) - G(\mathbf{B}) = \sum_{j=1}^{2} \big(2(A_j - B_j)\lambda_j + (A_j - B_j)A_j + B_j(A_j - B_j)\big),$$

from which it follows that:

$$\begin{aligned} \left| G(\mathbf{A}) - G(\mathbf{B}) \right| &\leq \sum_{j=1}^{2} \big(2\|A_j - B_j\| + (\|A_j\| + \|B_j\|) \cdot \|A_j - B_j\|\big) \\ &\leq (2 + |||\mathbf{A}||| + |||\mathbf{B}|||) \cdot |||\mathbf{A} - \mathbf{B}||| \,. \end{aligned}$$

Accordingly, G is continuous. Define $D : M_{n,2}^{sa}[\lambda] \to \mathbb{P}_{2n}$ by

$$D(p) : \lambda \mapsto \det(p(\lambda)), \qquad p(\lambda) \in M_{n,2}^{sa}[\lambda].$$

The norm of the polynomial $D(p) \in \mathbb{P}_{2n}$ is the sum of the moduli of its coefficients. By definition of the determinant, each coefficient of $D(p)$ arises as a finite sum of products of the entries of the coefficient matrices of $p(\lambda)$. Since the entries of a matrix depend continuously on the matrix, it follows that the moduli of the coefficients of $D(p)$ depend continuously on the coefficient matrices of $p(\lambda)$. So, D is continuous. Since $\Psi_n = D \circ G$, we conclude that Ψ_n is continuous. ■

Lemma 4.2. *Let $k, n \in \mathbb{N}$ and $p \in \mathbb{P}_k$ be fixed. The non-linear map $\Phi_p : [M_n^{sa}]^2 \to M_n^{sa}$ defined by $\Phi_p : \mathbf{A} \mapsto W_{\mathbf{A}}(p)$ is continuous.*

Proof. Suppose that

$$p(\lambda) = \sum_{k_1,\, k_2} \alpha_{k_1,k_2} \lambda_1^{k_1} \lambda_2^{k_2}, \qquad \lambda \in \mathbb{R}^2,$$

where the sum is over all integers $k_1, k_2 \geq 0$ with $k_1 + k_2 \leq k$. For each $\mathbf{A} \in [M_n^{sa}]^2$, linearity of $f \mapsto W_{\mathbf{A}}(f)$ yields

$$\Phi_p(\mathbf{A}) = W_{\mathbf{A}}(p) = \sum_{k_1,k_2} \alpha_{k_1,k_2} W_{\mathbf{A}}\big(\lambda_1^{k_1} \lambda_2^{k_2}\big).$$

According to (2e), in the formula for each term $W_{\mathbf{A}}(\lambda_1^{k_1}\lambda_2^{k_2})$ only finitely many matrix operations are performed. It follows that Φ_p is continuous. See also [1, theorem 2.9(c)]. ■

Lemma 4.3. *Fix $k, n \in \mathbb{N}$ and $\mathbf{A} \in [M_n^{sa}]^2$. The map ${W_{\mathbf{A}}}^{[k]} : \mathbb{P}_k \to M_n^{sa}$ defined by $p \mapsto W_{\mathbf{A}}(p)$ is linear and, hence, continuous. In particular,*

$$\|{W_{\mathbf{A}}}^{[k]}\| = \sup\{\|W_{\mathbf{A}}(p)\| : p \in \mathbb{P}_k,\ \|p\| \leq 1\} < \infty.$$

Our final preparatory result is the following:

Lemma 4.4. *Fix $k, n \in \mathbb{N}$. The non-linear map $(\mathbf{A}, p) \mapsto W_{\mathbf{A}}(p)$ is continuous from $[M_n^{sa}]^2 \times \mathbb{P}_k$ into M_n^{sa}.*

Proof. Fix $(\mathbf{A}, p) \in [M_n^{sa}]^2 \times \mathbb{P}_k$. Since

$$D_1(\mathbf{A}) := \big\{\mathbf{B} \in [M_n^{sa}]^2 : \|\!|\, \mathbf{A} - \mathbf{B} |\!\|\ \leq 1\big\}$$

is compact in $[M_n^{sa}]^2$, it follows from Lemma 4.2 that, for each $q \in \mathbb{P}_k$, the set $\Phi_q(D_1(\mathbf{A}))$ is compact in M_n^{sa} and, hence, that $\sup\{\|W_{\mathbf{B}}(q)\| : \mathbf{B} \in D_1(\mathbf{A})\} < \infty$. Accordingly, $\{W_{\mathbf{B}}^{[k]} : \mathbf{B} \in D_1(\mathbf{A})\}$ is a pointwise bounded subset of the Banach space $L(\mathbb{P}_k, M_n^{sa})$ consisting of all linear maps from $\mathbb{P}_k$ into M_n^{sa}. That is, for each $q \in \mathbb{P}_k$, we have $\sup\big\{\|W_{\mathbf{B}}^{[k]}(q)\| : \mathbf{B} \in D_1(\mathbf{A})\big\} < \infty$. By the uniform boundedness principle,

$$\gamma_{\mathbf{A}} := \sup\big\{\|W_{\mathbf{B}}^{[k]}\| : \mathbf{B} \in D_1(\mathbf{A})\big\} < \infty.$$

Let $\varepsilon > 0$. For each $(\mathbf{B}, q) \in [M_n^{sa}]^2 \times \mathbb{P}_k$ we have, by linearity of $W_{\mathbf{B}}^{[k]}$ and Lemma 4.3,

$$\begin{aligned}\|W_{\mathbf{A}}(p) - W_{\mathbf{B}}(q)\| &\le \|W_{\mathbf{A}}(p) - W_{\mathbf{B}}(p)\| + \|W_{\mathbf{B}}(p) - W_{\mathbf{B}}(q)\| \\ &\le \|W_{\mathbf{A}}(p) - W_{\mathbf{B}}(p)\| + \|W_{\mathbf{B}}^{[k]}\| \cdot \|p - q\|.\end{aligned}$$

By Lemma 4.2 there is $\delta_1 \in (0,1)$, such that $\|W_{\mathbf{A}}(p) - W_{\mathbf{B}}(p)\| < \frac{\varepsilon}{2}$ whenever $\mathbf{B}$ satisfies $|||\, \mathbf{A} - \mathbf{B} |||< \delta_1$. Then, for all $q \in \mathbb{P}_k$ satisfying $\|p - q\| < \delta_2 := \frac{\varepsilon}{2\gamma_{\mathbf{A}}}$ and all $\mathbf{B}$ satisfying $|||\, \mathbf{A} - \mathbf{B} ||| < \delta_1$, we have $\|W_{\mathbf{A}}(p) - W_{\mathbf{B}}(q)\| < \varepsilon$, thus proving the claim. ■

Proposition 4.5. *For each $n \in \mathbb{N}$, the non-linear map $\mathbf{A} \mapsto W_{\mathbf{A}}(c_{\mathbf{A}})$ is continuous from $[M_n^{sa}]^2$ into M_n^{sa}.*

PROOF. Fix $\mathbf{A} \in [M_n^{sa}]^2$ and let $\{\mathbf{A}_m\}_{m=1}^{\infty} \subseteq [M_n^{sa}]^2$ be a sequence such that $\mathbf{A}_m \to \mathbf{A}$ as $m \to \infty$. Then $c_{\mathbf{A}_m} \to c_{\mathbf{A}}$ in $\mathbb{P}_{2n}$ by Lemma 4.1. Therefore, Lemma 4.4 implies that $W_{\mathbf{A}_m}(c_{\mathbf{A}_m}) \to W_{\mathbf{A}}(c_{\mathbf{A}})$ in M_n^{sa} as $m \to \infty$. ■

5. Comparison of $\mathcal{C}_n$ and $\mathcal{Z}_n$

Throughout this section, $n \in \mathbb{N}$ is fixed and the subsets $\mathcal{C}_n$ and $\mathcal{Z}_n$ of $[M_n^{sa}]^2$ are given by (1.3) and (1.4), respectively. As noted in the Introduction above, the inclusion $\mathcal{C}_n \subseteq \mathcal{Z}_n$ always holds. The following invariance properties are straightforward for $\mathcal{C}_n$ and follow from Lemma 2.1 for $\mathcal{Z}_n$.

(5a) If $\mathbf{A} \in \mathcal{C}_n$ (resp., $\mathcal{Z}_n$), then $\alpha\mathbf{A} \in \mathcal{C}_n$ (resp., $\mathcal{Z}_n$) for all $\alpha \in \mathbb{R}$.
(5b) If $\mathbf{A} \in \mathcal{C}_n$ (resp., $\mathcal{Z}_n$), then $\alpha\widetilde{\mathbf{A}} \in \mathcal{C}_n$ (resp., $\mathcal{Z}_n$) for all $\alpha \in \mathbb{R}$, where $\widetilde{\mathbf{A}}$ denotes $(A_1, -A_2)$ or $(-A_1, A_2)$.
(5c) If $\mathbf{A} \in \mathcal{C}_n$ (resp., $\mathcal{Z}_n$), then $\mathbf{A} + \boldsymbol{\varepsilon} \in \mathcal{C}_n$ (resp., $\mathcal{Z}_n$) for all $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$.
(5d) Both $\mathcal{C}_n$ and $\mathcal{Z}_n$ are unitarily invariant in the sense of Lemma 2.1(iv).

Is there a topological criterion that distinguishes $\mathcal{C}_n$ and $\mathcal{Z}_n$?

Proposition 5.1. *Both $\mathcal{C}_n$ and $\mathcal{Z}_n$ are closed subsets of $[M_n^{sa}]^2$.*

PROOF. The fact that $\mathcal{C}_n$ is closed follows easily from joint continuity of matrix multiplication $(S,T) \to ST$ in M_n. The closedness of $\mathcal{Z}_n$ is immediate from Proposition 4.5. ■

Proposition 5.2. *Both $\mathcal{C}_n$ and $\mathcal{Z}_n$ are arcwise connected (hence, connected) subsets of $[M_n^{sa}]^2$.*

PROOF. Let $\mathbf{A} \in \mathcal{C}_n$. Then $t\mathbf{A} + (1-t)\mathbf{I} \in \mathcal{C}_n$ for every $t \in [0,1]$. So, every $\mathbf{A} \in \mathcal{C}_n$ is connected by a continuous path (within $\mathcal{C}_n$) to $\mathbf{I}$. Hence, any pair of elements from $\mathcal{C}_n$ can be connected by a continuous path within $\mathcal{C}_n$.

Let $\mathbf{A} \in \mathcal{Z}_n$. Then $t\mathbf{A} \in \mathcal{Z}_n$ for every $t \in [0,1]$; see (5a). With $\boldsymbol{\varepsilon} = (1-t, 1-t)$, it follows from (5c) that

$$t\mathbf{A} + \boldsymbol{\varepsilon} = t\mathbf{A} + (1-t)\mathbf{I} \in \mathcal{Z}_n, \qquad t \in [0,1].$$

Arguing as for $\mathcal{C}_n$, it follows that $\mathcal{Z}_n$ is also arcwise connected. ■

Proposition 5.3. *For each $n \geq 2$, the set $\mathcal{C}_n$ is nowhere dense in $[M_n^{sa}]^2$.*

Proof. Let $\mathbf{A} \in [M_n^{sa}]^2$. Since $[M_n^{sa}]^2$ is invariant under the map $\mathbf{A} \mapsto U\mathbf{A}U^{-1}$, for each unitary $U \in M_n$, we may assume that $A_1 = \operatorname{diag}(\mu_j)_{j=1}^n$ is a diagonal matrix and that (β_{jk}) is the matrix of A_2 relative to an orthonormal basis determined by the eigenvectors of A_1. According to [3, proposition 3] we have

$$A_1A_2 = A_2A_1 \text{ if and only if } \beta_{jk}(\mu_j - \mu_k) = 0 \quad \text{for all } j < k. \tag{5.1}$$

Now, fix $\mathbf{A} \in \mathcal{C}_n$.

Case (i). Suppose that $\beta_{jk} \neq 0$ for at least one pair $j < k$. There is no loss of generality in supposing that $\beta_{12} \neq 0$. Then (5.1) implies that $\mu_1 = \mu_2$. For $\varepsilon > 0$, define $\mathbf{A}^{(\varepsilon)} := (A_1^{(\varepsilon)}, A_2) \in [M_n^{sa}]^2$, where $A_1^{(\varepsilon)} := \operatorname{diag}(\mu_1 + \varepsilon, \mu_2, \ldots, \mu_n)$. Since $\beta_{12}\left((\mu_1 + \varepsilon) - \mu_2\right) = \beta_{12}\varepsilon \neq 0$, it follows from (5.1) that $\mathbf{A}^{(\varepsilon)} \notin \mathcal{C}_n$.

Case (ii). Suppose that $\beta_{jk} = 0$ for all $j < k$ (hence, also $\beta_{kj} = \overline{\beta_{jk}} = 0$). Then $A_2 = \operatorname{diag}(\beta_{jj})_{j=1}^n$. Suppose that at least two of the $(\mu_j)_{j=1}^n$ or two of the $(\beta_{jj})_{j=1}^n$ are distinct, say $\beta_{11} \neq \beta_{22}$ (without loss of generality). For $\varepsilon > 0$, define $\mathbf{A}^{(\varepsilon)} = (A_1^{(\varepsilon)}, A_2) \in [M_n^{sa}]^2$, where $A_1^{(\varepsilon)} = \left(\begin{smallmatrix} \mu_1 & \varepsilon \\ \varepsilon & \mu_2 \end{smallmatrix}\right) \oplus \operatorname{diag}(\mu_3, \ldots, \mu_n)$. Direct calculation shows that $\mathbf{A}^{(\varepsilon)} \notin \mathcal{C}_n$.

Case (iii). Suppose that $\mu_j = \mu$ and $\beta_{jj} = \beta$ for all $j \in \{1, 2, \ldots, n\}$, in which case $A_1 = \mu I$ and $A_2 = \beta I$. Define $\mathbf{A}^{(\varepsilon)} = (A_1^{(\varepsilon)}, A_2^{(\varepsilon)})$ for $\varepsilon \in \mathbb{R} \setminus \{0\}$, where $A_1^{(\varepsilon)} := \left(\begin{smallmatrix} \mu & \varepsilon \\ \varepsilon & \mu \end{smallmatrix}\right) \oplus \mu I_{n-2}$ and $A_2^{(\varepsilon)} := (\beta + \varepsilon)I_1 \oplus \beta I_{n-1}$. Direct calculation shows that $\mathbf{A}^{(\varepsilon)} \notin \mathcal{C}_n$.

In all three cases and for each $j \in \{1, 2\}$, we see that $A_j - A_j^{(\varepsilon)}$ is either the zero matrix, or a diagonal matrix with ε in precisely one position and zeroes elsewhere, or a permutation matrix of $\left(\begin{smallmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{smallmatrix}\right) \oplus 0_{n-2}$. It is then clear that $|||\, \mathbf{A} - \mathbf{A}^{(\varepsilon)} |||\, \to 0$ as $\varepsilon \to 0$. Accordingly, no open ball centred at $\mathbf{A}$ is contained in $\mathcal{C}_n$. ■

The proof of Proposition 5.3 is based on criterion (5.1), which characterises elements of $\mathcal{C}_n$. No such criterion is known for characterising $\mathcal{Z}_n$. Even with the various continuity properties available from Section 4, we have been unable to decide whether or not $\mathcal{Z}_n$ is nowhere dense in $[M_n^{sa}]^2$ (for $n \geq 4$). A positive answer would provide evidence for the Conjecture; a negative answer would show that the Conjecture is false.

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