ON THE TRIVIALITY OF CERTAIN WHITEHEAD GROUPS

By N. Grenier-Boley
Laboratoire de Mathématiques Nicolas Oresme,
Université de Caen BP 5186, 14032 Caen Cedex, France

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Abstract

Let K be a field of arbitrary characteristic and let q be a prime number different from the characteristic of K. If A is a central simple algebra over K whose index is a power of q, we show the triviality of the Whitehead groups $\mathrm{SK}_1(A)$ and $\mathrm{USK}_1(A)$ when the cohomological q-dimension of K is at most 2. We give a global version of this result and indicate what can be done in the case where the index of the algebra is a power of the characteristic of the field. Triviality results of the Whitehead group $\mathrm{K}_1\mathrm{Spin}(A)$ are easily derived.

1. Introduction

Let K be a field of arbitrary characteristic. If A is a (finite dimensional) central simple algebra over K and if $\mathrm{Nrd}_{A/K}$ is the reduced norm map from A to K, we set

$$\mathrm{SL}_1(A) = \{a \in A^* \mid \mathrm{Nrd}_{A/K}(a) = 1\},$$

and we denote by $[A^*, A^*]$ the commutator subgroup of A^*. By definition, the *reduced Whitehead group* of A is the factor group

$$\mathrm{SK}_1(A) = \mathrm{SL}_1(A)/[A^*, A^*].$$

It was conjectured (independently) by Tannaka and Artin (see [12] or [16]) that $\mathrm{SK}_1(A) = 1$ when A is different from $M_2(\mathbb{F}_2)$ and $M_2(\mathbb{F}_3)$. This conjecture is part of a more general conjecture from Kneser and Tits (see [16]) concerning simply connected algebraic groups: more precisely, the Tannaka–Artin Conjecture is equivalent to the Kneser–Tits Conjecture for algebraic groups of type A_n, see [16]. It turns out that these conjectures are false in general, by Platonov's results in [10]. In a series of papers (see references (8) to (14) cited in Yanchevskiĭ's paper [20]), Platonov consequently builds a reduced K-theory and highlights important connections between reduced Whitehead groups and algebraic groups. However, the reduced Whitehead group is trivial in many cases, and it is still interesting to find sufficient conditions over K or over A to guarantee this triviality. For example, an important result achieved by Yanchevskiĭ [19] asserts that $\mathrm{SK}_1(A)$ is trivial whenever the base field is a C_2^0-field.

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*E-mail: grenier@math.univ-paris13.fr
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Since 1973 Platonov and Yanchevskiĭ have been developing a reduced unitary K-theory in which the central object is the *reduced unitary Whitehead group* $\mathrm{USK}_1(A)$. This group is an analog of the reduced Whitehead group for central simple algebras endowed with a unitary involution, and its triviality is equivalent to the Kneser–Tits Conjecture for unitary groups. In [11], Platonov and Yanchevskiĭ show that this group is not always trivial. Nevertheless, Yanchevskiĭ proves that this group is trivial if the center of A is a C_2^0-field.

In the case where the central simple algebra is endowed with a symplectic involution, the analog of the reduced Whitehead group is the group $\mathrm{K}_1\mathrm{Spin}(A)$. In [8], Monastyrnyĭ and Yanchevskiĭ show that this group is not trivial in general, even if it is still the case over C_2^0-fields.

Many results concerning these three types of Whitehead groups are available in the literature and some of them are recalled in Section 2 below.

In [14, §24], as a consequence of their works about norm residue homomorphisms, Merkurjev and Suslin prove that the cohomological p-dimension of a field K is at most 2 if and only if K is a C_2^0-field locally at p, when p is different from the characteristic of the field (see Theorem 3.1 below). In particular, if F is a perfect field, the cohomological dimension of F is at most 2 if and only if F is a C_2^0-field (see Corollary 3.2). Section 3 is dedicated to the recall of these results and focuses on a proof of this last fact.

All these results make us believe that there are local cohomological criteria for the above Whitehead groups to be trivial. Our main result asserts that this is true, modulo a restriction on the index of the algebras:

Theorem 1.1. *Let K be a field of characteristic p (which can be zero) and let q be a prime number different from p. Suppose that $\mathrm{cd}_q(K) \leq 2$ and that A is a central simple algebra over K whose index is q-primary. Then, the Whitehead groups $\mathrm{SK}_1(A)$ and $\mathrm{USK}_1(A)$ are trivial.*

In Section 4, we prove our preliminary results. First, we are interested in scalar extension results (see Lemma 4.3). Then we adapt a result given by Bayer-Fluckiger and Serre in [2] to construct a perfect field satisfying certain properties starting from any field and from any prime number different from the characteristic of the field (see Proposition 4.4). In Section 5, we give the proof of Theorem 1.1. We consequently obtain a triviality result for the group $\mathrm{K}_1\mathrm{Spin}(A)$, stated as Corollary 5.2.

In [5], Gille defines the separable p-dimension of a field based on the notion of p-dimension initially defined by Kato. Using this notion, he completes Merkurjev and Suslin's result cited above in the case where the degree of the algebra is a power of the characteristic of the field (see Theorem 5.4 below). In the last part of Section 5, we indicate that, as a consequence of Gille's result, Theorem 1.1 is still true when $q = p = \mathrm{char}(F)$ if we suppose that the separable p-dimension of F is at most 2 (see Corollary 5.5). Finally, we obtain the following global result:

Theorem 1.2. (1) *Suppose that K is a field whose separable dimension is at most 2. Then, for any central simple algebra A over K, the Whitehead groups $\mathrm{SK}_1(A)$*

and $\mathrm{USK}_1(A)$ *are trivial.*
(2) *Suppose that* K *is a field whose separable* 2*-dimension is at most* 2*. Then, for any central simple algebra* A *over* K*, the Whitehead group* $\mathrm{K}_1\mathrm{Spin}(A)$ *is trivial.*

2. Whitehead groups

This section presents some basic properties and triviality results for the Whitehead groups under study. Other standard facts and results about the reduced Whitehead group can be found in [4, §23] or in [12, chapter 4, §2]. Concerning reduced unitary Whitehead groups, we refer to [11; 20; 21] or [12, chapter 4, §3] for more details. Standard references for the group $\mathrm{K}_1\mathrm{Spin}(A)$ are [8] and [21].

2.1. The group $\mathrm{SK}_1(A)$
Basic properties. Let K be a field and let A be a central simple algebra over K. From now on, G_K will denote the absolute Galois group of K. If p is a prime number, $\mathrm{cd}_p(K)$ is the cohomological p-dimension of G_K and $\mathrm{cd}(K)$ is the cohomological dimension of G_K. We refer to [13] for precise results about Galois cohomology.

If A is split, its reduced Whitehead group is trivial, except if $A = M_2(\mathbb{F}_2)$ or $A = M_3(\mathbb{F}_3)$ (see [4, §20, theorem 4]). In fact, one easily shows that $\mathrm{SK}_1(M_2(\mathbb{F}_2))$ (resp. $\mathrm{SK}_1(M_2(\mathbb{F}_3))$) is a cyclic group of order 2 (resp. of order 3). From now on, when speaking about the reduced Whitehead group of A, we will always implicitly assume that $A \neq M_2(\mathbb{F}_2),\ M_2(\mathbb{F}_3)$.

The study of the reduced Whitehead groups can be reduced to the case of division algebras of p-primary degree, p a prime number. More precisely, if D is a division algebra Brauer-equivalent to A of degree $\prod_{i=1}^n p_i^{n_i}$, it can be decomposed as $D = D_1 \otimes_K \cdots \otimes_K D_n$ where each D_i is a division algebra of degree $p_i^{n_i}$, and we have

$$\mathrm{SK}_1(A) \simeq \mathrm{SK}_1(D) \simeq \coprod_{i=1}^{n} \mathrm{SK}_1(D_i). \tag{2.1}$$

A proof of this result can be found in [4]. Another important property is the fact that the exponent of $\mathrm{SK}_1(A)$ divides the index of A (apply Lemma 4.1 below to a maximal commutative subfield of D).

Triviality results. The reduced Whitehead group is not always trivial. For a great number of fields, among which are C_2^0-fields, it is known that $\mathrm{SK}_1(A)$ is trivial:

Definition 2.1. A field K is a C_2^0-field if, for any finite field extension E/K and for any finite-dimensional division algebra D with center E, the reduced norm map $\mathrm{Nrd}_{D/E} : D^* \to E^*$ is surjective.

Remarks 2.2. (1) One can see that C_2-fields are C_2^0-fields (see [13, chapter 2, §4.5] for the definition of C_2-fields and [12, chapter II, §4.3] for a proof of this result). The class of C_2^0-fields is strictly bigger than the class of C_2-fields as it contains the field $\mathbb{Q}_2$, which is not a C_2-field by a result of Terjanian (see Serre [13, p. 98]).

Moreover, the cohomological dimension of a C_2^0-field is less than or equal to 2, see Corollary 3.2. We thus have:

$$K \text{ is a } C_2\text{-field} \Rightarrow K \text{ is a } C_2^0\text{-field} \Rightarrow \operatorname{cd}(K) \leq 2 \Rightarrow \operatorname{cd}_2(K) \leq 2 \Rightarrow I^3(K) = 0,$$

where $I(K)$ is the fundamental ideal of the Witt ring of K and $I^3(K) := \big(I(K)\big)^3$.
(2) Definition 2.1 does not change if we replace 'division algebra' by 'central simple algebra' (see [4, §22, theorem 3]).

The reduced Whitehead group is trivial in the following cases:

- if K is a p-adic field (Nakayama and Matsushima [9]);
- if K is a global field (Wang [17, theorem p. 329]);
- if K is a C_2^0-field (Yanchevskiĭ [19, theorem p. 492]);
- if the index of A is squarefree (Wang [17]); and
- if $\operatorname{cd}(K) \leq 3$ and if A is a biquaternion algebra (Rost [6, chapter 17] or [7, corollary p. 76]).

Note that the fourth result is no longer true if the index of the algebra is not squarefree as it is shown in [6, example 17.23]. Rost's result is optimal for biquaternion algebras since Platonov's original counter-example to the Tannaka–Artin Conjecture was built from a biquaternion algebra over a certain field K satisfying $\operatorname{cd}(K) \geq 4$. More generally, in [15], Suslin conjectures that $\mathrm{SK}_1(A) = 1$ whenever $\operatorname{cd}(K) \leq 3$.

2.2. The group $\mathrm{USK}_1(A)$

In this subsection, we suppose that A is endowed with a unitary involution σ. In this case, we will always denote by F the subfield of central invariant elements under σ and implicitly assume that K/F is a separable extension in the characteristic 2 case.

Basic properties. The *reduced unitary Whitehead group* of A is the factor group

$$\big(\mathrm{USK}_1\big)_\sigma(A) = \Sigma'_\sigma(A)/\Sigma_\sigma(A),$$

where $\Sigma_\sigma(A)$ (resp. $\Sigma'_\sigma(A)$) is the subgroup of A^* generated by the elements that are symmetric (resp. that have symmetric reduced norm) with respect to σ. We easily see that the definitions of these three groups depend only on the restriction of σ to K (see [18, lemma 1]) and are, respectively, denoted by $\mathrm{USK}_1(A)$, $\Sigma(A)$ and $\Sigma'(A)$.

It turns out that the reduced unitary Whitehead group of A has many properties in common with the reduced Whitehead group. For example, property (2.1) is still true when replacing SK_1 by USK_1 (see [18, lemma 2, lemma 3] and [20, proposition 2.7]) and its exponent divides the index of A. Again, the study of reduced unitary Whitehead groups can be reduced to the case of divison algebras whose degree is p-primary, p is a prime.

It can be shown that $[A^*, A^*] \subseteq \Sigma(A)$ (see [18, lemma 2]), which implies that $\mathrm{USK}_1(A)$ is abelian. This fact can also be used to deduce a link between $\mathrm{SK}_1(A)$ and $\mathrm{USK}_1(A)$. If $x \in \Sigma'(A)$, then we can write $x = \sigma(x)a$, where $a \in \mathrm{SL}_1(A)$.

The canonical surjection from $\mathrm{SL}_1(A)$ to $\mathrm{SK}_1(A)$ induces a group homomorphism $\Phi : \mathrm{USK}_1(A) \to \mathrm{SK}_1(A) : x \mapsto \sigma(x)^{-1}x$.

Lemma 2.3 (Yanchevskiĭ [20, lemma p. 183]). *The exponent of* $\ker \Phi$ *divides* 2. *In particular, if the index of A is odd,* Φ *is injective.*

Triviality results. The reduced unitary Whitehead group of a central simple algebra is not trivial in general (this was first proved by Platonov and Yanchevskii, see [11]). However, one can show that this group is trivial if K is a global field (Platonov and Yanchevskiĭ [11]), if the index of A is a prime number (Yanchevskiĭ [18, lemma 5]) or if K is a C_2^0-field (Yanchevskiĭ [18, theorem 1]).

2.3. The group $\mathrm{K}_1\mathrm{Spin}(A)$

In this subsection, the central simple algebra A is supposed to be endowed with a symplectic involution σ. Let $R(A) = \{a \in A^* \mid \mathrm{Nrd}_{A/K}(a) \in F^{*2}\}$. The involution σ being symplectic, $\Sigma_\sigma(A) \subset R(A)$ (see [6, proposition 2.9]). We may thus define

$$\mathrm{K}_1\mathrm{Spin}(A) = R(A)/\Sigma_\sigma(A)[A^*, A^*].$$

This group does not depend on the choice of σ, is 2-torsion abelian and is not trivial in general (see [8]). However, this group is trivial if $\deg A \leq 4$ by a result of Yanchevskiĭ (see [6, proposition 17.28]) or if K is a C_2^0-field (see Corollary 5.2 for a similar proof).

3. About a result of Merkurjev and Suslin

The purpose of this section is to state two important results that will be useful in the proof of the main Theorem.

Theorem 3.1 (Merkurjev-Suslin [14, theorem 24.8]). *Let K be a field and let p be a prime number different from the characteristic of K. Then the following are equivalent:*
(1) $\mathrm{cd}_p(K) \leq 2$.
(2) *For any finite field extension E/K and for any central simple algebra of center E, if the degree of A is a power of p, then the reduced norm map* $\mathrm{Nrd}_{A/E} : A^* \to E^*$ *is surjective.*

Corollary 3.2 (Merkurjev-Suslin [14, corollary 24.9]). *If K is a perfect field, the following are equivalent:*
(1) $\mathrm{cd}(K) \leq 2$.
(2) *K is a C_2^0-field.*

PROOF. The proof of (1) $\Rightarrow$ (2) is an easy consequence of 2.2(2) and Theorem 3.1. The proof of the converse is based on the fact that $\mathrm{Nrd}_{D/K}$ is surjective if $\mathrm{Nrd}_{D_i/K}$ is surjective for all the p_i-primary components of the division algebra D. For $p_i \neq \mathrm{char}(K)$ (resp. $p_i = \mathrm{char}(K)$), the surjectivity of $\mathrm{Nrd}_{D_i/K}$ follows from Theorem 3.1 (resp. from the perfectness of K). ■

Remark 3.3. Theorem 3.1 and Corollary 3.2 have many important consequences. For example, Theorem 3.1 shows that Serre's Conjecture II is true for special linear groups, and Corollary 3.2 has been used by Bayer-Fluckiger and Parimala to show this conjecture for special unitary groups, see [1, theorem 5.1.2]).

4. Preliminary results

4.1. Scalar extension
Let D be a division algebra over K. To prove our main theorem we also need some results on $D \otimes_K L$, with L/K a finite field extension.

Lemma 4.1 (Wang [17, lemma 3]). *Let L/K be a field extension of degree m. If $\alpha \in D$ is such that $\alpha \otimes 1 \in [(D \otimes_K L)^*, (D \otimes_K L)^*]$, then $\alpha^m \in [D^*, D^*]$.*

Suppose now that D is endowed with a unitary involution σ and that F is, as usual, the subfield of K fixed by σ. The following lemma is an analog of the previous one for USK_1.

Lemma 4.2 (Yanchevskiĭ [18, lemma 4]). *Let L/K be a field extension of degree m containing an element a such that $L = K(a)$ and $[L : F(a)] = 2$. If $\beta \in \Sigma'(D)$ is such that $\beta \otimes 1 \in \Sigma(D \otimes_K L)$, then $\beta^m \in \Sigma(D)$.*

As a consequence, we can prove:

Lemma 4.3. *Let H/F be a field extension of degree m that is linearly disjoint from the extension K/F, and let L be the composite of K and H. If $\beta \in \Sigma'(D)$ is such that $\beta \otimes 1 \in \Sigma(D \otimes_K L)$, then $\beta^m \in \Sigma(D)$.*

Proof. There exist $a_1, \cdots, a_n$ such that $H = F(a_1, \cdots, a_n)$ and $L = K(a_1, \cdots, a_n)$. For $i = 1, \cdots, n$, write $H_i = F(a_1, \cdots, a_i)$ and $L_i = K(a_1, \cdots, a_i)$. As K and H are linearly disjoint over F, $[L_i : K_i] = [K : F] = 2$ (see [3, chapitre V, §2, no. 3, proposition 6]). Consequently, the field extensions L_i/K_i are Galois (in the characteristic 2 case, recall that K/F and thus that the L_i/K_i's are separable). The central simple L-algebra $D \otimes_K L$ is endowed with the unitary involution $\sigma \otimes \theta$, where θ is the nontrivial automorphism of L over H. If $D' = D \otimes_K L_{n-1}$, there is an L-algebra isomorphism $D \otimes_K L \simeq D' \otimes_{L_{n-1}} L$. Note that, by definition of θ, $\theta|_{L_{n-1}}$ is the nontrivial automorphism of L_{n-1} over K_{n-1}. If $\sigma' = \sigma \otimes \theta|_{L_{n-1}}$, $\sigma' \otimes \theta$ is a unitary involution over $D' \otimes_{L_{n-1}} L$. Now $\beta \in \Sigma'(D')$ and $\beta \otimes 1 \in \Sigma(D' \otimes_{L_{n-1}} L)$. By Lemma 4.2, it follows that

$$(\beta \otimes 1)^{[L:L_{n-1}]} \in \Sigma(D') = \Sigma(D \otimes_K L_{n-1}).$$

By induction over n, we get that $\beta^m \in \Sigma(D)$, where $m = [L : L_{n-1}] \cdots [L_1 : K] = [L : K]$. ■

4.2. Construction of fields satisfying certain conditions
In this subsection, we construct a field extension of the base field K with specific properties. This field extension will be used in the proof of the main theorem.

Proposition 4.4. *Let K be a field and let q be a prime number different from* $\mathrm{char}(K) = p$ *(p can possibly be zero). There exists an algebraic extension K'/K having the following properties:*
(1) *K' is a filtered union of field extensions of K whose degree is prime to q.*
(2) *K' is a perfect field.*
(3) *$G_{K'}$ is a pro-q-group.*
Moreover $\mathrm{cd}(K') = \mathrm{cd}_q(K)$.

PROOF. When $q = 2$, this result is due to Bayer-Fluckiger and Serre in [2, proposition 2.3.1]. The proof of the general case we present here uses the same type of arguments.

Denote by $S_q(G_K)$ a q-Sylow of G_K (which exists in accordance with [13, chapitre I, §1, proposition 3]), and denote by K_q the subfield of K_{sep} fixed by $S_q(G_K)$. Then $K_q = \bigcup_{x\in K_q} K(x)$. The field extension K_q/K is thus a union of field extensions of K whose degree is prime to q. The Theorem of the Primitive Element implies that this is a filtered union. If $\mathrm{char}(K) = 0$, we choose $K' = K_q$.

Now suppose that $\mathrm{char}(K) = p \neq 0$, and denote by $K' = K_q^{p^{-\infty}}$. Then

$$K' = \bigcup_{L\in J} L, \tag{4.1}$$

where J is the set of finite degree field extensions L/K such that there exist $e \in \mathbb{N}$ and $x \in K_q$ with $L \subset \big(K(x)\big)^{p^{-e}}$. This union is a filtered union. Let $L/K \in J$. Then there exist $x \in K_q$ and $e \in \mathbb{N}$ such that $L \subset \big(K(x)\big)^{p^{-e}}$. As $x \in K_q$, $[K(x) : K]$ and q are coprime. Moreover, the field extension $LK(x)/K(x)$ is purely inseparable and, by [3, chapitre V, §8, proposition 6], its degree is a power of p. Consequently, $[L : K]$ and q are coprime hence K' satisfies condition (1). By definition, K' is the smallest perfect subfield of $\overline{K_q}$ containing K_q, so it satisfies condition (2).

By [13, chapitre II, §4.1], $G_{K'}$ can be identified to a subgroup of G_{K_q} and its index is equal to the separability degree $[K' : K_q]_s = 1$. Thus, $G_{K'} = G_{K_q} = S_q(G_K)$ is a pro-q-group and condition (3) follows. Finally, $\mathrm{cd}(K') = \mathrm{cd}_q(K)$ by [13, chapitre I, §3.3, corollaire 1]. ■

5. Proofs of the triviality results

In this section we give the proof of our results on the triviality of $\mathrm{SK}_1(A)$, $\mathrm{USK}_1(A)$ and $\mathrm{K}_1\mathrm{Spin}(A)$.

5.1. Case where the index is prime to the characteristic
Theorem. *Let K be a field of characteristic p (which can be zero) and let q be a prime number different from p. Suppose that* $\mathrm{cd}_q(K) \leq 2$ *and that A is a cen-*

tral simple algebra over K whose index is q-primary. Then, the Whitehead groups $\mathrm{SK}_1(A)$ *and* $\mathrm{USK}_1(A)$ *are trivial.*

PROOF. It suffices to prove the statement for division algebras of q-primary degree (see Section 2 above). Suppose that D is such an algebra.

We first show that the reduced Whitehead group of D is trivial. Let $b \in \mathrm{SL}_1(D)$. By Proposition 4.4, there exists a perfect field L that is a filtered union of field extensions whose degree is prime to q and such that $\mathrm{cd}(L) \leq 2$. By Corollary 3.2, L is thus a C_2^0-field, hence $b \otimes 1 \in \mathrm{SL}_1(D \otimes_K L) = [(D \otimes_K L)^*, (D \otimes_K L)^*]$ by Yanchevskiĭ's result (see Subsection 2.1). We can suppose that $b \otimes 1 \in [(D \otimes_K K_i)^*, (D \otimes_K K_i)^*]$ where $[K_i : K]$ and q are coprime. On the one hand, by Lemma 4.1, it follows that $b^{[K_i:K]} \in [D^*, D^*]$. On the other hand, $b^{\deg D} \in [D^*, D^*]$ (see Subsection 2.1). Finally, the degree of D being q-primary, $b \in [D^*, D^*]$, thus proving that $\mathrm{SK}_1(D)$=1.

Next, we show that the reduced unitary Whitehead group of D is trivial. Let σ be a unitary involution over D. If q is odd, $\mathrm{USK}_1(D) \hookrightarrow \mathrm{SK}_1(D)$ by Lemma 2.3, so $\mathrm{USK}_1(D) = 1$. Suppose now that $q = 2$ and that $p = \mathrm{char}(K) \neq 2$. Again, we apply Proposition 4.4 to find a perfect field L, which is a filtered union of field extensions of F of odd degree with $\mathrm{cd}(L) \leq 2$. Let $L = \bigcup_{i \in I} F_i$ be such that $[F_i : F]$ is odd. For $i \in I$, let $E_i = KF_i$ and let $E = \bigcup_{i \in I} E_i$. Thus, $[E_i : F_i] = 2$ and $[E_i : K]$ is odd for every $i \in I$, which means that E is also a filtered union of field extensions of K of odd degree. Moreover, E/L is a quadratic field extension with nontrivial automorphism τ, and $D \otimes_K E$ is a central simple algebra over E that is endowed with the unitary involution $\sigma \otimes \tau$. Let $a \in \Sigma'(D)$. Then $a \otimes 1 \in \Sigma'(D \otimes_K E)$. The field L is a C_2^0-field by Corollary 3.2, so we obtain that $a \otimes 1 \in \Sigma(D \otimes_K E)$ by Yanchevskiĭ's result (see Subsection 2.2). We can suppose that there exists $j \in I$ such that $a \otimes 1 \in \Sigma(D \otimes_K E_j)$, and, by Lemma 4.3, we deduce that $a^{[E_j:K]} \in \Sigma(D)$. Finally, let

$$b = a^{\deg D} (\mathrm{Nrd}_{D/K}(a))^{-1}.$$

One checks that $b \in \mathrm{SL}_1(D)$ so $b^{\deg D} \in [D^*, D^*]$. As $[D^*, D^*] \subset \Sigma(D)$, it follows that $a^{\deg D^2} \in \Sigma(D)$. The degree of D being even, $\mathrm{USK}_1(D) = 1$. ■

Remark 5.1. The proof of the preceding theorem for $\mathrm{USK}_1(A)$ in the case where $q \neq 2$ can be obtained independently from considerations about $\mathrm{SK}_1(A)$ by applying Lemma 4.3.

As an application, we obtain the following triviality result for $\mathrm{K}_1\mathrm{Spin}$:

Corollary 5.2. *Let K be a field of characteristic different from* 2 *satisfying* $\mathrm{cd}_2(K) \leq 2$. *If A is a central simple algebra over K, then* $\mathrm{K}_1\mathrm{Spin}(A) = 1$.

PROOF. This proof is analogous to the case where the field is supposed to be a C_2^0-field. Suppose that σ is a symplectic involution over A. In this case, it is well known that the index of A must be a power of 2 (see [6, corollary 2.8]). Let

$a \in R(A)$ and $\mathrm{Nrd}_{A/K}(a) = \alpha^2$, $\alpha \in K^*$. By Theorem 3.1, there exists $b \in A^*$ such that $\mathrm{Nrd}_{A/K}(b) = \alpha$, and we can write $a = b^2c$ with $c \in \mathrm{SL}_1(A)$. By Theorem 1.1, $c \in [A^*, A^*]$. It suffices to show that $b^2 \in \Sigma(A)[A^*, A^*]$. By [6, proposition 4.17], there exists $g \in A^*$ such that the involution $\mathrm{Int}(g) \circ \sigma$ leaves b invariant. Finally,

$$b\sigma(b) = b^2(b^{-1}gbg^{-1}) \in [A^*, A^*],$$

thus $\mathrm{K}_1\mathrm{Spin}(A) = 1$. ∎

5.2. Case where the index is a power of the characteristic

We want to state analogs of Theorem 1.1 and Corollary 5.2 in the case where the index of A is a power of the characteristic of K. To do this, we need the notion of separable p-dimension introduced by Gille in [5]. This notion is a separable version of the p-dimension initially defined by Kato. More precisely:

Definition 5.3. Let p be a prime number. The *separable p-dimension* of K denoted by $\dim_{\mathrm{p}}^{\mathrm{sep}}(K)$ is defined as follows. If $p \neq \mathrm{char}(K)$, then $\dim_{\mathrm{p}}^{\mathrm{sep}}(K) = \mathrm{cd}_p(K)$. If $p = \mathrm{char}(K)$, then $\dim_{\mathrm{p}}^{\mathrm{sep}}(K)$ is defined as the infimum of the integers r, such that $H_p^{r+1}(K') = 0$ for every finite separable field extension K'/K (where $H_p^{r+1}(K')$ is the corresponding Kato cohomology group of K', see [5, §1.1 (a)]). The supremum of the separable p-dimensions of K where p runs over all prime numbers is called the *separable dimension* of K.

The following theorem extends Merkurjev and Suslin's result 3.1 to every prime number:

Theorem 5.4 (Gille [5, theorem 7]). *Let K be a field and let p be a prime number. Then, the following are equivalent:*
(1) $\dim_{\mathrm{p}}^{\mathrm{sep}}(K) \leq 2$.
(2) *For any finite separable field extension E/K and for any central simple algebra A over E, if the degree of A is a power of p then the reduced norm map $\mathrm{Nrd}_{A/E} : A^* \to E^*$ is surjective.*

We can now be more precise about the missing cases in Theorem 1.1 and Corollary 5.2:

Corollary 5.5. (1) *Let K be a field of characteristic $p \neq 0$ and suppose that $\dim_{\mathrm{p}}^{\mathrm{sep}}(K) \leq 2$. Let A be a central simple algebra over K whose index is p-primary. Then the Whitehead groups $\mathrm{SK}_1(A)$ and $\mathrm{USK}_1(A)$ are trivial.*
(2) *Let K be a field of characteristic 2. Suppose that $\dim_2^{\mathrm{sep}}(K) \leq 2$ and that A is a central simple algebra over K. Then $\mathrm{K}_1\mathrm{Spin}(A) = 1$.*

Proof. (1) The result is obtained by mimicking Yanchevskiĭ's proofs of the triviality of $\mathrm{SK}_1(A)$ (resp. $\mathrm{USK}_1(A)$) over C_2^0-fields, see [19] (resp. [18]). The only difference is to use Theorem 5.4(2) whenever the surjectivity of a reduced norm

map is needed in the original proofs.
(2) The proof is analoguous to Corollary 5.2 using (1) and Theorem 5.4. ■

5.3. Global results
We summarize all the previous results in a unified way using Theorem 1.1 and Corollary 5.5:

Theorem. (1) *Suppose that K is a field whose separable dimension is at most* 2. *Then, for any central simple algebra A over K, the Whitehead groups* $\mathrm{SK}_1(A)$ *and* $\mathrm{USK}_1(A)$ *are trivial.*
(2) *Suppose that K is a field whose separable* 2*-dimension is at most* 2. *Then, for any central simple algebra A over K, the Whitehead group* $\mathrm{K}_1\mathrm{Spin}(A)$ *is trivial.*

Remarks 5.6. (1) Note that, in (1), the assertion is still true under the weaker hypothesis that $\mathrm{cd}(K) \leq 2$ if we suppose further that the index of A is prime to the characteristic of K.
(2) In [22], Yanchevskiĭ has proved that if $\mathrm{char}(K) = 0$ and if $\mathrm{vcd}(K) := \mathrm{cd}\big(K(\sqrt{-1})\big) \leq 2$, then $\mathrm{SK}_1(A) = \mathrm{USK}_1(A) = \mathrm{K}_1\mathrm{Spin}(A) = 1$ for every central simple algebra A over K.

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References

[1] E. Bayer-Fluckiger and R. Parimala, Galois cohomology of the Classical groups over fields of cohomological dimension $\leqq 2$, *Inventiones Mathematicae* **122** (1995), 195–229.
[2] E. Bayer-Fluckiger and J.-P. Serre, Torsions quadratiques et bases normales autoduales, *American Journal of Mathematics* **116** (1994), 1–63.
[3] N. Bourbaki, Corps commutatifs, in Bourbaki, *Algèbre*, Chapitre 5, Hermann, Paris, 1950.
[4] P.K. Draxl, *Skew fields*, London Mathematical Society Lecture Note Series, vol. 81, Cambridge University Press, Cambridge, 1983.
[5] P. Gille, Invariants cohomologiques de Rost en caractéristique positive (French) [Rost's cohomological invariants in positive characteristic], *K-Theory* **21** (2000), 57–100.
[6] M.-A. Knus, A.S. Merkurjev, M. Rost and J.-P. Tignol, *The book of involutions*, Colloquium Publications 44, American Mathematical Society, Providence, R.I., 1998.
[7] A.S. Merkurjev, K-theory of simple algebras, *Proceedings of Symposia in Pure Mathematics*, **58.1** (1995), 65–83.
[8] A.P. Monastyrnyĭ and V. Yanchevskiĭ, Whitehead groups of spinor groups (Russian), *Izvestija Akademii Nauk SSSR Seriya Matematicheskaya* **54** (1990), 60–96, 221; English translation: *Mathematics of the USSR Izvestija* **36** (1991), 61–100
[9] T. Nakayama and Y. Matsushima, Über die multiplikative Gruppe einer p-adischen Divisionsalgebra, *Proceedings of the Imperial Academy of Tokyo* **19** (1943), 622–8.
[10] V. P. Platonov, On the Tannaka–Artin problem, *Doklady Akademii Nauk SSSR* **221** (1975), 1038–41; English translation: *Soviet Mathematics Doklady* **16** (1975), 468–73.
[11] V.P. Platonov and V. Yanchevskiĭ, The Kneser–Tits conjecture for unitary groups, *Doklady*

Akademii Nauk SSSR **225** (1975), 48–51; English translation: *Soviet Mathematics Doklady* **16** (1975), 1456–60.

[12] V.P. Platonov and V. Yanchevskiĭ, *Finite-dimensional division algebras*, Algebra IX, Finite groups of Lie Type, Finite-Dimensional Division Algebras, in A.I. Kostrikin and I.R. Shafarevich, eds, *Encyclopaedia of Mathematical Science*, vol. 77, 121–233, Springer-Verlag, Berlin, 1996.

[13] J.-P. Serre, *Cohomologie galoisienne*, Lecture notes in Mathematics **5**, Springer-Verlag, Berlin, 1994.

[14] A.A. Suslin, Algebraic K-theory and the norm residue homomorphism, *Journal of Soviet Mathematics* **30** (1985), 2556–611.

[15] A.A. Suslin, SK_1 of division algebras and Galois cohomology, *Advances in Soviet Mathematics* **4** (1991), 75–99.

[16] J. Tits, Groupes de Whitehead de groupes algèbriques simples sur un corps, *Séminaire Bourbaki*, **29** (1976–77), 218–36.

[17] S. Wang, On the commutator group of a simple algebra, *American Journal of Mathematics* **72** (1950), 323–34.

[18] V. Yanchevskiĭ, Simple algebras with involution and unitary groups, *Matematicheskii Sbornik* **93** (1974), 368–80; English translation: *Mathematics of the USSR Sbornik*, **22** (1974), 372–84.

[19] V. Yanchevskiĭ, The commutator subgroups of simple algebras with surjective reduced norms, *Doklady Akademii Nauk SSSR* **221** (1975), 1056–8; English translation: *Soviet Mathematics Doklady* **16** (1975), 492–5.

[20] V. Yanchevskiĭ, Reduced unitary K-theory and division rings over discretely valued Hensel fields, *Izvestija Akademii Nauk SSSR Seriya Matematicheskaya* **42** (1978), 879–918; English translation: *Mathematics of the USSR Izvestija* **13** (1979), 175–213.

[21] V. Yanchevskiĭ, Symmetric and skew-symmetric elements of involutions, associated groups and the problem of decomposability of involutions, *Proceedings of Symposia in Pure Mathematics* **58.2** (1995), 431–44.

[22] V. Yanchevskiĭ, Whitehead groups and groups of R-equivalence classes of linear algebraic groups of non-commutative classical type over some virtual fields, in S.G. Dani (ed.) *Algebraic groups and arithmetic*, Proceedings of the international conference, Mumbai, India, 17–22 December 2001, Tata Institute of Fundamental Research, 491–505, Narosa Publishing, New Delhi, 2004.