ON δ^* SETS AND THE GENERATED IDEAL

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Abstract

In this paper, the concept of δ^*-sets is introduced and the ideal Ω generated by all δ^*-sets in a topological space is considered. Some properties of δ^*-sets, the ideal Ω and their relations with resolvability, irresolvability, the Volterra property and other ideals are explored. It is shown that Ω is compatible with the topology for any given space, which is an analogue to the Banach Category theorem.

1. Introduction

A nonempty collection $\mathbf{I}$ of subsets on a topological space (X, τ) is called a topological ideal on (X, τ) if it satisfies the following two conditions :
(1) If $A \in \mathbf{I}$ and $B \subseteq A$ then $B \in I$ (heredity).
(2) If $A \in \mathbf{I}$ and $B \in \mathbf{I}$ then $A \bigcup B \in \mathbf{I}$ (finite additivity).
If (2) is replaced by $\bigcup_{n<\omega} A_n \in \mathbf{I}$ for any sequence $\{A_n : n < \omega\}$ in $\mathbf{I}$, then $\mathbf{I}$ is called a σ-ideal. An ideal $\mathbf{I}$ is called proper if $X \notin \mathbf{I}$.

The following collections form important ideals in a topological space (X, τ).
The ideal of all finite subsets.
The ideal of all countable subsets.
The ideal of all closed and discrete subsets.
The ideal of all nowhere dense subsets (denoted by $\mathbf{I_n}$).
The σ-ideal of all meager subsets (denoted by $\mathbf{I_m}$).
The ideal of all scattered sets (here X must be a T_D space [7]).
The σ-ideal of all Lebesgue measure zero sets in R_n.
The ideal generated by all σ-nowhere dense sets.

Ideals in general topological spaces were considered in [16] and a more modern study can be found in [14].

Chattopadhyay and Bandyopadhyay [3] introduced the concept of δ-sets in a topological space X. A set $B \subset X$ is a δ-set if *int cl* $A \subset$ *cl int* A. δ-sets possess the following properties:
(1) $A \subset X$ is a δ-set iff $A = O \bigcup N$, where O is open and N is nowhere dense in X.
(2) The finite union of δ-sets is a δ-set.
(3) The complement of a δ-set is a δ-set.
(4) The finite intersection of δ-sets is a δ-set.
(5) Every nowhere dense set in X is a δ-set.

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Recall that a space is resolvable if it contains two disjoint dense subsets. Otherwise it is called irresolvable [13]. A space X is called strongly irresolvable or open hereditarily irresolvable [4] if every open subspace of X is irresolvable. Hewitt [13] started the study of resolvable and irresolvable spaces in 1943. Hewitt proved that every topological space X has a unique representation $X = F \bigcup G$, where F is closed and resolvable and G is open and hereditarily irresolvable and $F \bigcap G = \emptyset$. Later on, Elkin [9], Ganster [10; 11] with his coauthors and Chattopadhyay [4; 5 and 6] with his coauthors studied resolvable and irresolvable spaces to some extent. The study of resolvability and irresolvability from different angles has become an important area of research in general topology. Recently, Dontchev, Ganster and Rose [8] studied ideal resolvability. The following interesting result, which will be useful in this paper, has been proved in [4].

Theorem 1.1. *(See [4]) A space (X, τ) is open hereditarily irresolvable iff τ^δ forms the discrete topology on X, where τ^δ denotes the collection of all δ-sets in (X, τ).*

Let us now define a δ^*-set in a topological space (X, τ).
A set $B \subset X$ is said to be a δ^*-set if $B \in \tau^\delta$ and for every $P \subset B$, $P \in \tau^\delta$. Note that every nowhere dense set is a δ^*-set. Let Ω denote the collection of all δ^*-sets in (X, τ). Clearly, $I_n \subset \Omega$. Since the σ-ideal I_m plays an important role in the study of Baire spaces and other related properties, and in particular the Banach Category theorem can be formulated by using the σ-ideal I_m, it is natural to consider which ideal might play a similar role in the study of irresolvable spaces, and whether or not there exists an ideal analogue to the Banach Category theorem for the property of irresolvability. In this context, the papers [14] and [15] may be referred to.

We will show in this paper that Ω is such an ideal. Along with some basic properties of Ω in Section 2, relations between the irresolvability property and Ω are investigated; and in Section 3, the compatibility of Ω with the topology of any given space is established.
Let 'int' and 'cl' denote interior and closure w.r.t the whole space (X, τ), and if $A \subset X$, then $'int'_A$ and $'cl'_A$ denote interior and closure w.r.t the subspace $(A, \tau/A)$.

2. Basic properties of Ω

Theorem 2.1. *Let (X, τ) be a topological space. Then the set Ω of all δ^*-sets in (X, τ) forms an ideal.*

Proof. The proof follows from the definition of a δ^*-set and the fact that finite union of δ-sets is a δ-set. ■

Noting the fact that every open subspace of a resolvable space is resolvable, we have:

Theorem 2.2. *A topological space (X, τ) is resolvable iff $I_n = \Omega$.*

Proof. Let (X, τ) be resolvable. We already have $I_n \subset \Omega$. Let $A \in \Omega$. If possible, let $A \notin I_n$. Then A is not nowhere dense in τ and $A \in \tau^\delta$. So, $intclA \neq \emptyset$ and $intclA \subset clintA$. Thus, $intA \neq \emptyset$. Let $B = intA$. Since (X, τ) is resolvable, so is (B, τ_B). So there are disjoint sets, say D_1, D_2, dense in (B, τ_B) such that $D_1 \bigcup D_2 = B$. We claim that $D_1 \notin \tau^\delta$. On the contrary, suppose that $int\ clD_1 \subset cl\ intD_1$; then, since $cl_B D_1 = B \cap clD_1 = B$, $clD_1 \supset B$. That is, $intclD_1 \supset B$ (B being an open set in τ). So $intD_1 \neq \emptyset$, and thus $int_B D_1 = B \bigcap intD_1 \neq \emptyset$ (since $D_1 \subset B$), which is a contradiction to the fact that D_2 is dense in $(B, \tau/B)$. Thus $D_1 \notin \tau^\delta$, and hence it follows that $A \notin \Omega$, which is a contradiction. Hence, if (X, τ) is resolvable, $I_n = \Omega$. Conversely, let $I_n = \Omega$. If possible, let(X, τ) be irresolvable. Then, by Hewitt's representation theorem, X can be represented uniquely as $X = F \bigcup G$, where F is closed and resolvable and G is open and hereditarily irresolvable in (X, τ) where $F \bigcap G = \emptyset$. So $G \neq \emptyset$.

Now we have the following lemmas:

Lemma 2.1. *Let A be open in (X, τ) and $B \subset A$. If B is a δ-set in $(A, \tau/A)$, then $B \in \tau^\delta$.*

PROOF. (1) Let $int_A\ cl_A\ B \subset cl_A\ int_A\ B$.
We show that $int\ cl\ B \subset cl\ int\ B$. Let $H \in \tau$ such that $H \subset cl\ B$. It suffices to show that $H \bigcap int\ B \neq \emptyset$. Now $cl\ B = A \bigcap cl\ B$. Also $H \bigcap B \neq \emptyset \Rightarrow H \bigcap A \neq \emptyset$ and $A \in \tau \Rightarrow H \bigcap A \in \tau$. Now $int_A\ cl_A B = int_A(A \bigcap cl\ B)$. Therefore, $H \bigcap A \subset cl\ B, H \bigcap A \subset A \Rightarrow H \bigcap A \subset A \bigcap cl\ B \Rightarrow H \bigcap A \subset int_A(A \bigcap cl\ B)$(since $H \bigcap A \in \tau$ and $H \bigcap A \subset A$).
Thus, (2) $H \bigcap A \subset int_A cl_A B$.
Now (1), (2) and $H \bigcap A \in \tau/A \Rightarrow H \bigcap A \bigcap int_A B \neq \emptyset \Rightarrow H \bigcap int\ B \neq \emptyset$. Hence, the lemma is proved. ■

Lemma 2.2. *If $(A, \tau/A)$ is an open and hereditarily irresolvable subspace of (X, τ), then $A \in \Omega$.*

PROOF. Since A is open in τ, $A \in \tau^\delta$. By Theorem 1.1, since $(A, \tau/A)$ is hereditarily irresolvable, every subset B of A is a δ set in $(A, \tau/A)$. Since A is open in τ, by Lemma 2.1, for every subset B of A, $B \in \tau^\delta$. Thus $A \in \Omega$. Hence, the lemma is proved. ■

It now evidently follows that $G \in \Omega$. Since $G \neq \emptyset$ and G is open in τ, it follows that $G \notin I_n$; which is a contradiction to the fact that $I_n = \Omega$. This completes the proof of the theorem. ■

Note 2.3. It follows that a topological space (X, τ) is irresolvable iff $I_n \neq \Omega$. Now in [1], Anderson proved a general existence theorem on connected irresolvable Hausdorff spaces, which shows that such spaces are numerous. Thus, it follows that $I_n \neq \Omega$ for topological spaces, which exist enormously.

Theorem 2.4. *Ω is a proper ideal iff (X, τ) has a nonempty open resolvable subspace.*

Proof. Let Ω be proper. Then $X \notin \Omega \Rightarrow X$ is not a δ^*-set in (X, τ); that is, there exists $A \subset X$ such that $A \notin \tau^\delta$. This implies that τ^δ is not the discrete topology on X. Hence, by Theorem 1.1, X is not open hereditarily irresolvable. So there exists a nonempty open resolvable subspace of X. Conversely, if $X \in \Omega$, then every subset of X is a δ-set in (X, τ). Hence, by Theorem 1.1, (X, τ) is open hereditarily irresolvable. Hence the theorem. ■

In [2] and [12], Cao, Gauld, Greenwood and Piotrowski studied the Volterra property. A topological space (X, τ) is called a Volterra space if the intersection of any two dense G_δ-sets is dense. The class of Volterra spaces is closely related to the class of Baire spaces. We now relate Ω with the Volterra property of a topological space.

Theorem 2.5. *If Ω is not proper, then (X, τ) is Volterra.*

Proof. If Ω is not proper, then, by Theorem 2.3, (X, τ) has no nonempty open resolvable subspace. So (X, τ) is open hereditarily irresolvable. Now consider two dense G_δ-sets A and B in (X, τ). By Theorem 1.1, $A \in \tau^\delta, B \in \tau^\delta$. Therefore, $int\ A, int\ B$ are dense in (X, τ). So $int(A \bigcap B)$ is dense in (X, τ); that is, $A \bigcap B$ is dense in (X, τ). Hence (X, τ) is Volterra. ■

3. Compatibility of Ω and τ for a topological space (X, τ)

In this section we establish the compatibility of Ω and τ for any topological space (X, τ).

Definition 3.1. Let (X, τ) be a topological space. An ideal I on X is said to be compatible with τ, denoted by $I \sim \tau$, if, for each $A \subset X$ and for every point $x \in A$, there is an open neighbourhood U of x such that $U \bigcap A \in I$, then $A \in I$.

It is easy to see that $I_n \sim \tau$ for any topological space (X, τ). The next theorem is called the Banach category theorem in the literature (see [16]).

Theorem 3.1. *(See [16]) For any topological space (X, τ), $I_m \sim \tau$.*

Several equivalent versions and generalizations via ideals of Theorem 3.1 are given in [15]. Also in [15], it is shown that there is a space (X, τ) and an ideal between I_n and I_m on X that is not compatible with τ. However, Ω is a compatible ideal, as the next theorem shows.

Theorem 3.2. *Let (X, τ) be a topological space. Then $\Omega \sim \tau$.*

PROOF. Let $A \subset X$ be such that for each $x \in A$, $\exists$ an open neighbourhood U of x such that $U \bigcap A \in \Omega$. We shall show that $A \in \Omega$. First, we show that $A \in \tau^\delta$. If possible, let $A \notin \tau^\delta$. Then $int\ cl\ A$ is not a subset of $cl\ int\ A$. Let $x \in int\ cl\ A$ but $x \notin cl\ int\ A$. So $\exists$ an open set G_x containing x, such that $G_x \bigcap int\ A = \emptyset$. Now consider any open neighbourhood U of x. We assert that $A \bigcap U \notin \Omega$. Let $G_x \bigcap U = H_x$. Then H_x is an open set, $x \in H_x$ and $H_x \bigcap int\ A = \emptyset$. Now $x \in int\ cl\ A \Rightarrow \exists$ an open set P_x containing x, such that $P_x \subset cl\ A$. So $P_x \bigcap H_x \subset cl\ A \Rightarrow P_x \bigcap H_x \bigcap A \neq \emptyset \Rightarrow H_x \bigcap A \neq \emptyset$. Then $H_x \bigcap A \subset U \bigcap A$. We now show that $H_x \bigcap A \notin \tau^\delta$. Set $S = H_x \bigcap A$. Then:

(1) $int\ S = int\ (H_x \bigcap A) = H_x \bigcap int\ A = \emptyset$.

But $cl\ S = cl(H_x \bigcap A)$. Now let $P_x \bigcap H_x = T_x$. Then $x \in T_x$ and T_x is an open set and $T_x \subset cl\ A \Rightarrow T_x \subset cl(T_x \bigcap A)$. Therefore, $T_x \subset cl\ (P_x \bigcap H_x \bigcap A) \subset cl\ (H_x \bigcap A) = cl\ S$ so that:

(2) $int\ cl\ S \neq \emptyset$.

Thus, from (1) and (2) it is clear that $S \notin \tau^\delta$. Hence, $S = H_x \bigcap A \subset U \bigcap A$ and $S \notin \tau^\delta \Rightarrow U \bigcap A \notin \Omega$. So, under the supposition, it follows that $A \in \tau^\delta$.

We now show that if $B \subset A$ then $B \in \tau^\delta$. Let $B \subset A$. If possible, let $B \notin \tau^\delta$. Then, by a similar argument as given above, we can show that

$$S' = H_x \bigcap B \subset U \bigcap B \subset U \bigcap A,$$

where $S' \neq \emptyset$ and $S' \notin \tau^\delta$. This proves that $U \bigcap A \notin \Omega$. Hence, under the supposition, $B \subset A \Rightarrow B \in \tau^\delta$. Thus, finally it follows under the supposition that $A \in \Omega$. Hence $\Omega \sim \tau$. ■

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