ON THE RANGE OF THE ELEMENTARY OPERATOR $X \longmapsto AXA - X$

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ABSTRACT

Let L(H) denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space H into itself. Given $A \in L(H)$, we define the elementary operator $\Delta_A : L(H) \longrightarrow L(H)$ by $\Delta_A(X) = AXA - X$. In this paper, we initiate the study of the class of operator A for which $\overline{R(\Delta_A)} = \overline{R(\Delta_{A^*})}$, where $\overline{R(\Delta_A)}$ denotes the norm closure of the range of Δ_A. We call such operators quasi-adjoint. We give a characterization and some basic results concerning this class of operator.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H into itself. Given $A, B \in L(H)$, we define the elementary operator $\Delta_{A,B}$ as follows:

$$\begin{aligned} \Delta_{A,B} : L(H) &\longrightarrow L(H) \\ X &\longmapsto \quad \Delta_{A,B}(X) = AXB - X. \end{aligned}$$

Taking $A = B$, we denote $\Delta_{A,A} = \Delta_A$. The properties of elementary operators, their spectrum (see [7; 9; 10 and 11]) and ranges (see [1; 2; 3; 4; 5; 8; 12; 13; 14 and 15]) have been much studied, and many of their problems remain also open. Our aim in this paper is a modest one. Here, the particular classes that have received a lot of attention are those consisting of operators A for which $\overline{R(\Delta_A)} = \overline{R(\Delta_{A^*})}$, where $\overline{R(\Delta_A)}$ is the closure of the range, $R(\Delta_A)$ of Δ_A in the norm topology. Such

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operators are called quasi-adjoint. We give a characterization of and some basic properties about this class of operators. Finally, we pose an open problem.

Preliminaries. Let $C_1(H)$ be the ideal of trace class operators, that is, all compact operators $A \in L(H)$ for which the eigenvalues of $(T^*T)^{\frac{1}{2}}$, counted according to multiplicity, are summable. The ideal $C_1(H)$ admits a complex valued function $tr(T)$, which has the characteristic properties of the trace of matrices. The trace function is defined by $tr(T) = \sum_n (Te_n, e_n)$, where (e_n) is any complete orthonormal sequence in H. As a Banach space, $C_1(H)$ may be identified with the conjugate space of the ideal $K(H)$ of compact operators by means of the linear isometry $T \longmapsto f_T$, where $f_T(X) = tr(XT)$. Moreover, $L(H)$ is the conjugate space of $C_1(H)$. The ultra-weak continuous linear functionals on $L(H)$ are those of the form f_T for some $T \in C_1(H)$, and the weak continuous linear functionals on $L(H)$ are those of the form f_T where T is of finite rank. In addition to the notation already introduced, we shall also use the following notation. Given $A \in L(H)$, we shall denote the kernel, the weak closure and the ultra-weak closure of the elementary operator Δ_A by $\ker(\Delta_A)$, $\overline{R(\Delta_A)}^{W}$ and $\overline{R(\Delta_A)}^{W^*}$, respectively. Recall that for $x, y \in H$, the operator $x \otimes y \in L(H)$ is defined by $(x \otimes y)z = (z, y)x$ for $z \in H$.

2. Main results

Definition 2.1. *Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$. We say that a is quasi-adjoint if $\overline{R(\Delta_a)} = \overline{R(\Delta_{a^*})}$.*

Remark. *Let $A \in L(H)$, then A is quasi-adjoint if and only if $\overline{R(\Delta_A)}$ is a self-adjoint subspace of $L(H)$. Equivalently, $R(\Delta_A)^\circ$ the annihilator of $R(\Delta_A)$ is a self-adjoint subspace of $L^{'}(H)$ in the sense that $f \in R(\Delta_A)^\circ$ implies $f^* \in R(\Delta_A)^\circ$.*

Theorem 2.2. *[16] Let E, F be Banach spaces and let $S \in L(E, F)$ be a bounded operator. Then*

$$R(S^{**})^\circ = (R(S^{**})^\circ \cap F^\circ) \oplus \ker(S^*).$$

For $A \in L(H)$, if we take $S = \Delta_A$ and $E = F = K(H)$ we easily obtain the following result.

Corollary 2.3. *Let $A \in L(H)$, then*

$$R(\Delta_A)^\circ = R(\Delta_A)^\circ \cap K(H)^\circ \oplus \ker(\Delta_A) \cap C_1(H).$$

Theorem 2.4. *If $A \in L(H)$ the following statements are equivalent:*
(1) A is quasi-adjoint.
(2) (i) the element $[A]$ of the Calkin algebra is quasi-adjoint, and
(ii) for $T \in C_1(H)$, $ATA = T$ implies $A^*TA^* = T$.
(3) (i)$\overline{R(\Delta_A)}^{W^*} = \overline{R(\Delta_{A^*})}^{W^*}$, and
(ii) $[A]$ is quasi-adjoint.

PROOF. (1) $\Longrightarrow$ (2). Suppose that A is quasi-adjoint.
(i) Let $\psi \in R(\Delta_{[A]})^\circ$. We define the bounded linear functional f on $L(H)$ by $f(X) = \psi([X])$. It is clear that $f \in R(\Delta_A)^\circ$ if and only if $\psi \in R(\Delta_{[A]})^\circ$. Since A is quasi-adjoint, it follows from the above remark that $f^* \in R(\Delta_A)^\circ$ and consequently $\psi^* \in R(\Delta_{[A]})^\circ$. Then $[A]$ is quasi-adjoint.
(ii) If $ATA = T$ and $T \in C_1(H)$, then Corollary 2.3 implies that $f_T \in R(\Delta_A)^\circ$. Since A is quasi-adjoint, it follows that $(f_T)^* = f_{T^*} \in R(\Delta_A)^\circ$, from which we get $A^*TA^* = T$.
(2) $\Longrightarrow$ (1). Let $f \in R(\Delta_A)^\circ$. We can write $f = f_\circ + f_T$, where $f_\circ \in R(\Delta_A)^\circ \cap K(H)^\circ$ and $T \in \ker(\Delta_A) \cap C_1(H)$. By using (ii), one obtains $A^*TA^* = T$, that is $f_{T^*} \in R(\Delta_A)^\circ$. It remains to show that $f_\circ^* \in R(\Delta_A)^\circ$. Let φ be the linear functional on the Calkin algebra defined by $\varphi([X]) = f_\circ(X)$. Since $f_\circ$ vanishes on $K(H)$, it follows that φ is well defined. From (i), $[A]$ is quasi-adjoint, then $\varphi \in R(\Delta_{[A]})^\circ$ implies that $\varphi^* \in R(\Delta_{[A]})^\circ$; that is, $f_\circ^* \in R(\Delta_A)^\circ$. Thus, we have shown that $f^* = f_\circ^* + f_{T^*} \in R(\Delta_A)^\circ$; consequently A is quasi-adjoint.
(2) $\Longleftrightarrow$ (3). $\overline{R(\Delta_A)}^{W^*} = \overline{R(\Delta_{A^*})}^{W^*}$ if and only if, $f \in R(\Delta_A)^\circ \cap L^{'}(H)^{W^*}$ implies $f^* \in R(\Delta_A)^\circ \cap L^{'}(H)^{W^*}$. Then it follows from Corollary 2.3 that

$$R(\Delta_A)^\circ \cap L^{'}(H)^{W^*} \cong \ker(\Delta_A) \cap C_1(H).$$

This completes the proof. ■

Proposition 2.5. *Let* $V \in L(H)$. *If* V *is an isometry then* V *is quasi-adjoint.*

PROOF. Let V be an isometry. We consider the operator P defined by $P = I - VV^*$. It obvious that $\Delta_{V^*}(X) = \Delta_V(-V^*XV^*) - PX$ for all X in $L(H)$. Hence, it suffices to show that $PL(H) \subseteq \overline{R(\Delta_V)}$. Let $(T_n)_n$ be a sequence of operators defined by

$$T_n = \sum_{k=0}^{n-1} \frac{k-n}{n} V^k PXV^k.$$

We have

$$\Delta_V(T_n) - PX = -\frac{1}{n} \sum_{k=1}^{n} V^k PXV^k.$$

But it is easy to see that $< V^k Px, V^j Py >= 0$ for every $x, y \in H$ and for all $k \neq j$. Hence, it follows that

$$\Big\| \sum_{k=1}^{n} V^k PXV^k x \Big\|^2 = \sum_{k=1}^{n} \|V^k PXV^k x\|^2 \leq n\|PX\|^2\|x\|^2$$

for all $x \in H$. This implies that $\|\Delta_V(T_n) - PX\| \leq \frac{1}{\sqrt{n}}\|PX\|$, and consequently we obtain $PX \in \overline{R(\Delta_V)}$. Then, V is quasi-adjoint. ■

Proposition 2.6. *Let A and B be quasi-adjoint operators. If $1 \notin \sigma(A)\sigma(B)$, then $A \oplus B$ is quasi-adjoint.*

Proof. Let X be an operator on $H \oplus H$. It is obvious to check that

$$R(\Delta_{A\oplus B}) = \begin{pmatrix} R(\Delta_A) & R(\Delta_{A,B}) \\ R(\Delta_{B,A}) & R(\Delta_B) \end{pmatrix}.$$

Under the hypotheses $1 \notin \sigma(A)\sigma(B)$, it follows from [6, theorem 3.2] that $\Delta_{A,B}$ and $\Delta_{B,A}$ are invertible. Hence,

$$R(\Delta_{A\oplus B}) = \begin{pmatrix} R(\Delta_A) & L(H) \\ L(H) & R(\Delta_B) \end{pmatrix}.$$

Since A and B are quasi-adjoint, then we have $X \in \overline{R(\Delta_{A\oplus B})}$ implies $X^* \in \overline{R(\Delta_{A\oplus B})}$. Consequently, $A \oplus B$ is quasi-adjoint. ■

Proposition 2.7. *Let $A \in L(H)$. If there exists $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta = 1$ and nonzero vectors $f, g \in H$ such that,*
(i) $Af = \alpha f$ and $\|A^*f\| \neq \|\alpha f\|$,
(ii) $A^*g = \bar{\beta}g$,
then A is not quasi-adjoint.

Proof. We must show that $\overline{R(\Delta_A)} \neq \overline{R(\Delta_{A^*})}$. Suppose first that $A^*f \neq 0$. Let us consider the operator $T = g \otimes A^*f$. It is clear that $< (AYA - Y)f, g >= 0$ for all $Y \in L(H)$.
On the other hand, we have

$$< (A^*TA^* - T)f, g >= \bar{\beta}(\|A^*f\|^2 - \|\alpha f\|^2)\|g\|^2.$$

If $A^*TA^* - T \in \overline{R(\Delta_A)}^{W^*}$, then there exists a sequence $(X_n)_n$ in $L(H)$ such that

$$AX_nA - X_n \longrightarrow A^*TA^* - T,$$

which gives

$$0 =< (AX_nA - X_\alpha)f, g > \longrightarrow < (A^*TA^* - T)f, g >= \bar{\beta}(\|A^*f\|^2 - \|\alpha f\|^2)\|g\|^2.$$

It follows that $\bar{\beta}(\|A^*f\|^2 - \|\alpha f\|^2)\|g\|^2 = 0$, which is absurd.
If $A^*f = 0$, we consider the operator $T = g \otimes f$. By repeating the same argument we get the result. ■

Let $(e_k)_{k\in\mathbb{Z}}$ be an orthonormal basis for H and let S be the bilateral weighted shift $Se_n = \omega_n e_{n+1}$, $n \in \mathbb{Z}$, with nonzero weights ω_n. By taking a unitarily equivalent weighted shift, we may assume that $\omega_n = |\omega_n| > 0$.

Proposition 2.8. *Let* S *be the bilateral shift* $Se_i = \omega_i e_{i+1}$*, such that* $\omega_i \geq 0$ *for all* $i \in \mathbb{Z}$*. Then,* $K(H) \subseteq \overline{R(\Delta_S)}$ *implies that* $\sum_{j\in\mathbb{Z}} \omega_{j-n}\omega_{j-n+1}\cdots\omega_{j+n-1} = \infty$ *for every* $n \in \mathbb{N}$*.*

Proof. Assume $K(H) \subseteq \overline{R(\Delta_S)}$. It follows from Theorem 2.2 that $\ker(\Delta_S) \cap C_1(H) = \{0\}$. Let X be a nonzero operator in $\ker(\Delta_S)$. Then $SXS = X$ implies that $S^n X S^n = X$ for all $n \in \mathbb{N}$. If we set $Xe_j = \sum_{k\in\mathbb{Z}} b_{k,j} e_k$ for $j \in \mathbb{Z}$, then a simple calculation shows that

$$Xe_j = \sum_{k\in\mathbb{Z}} \omega_{k-n}\cdots\omega_{k-1}\omega_j\cdots\omega_{j+n-1} b_{k-n,j+n} e_k,$$

from where we get

$$b_{k,j} = \omega_{k-n}\cdots\omega_{k-1}\omega_j\cdots\omega_{j+n-1} b_{k-n,j+n}$$

for all $j, k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

$$\begin{aligned}\sum_j |b_{j,j}| &= \sum_j \omega_{j-n}\cdots\omega_{j+n-1}|b_{j-n,j+n}| \\ &\leq \|X\| \sum_j \omega_{j-n}\cdots\omega_{j+n-1}.\end{aligned}$$

Thus, $\sum_j \omega_{j-n}\cdots\omega_{j+n-1} = \infty$ for every $n \in \mathbb{N}$. ■

Open questions:

(1) The above proposition suggests the following question:
$\sum_{j\in\mathbb{Z}} \omega_{j-n}\omega_{j-n+1}\cdots\omega_{j+n-1} = \infty$ for all $n \in \mathbb{N}$ if and only if $K(H) \subseteq \overline{R(\Delta_S)}$?

(2) Is every normal operator quasi-adjoint?

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