REPRESENTATION OF MAGNETIC FIELDS BY JUMP THEOREM FOR HARMONIC FORMS

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[Accepted 13 June 2007. Published 28 January 2008.]

Abstract

It has previously been shown that a surface current density J on a closed surface Σ of class C^∞ in $\mathbb{R}^3$ induces a static magnetic field B_J in $\mathbb{R}^3 \setminus \Sigma$, which has some discontinuity along Σ. In this note, we represent B_J by use of jump theorem for harmonic forms in the case where Σ is of class C^ω. We then apply this result to prove the existence of a surface current density J, which induces the nonzero magnetic field B_J such that $B_J \equiv 0$ inside (or outside) of the domain bounded by Σ in $\mathbb{R}^3$. This has previously been called the equilibrium magnetic field for Σ.

1. Introduction

A C_0^∞ vector field J in $\mathbb{R}^3$ with div $J = 0$ is called a *volume current density* in $\mathbb{R}^3$. Then J induces the smooth static magnetic field B_J in $\mathbb{R}^3$. Let D be a domain in $\mathbb{R}^3$ bounded by a finite number of C^ω smooth closed surfaces Σ that has positive orientation with respect to D, namely, $\partial D = \Sigma$. We put $D^+ = D$ and $D^- = \mathbb{R}^3 \setminus (D \cup \Sigma)$. We denote by dx the volume element of $\mathbb{R}^3$ at x. We denote by dS_x and n_x, respectively, the surface area element and the unit outer normal vector of Σ at x. In [8], a C^∞ vector field J on Σ is called *surface current density* on Σ, if there exists a sequence of volume current densities J_n, $n = 1, 2, \ldots$ such that

$$\lim_{n\to\infty} J_n \, dx = J dS_x, \qquad \text{weakly.}$$

Then J induces the magnetic field B_J in $\mathbb{R}^3 \setminus \Sigma$:

$$B_J(x) = \operatorname{rot}\left(\frac{1}{4\pi}\int_\Sigma \frac{J(y)}{\|y-x\|} dS_y\right), \quad x \in \mathbb{R}^3 \setminus \Sigma.$$

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doi:10.3318/PRIA.2008.108.1.7

Cite as follows: U. Cegrell and H. Yamaguchi, Representation of magnetic fields by jump theorem for harmonic forms, *Mathematical Proceedings of the Royal Irish Academy* **108**A (2008), 7–17; doi:10.3318/PRIA.2008.108.1.7.

Thus $B_J(x) = O(1/\|x\|)$ at $x = \infty$ and B_J has the following discontinuity along Σ. If we put $B_J(x) = B_J^{\pm}(x),\ x \in D^{\pm}$, respectively, then $B_J^{\pm}$ are C^1 smoothly extended to Σ from $D^{\pm}$, respectively, and satisfy

$$B_J^{+}(x) - B_J^{-}(x) = n_x \times J(x), \quad x \in \Sigma. \tag{1.1}$$

In [4], the surface current density is defined on some symmetric surfaces and discontinuity (1.1) is studied. In [8], we also showed that there exists a surface current density $\mathcal{J}$ on Σ that induces a nonzero magnetic field $B_{\mathcal{J}}$ such that:

$$B_{\mathcal{J}}^{-} \equiv 0 \quad \text{in } D^{-}.$$

This is the generalisation of the magnetic field induced by the usual symmetric solenoid. We call such $\mathcal{J}$ and $B_{\mathcal{J}}$, respectively, the *equilibrium* surface current density and the equilibrium magnetic field for Σ. In [3], we showed a canonical algorithm to reach such $\mathcal{J}$ from an arbitrarily given surface current density on Σ.

In this paper, we treat the case when Σ and a surface current J on Σ are of class C^{ω}. We represent B_J by use of the jump theorem for harmonic forms. Using this representation, we show the existence of the surface current density $\mathcal{J}$ on Σ, which induces a nonzero magnetic field $B_{\mathcal{J}}$ such that $B_{\mathcal{J}} \equiv 0$ in D^{+}, as was remarked on in [6]. Recently, S. Hamano [5] applied this method to get a remarkable extension theorem for separately harmonic functions.

2. Jump theorem for harmonic forms

Let D, Σ and $D^{\pm}$ be as in Section 1, such that Σ is of class C^{ω} in $\mathbb{R}^3$. We consider the signed distance function $R(x)$ for Σ as follows: given $x \in \mathbb{R}^3$ close to Σ, we find a unique point $y = y(x) \in \Sigma$ and $R(x) \in \mathbb{R}$ such that $x - y = R(x)\, n_y$. Then, $R(y)$ is of class C^{ω} in a neighborhood U of Σ in $\mathbb{R}^3$, with $R(x) = 0$ iff $x \in \Sigma$ and $n_x = \operatorname{grad} R(x)$ on Σ. We take a sufficiently small $a > 0$ such that:

$$V := \{x \in U \mid -a < R(x) < a\} \Subset U,$$

and put

$$D_1 := D^{+} \cup V \quad \text{and} \quad D_2 := D^{-} \cup V.$$

By putting $\Sigma^{\pm} := \{R(x) = \pm a\} \Subset D^{\mp}$, respectively, we have $\partial V = \Sigma^{+} - \Sigma^{-}$, and $\partial D_1 = \Sigma^{+},\ \partial D_2 = -\Sigma^{-}$. Then we have:

Proposition 2.1. *Let h be a harmonic function in V. Then there uniquely exist harmonic functions $h_i,\ i = 1, 2$ in D_i such that:*

(1) $h_2 - h_1 = h$ in V;
(2) $h_2(x) = O(1/\|x\|)$ at $x = \infty$.

The proof is the same as Hörmander's proof for holomorphic functions (see [2]). Since we need the proof itself for this paper, we provide the brief proof here.

PROOF. Let $\chi_1,\ \chi_2 \in C_0^\infty(\mathbb{R}^3)$ such that $\chi_1 + \chi_2 \equiv 1$ in $\mathbb{R}^3$ and Supp $\chi_1 \Subset D_2$ and Supp $\chi_2 \Subset D_1$, where Supp χ_i $(i = 1, 2)$ means the support of the function χ_i in $\mathbb{R}^3$. We may assume $\chi_1 h \in C^\infty(D_1)$ by extending $\equiv 0$ in D_2^c. Similarly, $\chi_2 h \in C^\infty(D_2)$. Since h is harmonic in V, by extending $\equiv 0$ in $\mathbb{R}^3 \setminus V$, we may assume $\Delta\chi_i h \in C_0^\infty(\mathbb{R}^3)$. Thus,

$$\chi_1\ h + \chi_2\ h = h \ \text{ in } V; \qquad \Delta\chi_1\ h + \Delta\chi_2\ h \equiv 0 \ \text{ in } \mathbb{R}^3.$$

Putting

$$h_1(x) := -\left(\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\Delta\chi_1\, h)(y)}{\|y-x\|}\, dy + \chi_1 h(x)\right), \quad x \in D_1,$$
$$h_2(x) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\Delta\chi_2\, h)(y)}{\|y-x\|}\, dy + \chi_2\, h(x), \quad x \in D_2.$$

We see from the Poisson equation for Newton potential that h_i is a harmonic function in D_i, and satisfies (1) and (2) in the proposition. Uniqueness is clear from the maximum principle for harmonic functions in $\mathbb{R}^3$. ■

Under the same notations in Proposition 2.1 we have the following jump theorem for harmonic forms:

Theorem 2.2. *Let* $\omega = \alpha\, dx + \beta\, dy + \gamma\, dz$ *be a harmonic 1-form in* V*. Then there uniquely exist harmonic 1-forms* $\omega_i,\ i = 1, 2$ *in* D_i *such that*

(1) $\omega_2 - \omega_1 = \omega$ *in* V*;*
(2) $\omega_2(x) = O(1/\|x\|)$ *at* $x = \infty$*.*

PROOF. The uniqueness follows from the fact that a harmonic 1-form Ω in $\mathbb{R}^3$, such that $\Omega(x) = O(1/\|x\|)$ at $x = \infty$, is 0. To prove its existence, we note that α, β, γ are necessarily harmonic functions in V. It follows from Proposition 2.1 that we find harmonic functions $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ in D_i such that:

(i) $\alpha_2 - \alpha_1 = \alpha, \quad \beta_2 - \beta_1 = \beta, \quad \gamma_2 - \gamma_1 = \gamma$ in V;
(ii) $\alpha_2(x),\ \beta_2(x),\ \gamma_2(x) = O(1/\|x\|)$ at $x = \infty$.

We put $\omega_i := \alpha_i\, dx + \beta_i\, dy + \gamma_i\, dz$ in D_i $(i = 1, 2)$. We shall show that this ω_i is a harmonic 1-form in D_i that satisfies (1) and (2) in the theorem.

In fact, it remains to prove that ω_i is harmonic in D_i; that is,

$$\text{rot } (\alpha_i, \beta_i, \gamma_i) = 0 \ \text{ in } D_i \qquad \text{and} \qquad \text{div } (\alpha_i, \beta_i, \gamma_i) = 0 \ \text{ in } D_i.$$

Since the other cases are similarly proved, we only prove the case of the first element of rot $(\alpha_1, \beta_1, \gamma_1)$:

$$\mathcal{R}_1(x) := \frac{\partial\gamma_1}{\partial x_2}(x) - \frac{\partial\beta_1}{\partial x_3}(x) = 0 \quad \text{for } x \in D_1, \tag{2.1}$$

where $x = (x_1, x_2, x_3) = (x, y, z)$. By the representations of γ_1 and β_1 in Proposition 2.1 we have:

$$\begin{aligned}
\beta_1(x) &= -\left(\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\Delta\chi_1\,\beta)(y)}{\|y-x\|}\,dy \ + \chi_1\,\beta(x)\right) \quad \text{for } x \in D_1;\\
\gamma_1(x) &= -\left(\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\Delta\chi_1\,\gamma)(y)}{\|y-x\|}\,dy \ + \chi_1\,\gamma(x)\right) \quad \text{for } x \in D_1.
\end{aligned}$$

We take $\eta : 0 < \eta < a$ such that, if we put $V_\eta = \{x \in \mathbb{R}^3 \mid -\eta < R(x) < \eta\} \Subset V$ and $E_1 = (V \setminus V_\eta) \cap D^+,\ E_2 := (V \setminus V_\eta) \cap D^-$; then $\chi_1 \equiv 0$ (hence, $\chi_2 \equiv 1$) on E_1 and $\chi_2 \equiv 0$ ($\chi_1 \equiv 1$) on E_2. Since $\Delta\chi_1\beta$ and $\Delta\chi_1\gamma \in C_0^\infty(\mathbb{R}^3)$, we have, for $x \in D_1$,

$$\begin{aligned}
\mathcal{R}_1(x) = \frac{1}{4\pi}\left(\int_{\mathbb{R}^3} \frac{\frac{\partial}{\partial y_3}(\Delta\chi_1\,\beta)(y)}{\|y-x\|}\,dy - \int_{\mathbb{R}^3} \frac{\frac{\partial}{\partial y_2}(\Delta\chi_1\,\gamma)(y)}{\|y-x\|}\,dy\right)&\\
+\frac{\partial(\chi_1\beta)}{\partial x_3}(x) - \frac{\partial(\chi_1\,\gamma)}{\partial x_2}(x).&
\end{aligned}$$

Since $\operatorname{Supp}\Delta\chi_1\beta,\ \operatorname{Supp}\Delta\chi_1\gamma \Subset V$ and $\chi_i\beta, \chi_i\gamma \in C^\infty(V)$, we have:

$$\mathcal{R}_1(x) = \frac{1}{4\pi}\int_V \frac{1}{\|y-x\|}\,\Delta\left(\frac{\partial(\chi_1\,\beta)}{\partial y_3} - \frac{\partial(\chi_1\,\gamma)}{\partial y_2}\right)dy + \frac{\partial(\chi_1\beta)}{\partial x_3}(x) - \frac{\partial(\chi_1\,\gamma)}{\partial x_2}(x).$$

Let $x_0 \in E_2$ and take a small ball $B_\varepsilon = \{\|x - x_0\| < \varepsilon\} \Subset E_2 \subset V$. We want to show that $\mathcal{R}_1(x_0) = 0$.

In fact, since $\chi_1\beta = \beta$ and $\chi_1\gamma = \gamma$ on E_2, it follows from $d\omega = 0$ in V that:

$$\frac{\partial(\chi_1\beta)}{\partial x_3}(x_0) - \frac{\partial(\chi_1\,\gamma)}{\partial x_2}(x_0) = \frac{\partial\beta}{\partial x_3}(x_0) - \frac{\partial\gamma}{\partial x_2}(x_0) = 0.$$

Further, Green's formula implies that

$$\begin{aligned}
&\int_V \frac{1}{\|y-x_0\|}\,\Delta\left(\frac{\partial(\chi_1\,\beta)}{\partial y_3} - \frac{\partial(\chi_1\,\gamma)}{\partial y_2}\right)dy\\
&= \lim_{\varepsilon\to 0}\int_{\Sigma^+ - \Sigma^- - \partial B_\varepsilon} \frac{1}{\|y-x_0\|}\,\frac{\partial}{\partial n_y}\left(\frac{\partial(\chi_1\,\beta)}{\partial y_3} - \frac{\partial(\chi_1\,\gamma)}{\partial y_2}\right)dS_y\\
&\quad -\lim_{\varepsilon\to 0}\int_{\Sigma^+ - \Sigma^- - \partial B_\varepsilon}\left(\frac{\partial(\chi_1\,\beta)}{\partial y_3} - \frac{\partial(\chi_1\,\gamma)}{\partial y_2}\right)\frac{\partial}{\partial n_y}\left(\frac{1}{\|y-x_0\|}\right)dS_y.
\end{aligned}$$

Since $\chi_1\beta = \chi_1\gamma \equiv 0$ on E_1; $\chi_1\beta = \beta$ and $\chi_1\gamma = \gamma$ in E_2; and $\frac{\partial\beta}{\partial x_3} - \frac{\partial\gamma}{\partial x_2} \equiv 0$ in $V(\supset \Sigma^{\pm})$ by $d\omega = 0$ in V, the right-hand side integrals reduce to

$$\begin{aligned}
&= 0 + \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon}\left(\frac{\partial\beta}{\partial y_3} - \frac{\partial\gamma}{\partial y_2}\right)\frac{\partial}{\partial n_y}\left(\frac{1}{\|y - x_0\|}\right) dS_y \\
&= -4\pi\left(\frac{\partial\beta}{\partial y_3}(x_0) - \frac{\partial\gamma}{\partial y_2}(x_0)\right) \\
&= 0,
\end{aligned}$$

so that $\mathcal{R}_1(x_0) = 0$ and, hence, $\mathcal{R}_1(x) \equiv 0$ in E_2. Since E_2 is an open subset of D_1 and since γ_1 and β_1 are harmonic in D_1, we have formula (2.1). ■

3. Representation of B_J by jump theorem

For a vector field $A = (a, b, c)$ in a domain or a surface K in $\mathbb{R}^3$ and the euclidean coordinates $X = (x, y, z)$, we use the following notations:

$$dX = (dx, dy, dz), \quad *dX = (dy\wedge dz, dz\wedge dx, dx\wedge dy), \quad *(*dX) = dX,$$

and

$$\begin{aligned}
A\cdot dX &= adx + bdy + cdz; \\
A\cdot *dX &= ady\wedge dz + bdz\wedge dx + cdx\wedge dy,
\end{aligned}$$

where $A\cdot dX$ and $A\cdot *dX$ are 1-form and 2-form in K, respectively.

We showed in [3] and [8], given a C^∞ vector-valued function $\mathbf{f} = (f_1, f_2, f_3)$ on Σ, $\mathbf{f}$ is a surface current density on Σ if and only if:

1. $\mathbf{f}$ is a tangential vector on Σ;
2. $\sigma_{\mathbf{f}} := (n_x \times \mathbf{f})\cdot dX$ is closed as a 1-form on the surface Σ.

Lemma 3.1 (Extension Lemma). *Let $\sigma = A\cdot dX$ be a C^ω closed 1-form on Σ such that A is a tangent vector on Σ. Then there exists a unique harmonic 1-form $\widehat{\sigma} = \widehat{A}\cdot dX$ in a tubular neighborhood V of Σ in $\mathbb{R}^3$, such that $\widehat{A} = A$ on Σ.*

PROOF. Let $p \in \Sigma$ and let S_p be a simply connected surface in Σ centered at p. If we consider

$$f_p(x) = \int_p^x \sigma, \quad x \in S_p,$$

then f_p is a C^ω function on S_p, since σ is C^ω closed on Σ. By Cauchy–Kowalevsky's theorem we have a unique harmonic function F_p in a neighborhood U_p of S_p in $\mathbb{R}^3$ such that:

$$F_p = f_p \ \text{ on } S_p \qquad \text{and} \qquad \frac{\partial F_p}{\partial n_x} \equiv 0 \ \text{ on } S_p.$$

Hence, $\widehat{\sigma}_p := (\text{grad } F_p)\cdot dX =: \widehat{A}_p\cdot dX$ is a harmonic 1-form in U_p, such that $A = \widehat{A}_p$

on S_p. Let $p, q \in \Sigma$ such that $S_p \cap S_q \neq \emptyset$. Then, $f_q - f_p$ is a constant on each connected component of $S_p \cap S_q$. By Cauchy–Kowalevsky's uniqueness theorem, $F_q - F_p$ is a constant in each connected component of $U_p \cap U_q$, so that $\widehat{A}_p = \widehat{A}_q$ in $U_p \cap U_q$. We thus find a tubular neighborhood V of Σ in $\mathbb{R}^3$ and a harmonic 1-form $\widehat{\sigma} = \widehat{A} \cdot dX$ in V such that $\widehat{\sigma} = \widehat{\sigma}_p$ in $V \cap U_p$ for any $p \in \Sigma$. This $\widehat{\sigma}$ satisfies the lemma's condition. Uniqueness comes from Cauchy–Kowalevsky's theorem. ■

Let $J = (f_1, f_2, f_3)$ be a C^ω surface current density on Σ, and let B_J be the magnetic field in $\mathbb{R}^3 \setminus \Sigma$ induced by $J dS_x$. We put $B_J = B_J^\pm$ on $D^\pm$. As noted in (1.1), $B_J^\pm$ in $D^\pm$ is C^1 smoothly extended to Σ from $D^\pm$, respectively, and they have the discontinuity along Σ such that $B_J^+ - B_J^- = n_x \times J(x)$ for $x \in \Sigma$. If we simply put $(g_1, g_2, g_3) := n_x \times J(x)$ for $x \in \Sigma$, then we see by condition 2 that

$$\sigma_J := g_1 dx + g_2 dy + g_3 dz \quad \text{on } \Sigma$$

is a C^ω closed 1-form on Σ. We thus find, by Lemma 3.1, a tubular neighborhood V of Σ in $\mathbb{R}^3$ and a harmnoic 1-form

$$\widehat{\sigma}_J = \widehat{g}_1 dx + \widehat{g}_2 dy + \widehat{g}_3 dz \quad \text{in } V,$$

such that $\widehat{g}_i = g_i, i = 1, 2, 3$ on Σ. By Theorem 2.2 we uniquely find harmonic 1-forms $\mathcal{G}^\pm$ on $D^\pm \cup V$ such that:

$$\begin{aligned} &(i) \quad \mathcal{G}^+ - \mathcal{G}^- = \widehat{\sigma}_J \quad \text{in } V; \\ &(ii) \quad \mathcal{G}^-(x) = O(1/\|x\|) \quad \text{at } x = \infty. \end{aligned} \tag{3.1}$$

If we put $\mathcal{G}^\pm = G^\pm \cdot dX$ in $D^\pm \cup V$, then the magnetic field $B_J^\pm$ is represented by this vector field $G^\pm$ as follows:

Theorem 3.2. $B_J^\pm = G^\pm$ *in* $D^\pm \cup V$, *respectively.*

Proof. If we put $\mathcal{B}^\pm(x) := B_J^\pm \cdot dX$ in $D^\pm$, then the theorem is identical to $\mathcal{B}^\pm = \mathcal{G}^\pm$ as a 1-form in $D^\pm$. We have proved in [3], as a generalisation of the Maxwell equation in static electro-magnetism, that B_J is a harmonic field in $\mathbb{R}^3 \setminus \Sigma$, i.e., $\mathcal{B}^\pm$ is a harmonic 1-form in $D^\pm$. We put

$$\tau := \tau_1 dx + \tau_2 dy + \tau_3 dz := \mathcal{B}^\pm - \mathcal{G}^\pm \quad \text{in } D^\pm, \quad \text{respectively,}$$

which is a harmonic 1-form in $D^+ \cup D^- = \mathbb{R}^3 \setminus \Sigma$, such that $\tau(x) = O(1/\|x\|)$ at $x = \infty$. It suffices for the theorem to prove that

$$\tau \equiv 0 \quad \text{in } \mathbb{R}^3 \setminus \Sigma. \tag{3.2}$$

In fact, since $B_J^+ - B_J^- = (g_1, g_2, g_3) = (\widehat{g}_1, \widehat{g}_2, \widehat{g}_3) = G^+ - G^-$ on Σ, it follows that τ becomes a continuous 1-form in all $\mathbb{R}^3$, such that τ is harmonic in $\mathbb{R}^3 \setminus \Sigma$. We shall show that τ is a continuous closed 1-form in $\mathbb{R}^3$.

In fact, let γ be any closed curve in $\mathbb{R}^3$ and let δ be a surface (2-chain) in $\mathbb{R}^3$

such that $\partial\delta = \gamma$. We put $\gamma^{\pm} = \gamma \cap D^{\pm}$ and $l = \delta \cap \Sigma$, so that $\partial\delta^{+} = \gamma^{+} + l$ and $\partial\delta^{-} = \gamma^{-} - l$. We thus have:

$$\int_{\gamma} \tau = \int_{\partial\delta^{+}} \tau + \int_{\partial\delta^{-}} \tau = \iint_{\delta^{+}} d\tau + \iint_{\delta^{-}} d\tau = 0,$$

where the first equality comes from the continuity of τ in $\mathbb{R}^3$; the second is from Stokes's formula and the third is from the harmonicity of τ in $D^{\pm}$. Thus, τ is closed in $\mathbb{R}^3$.

We consider

$$h(x) = \int_{0}^{x} \tau, \qquad x \in \mathbb{R}^3,$$

which is a C^1 (single-valued) function in $\mathbb{R}^3$ such that $dh = \tau$ in $\mathbb{R}^3$. Since τ is harmonic in $\mathbb{R}^3 \setminus \Sigma$, it turns out that $h \in C^1(\mathbb{R}^3)$ and h is harmonic in $\mathbb{R}^3 \setminus \Sigma$. We want to prove that h is harmonic in all $\mathbb{R}^3$.

In fact, let $\delta = \delta_r(a) := \{\|x - a\| < r\}$ be any ball in $\mathbb{R}^3$, and put $\delta^{\pm} = \delta \cap D^{\pm}$ and $\partial\delta^{\pm} = [(\partial\delta) \cap D^{\pm})] \pm [\delta \cap \Sigma]$, respectively. Then,

$$\begin{aligned}\int_{\partial\delta} \frac{\partial h}{\partial n_x}\, dS_x &= \int_{\partial\delta^{+}} \frac{\partial h}{\partial n_x}\, dS_x + \int_{\partial\delta^{-}} \frac{\partial h}{\partial n_x}\, dS_x \\ &= \iint_{\delta^{+}} \Delta\, h\, dx + \iint_{\delta^{-}} \Delta\, h\, dx \\ &= 0,\end{aligned}$$

where the first equality comes from $h \in C^1(\mathbb{R}^3)$; the second is from Green's formula; and the third is from the harmonicity of h in $D^{\pm}$. Then, by the standard method, we see that h satisfies the mean value theorem, so that h is a harmonic function in $\mathbb{R}^3$ and hence $\tau = dh$ is a harmonic 1-form in $\mathbb{R}^3$. Since $\tau(x) = O(1/\|x\|)$ at $x = \infty$, it follows from the maximum principle for harmonic functions that $\tau \equiv 0$ in $\mathbb{R}^3$, which proves (3.2). ■

4. Equilibrium magnetic fields in D^{-}

Let J be a volume current density in $\mathbb{R}^3$. Let γ be a closed curve in $\mathbb{R}^3$. We take a 2-chain δ in $\mathbb{R}^3$ such that $\partial\delta = \gamma$ and consider $J[\gamma] := \int_{\delta} J \cdot n_x\, dS_x$, which is independent of the choice of δ. We call $J[\gamma]$ the *flux* of J for γ. Let J be a surface current density on Σ, and let J_n, $n = 1, 2, \ldots$ be volume current densities in $\mathbb{R}^3$, such that $\lim_{n\to\infty} J_n dV_x = J dS_x$ weakly. For a closed curve γ in $\mathbb{R}^3 \setminus \Sigma$, we define

$$J[\gamma] := \lim_{n\to\infty} J_n[\gamma],$$

which is called the *flux* of J for γ. In [8] we showed

$$J[\gamma] = \int_{\gamma} B_J \cdot dX. \tag{4.1}$$

We put $G = D^{+}$ or D^{-}. Let $\{\gamma_j\}_{j=1,\ldots,q}$ be a 1-dimensional homology base in G.

Then we shall show the following existence theorem of equilibrium magnetic fields for Σ by use of Theorem 3.2.

Theorem 4.1. *Fixed* γ_i $(i = 1, \ldots, q)$, *there exists a unique equilibrium surface current density* $\mathcal{J}_i$ *on* Σ *such that:*

(1) $B_{\mathcal{J}_i} \equiv 0$ *in* $\mathbb{R}^3 \setminus \overline{G}$;

(2) $\mathcal{J}_i[\gamma_j] = \delta_{ij}, \ \ j = 1, \ldots, q.$

This result in the case where $G = D^+$ is proved in [8], and T. Harada remarked on it in relation to the case where $G = D^-$ in [6]. We showed in [3] a canonical algorithm to reach $\mathcal{J}_i$ from an arbitrarily given surface current density J on Σ with $J[\gamma_j] = \delta_{ij}$. Here, we give the proof in the case where $G = D^-$.

To give the proof, we recall one of Weyl's theorems. We set, for $i = 1, 2$,

$$\begin{aligned}
C_i^k(G) &= \text{the set of all } i\text{-forms of class } C^k \text{ in } G;\\
\Gamma_i^2(G) &= \text{the Hilbert space of square integrable } i\text{-forms in } G;\\
Z_i(G) &= \{\omega \in \Gamma_i^2(G) : \omega \text{ is closed in } G\};\\
B_i(G) &= Cl\{\, d\sigma \in \Gamma_i^2(G) : \sigma \in C_{i-1}^{\infty}(\overline{G})\};\\
H_i(G) &= \{\omega \in \Gamma_i^2(G) : \omega \text{ is harmonic in } G\};\\
B_{i0}(G) &= Cl\{d\sigma_0 \in \Gamma_i^2(G) : \sigma_0 \in C_{i-1}^{\infty}(G) \text{ with } \operatorname{Supp}\sigma_0 \Subset G\};\\
H_{1e}(G) &= \{du \in L_1^2(G) : u \text{ is a harmonic function in } G\};\\
H_{20}(G) &= \{\omega \in H_2(G) \cap C_2^1(\overline{G}) : \omega = 0 \text{ along } \Sigma\}.
\end{aligned}$$

Hilbert space $\Gamma_i^2(G)$ is equipped with the usual inner product: $(\omega, \sigma) = \int_G \omega \wedge {*\sigma}$ for $\omega, \sigma \in \Gamma_i^2(G)$; $Cl[S]$ means the closure of S in $\Gamma_i^2(G)$; and, for $\omega = A \cdot {*dX} \in C_2^1(V)$ where $V \supset \Sigma$, '$\omega = 0$ *along* Σ' means that A is tangential on Σ. Then Weyl's orthogonal decomposition theorem holds:

$$\Gamma_1^2(G) = Z_1(G) \dotplus {*B_{20}(G)}; \quad Z_1(G) = B_1(G) \dotplus {*H_{20}(G)}. \tag{4.2}$$

We note that

$$H_{1e}(G) \perp {*H_{20}(G)} \quad \text{in } \ \Gamma_1^2(G).$$

Let $\zeta \in \Sigma$ and let V be a small ball centered at ζ in $\mathbb{R}^3$. Let $h(x)$ be a harmonic function in $V_0 := V \cap G$ and of class C^1 on $V_0 \cup (\Sigma \cap \partial V_0)$. Then, it is known that if $h = 0$ or $\frac{\partial h}{\partial n_x} = 0$ on $\Sigma \cap \partial V_0$, then h is harmonically exended to a neighborhood U of $\Sigma \cap \partial V_0$ in V. Therefore, it holds that $H_{20}(G) = H_{20}(\overline{G})$, i.e., for $\omega \in H_{20}(G)$, there exists a domain E with $E \supset G \cup \Sigma$ in $\mathbb{R}^3$ such that $\omega \in H_2(E)$ and $\omega = 0$ along Σ.

We put $Z_1^{\infty}(G) = Z_1(G) \cap C_1^{\infty}(G)$. Let γ be a simple closed curve in G. Following [7], we consider the linear functional

$$\omega \in Z_1^{\infty}(G) \to \int_{\gamma} \omega \in \mathbb{R}.$$

Then there exists $M > 0$ such that $|\int_\gamma \omega| \le M\|\omega\|_D$ for any $\omega \in Z_1^\infty(G)$. It follows from Riesz's theorem that there uniquely exists $*\Omega_\gamma \in Z_1(G)$ such that:

$$(\omega, *\Omega_\gamma)_G = \int_\gamma \omega \quad \text{for any } \omega \in Z_1^\infty(G).$$

We call $*\Omega_\gamma$ the *reproducing form* for γ in G (cf: [1]). By use of Weyl's decomposition theorem we have $\Omega_\gamma \in H_{20}(G)$. As mentioned above, there exists a domain E with $E \supset G \cup \Sigma$ in $\mathbb{R}^3$, such that Ω_γ is harmonic in E and $\Omega_\gamma = 0$ along Σ. In the case where $G = D^-$, we have $\Omega_\gamma(x) = O(1/\|x\|)$ at $x = \infty$, since $\Omega_\gamma \in \Gamma_2^2(D^-)$.

PROOF. We now provide the proof of Theorem 4.1 in the case where $G = D^-$. To prove the uniqueness, let $J^{(1)}$ and $J^{(2)}$ be surface current densities on Σ such that, for $k = 1, 2$,

$$(1)\ B_{J^{(k)}} \equiv 0 \ \text{ in } D^+; \qquad (2)\ J^{(k)}[\gamma_j] = \delta_{ij},\ j = 1, \dots, q.$$

We simply put $B_k^\pm = B_{J^{(k)}}^\pm$ on $D^\pm$, and consider $\sigma_k = B_k^- \cdot dX$ in D^-, so that $B_k^+ \equiv 0$ in D^+ and σ_k is harmonic in D^- and of class C^1 on $\overline{D^-} = D^- \cup \Sigma$. By the discontinuity property of B_k and (1) for B_k, we have

$$-B_k^- = B_k^+ - B_k^- = n_x \times J^{(k)} \ \text{ on } \Sigma.$$

Since $n_x \times J^{(k)}$ is tangential on Σ, it follows that $*\sigma_k$ is a harmonic 2-form on D^-, which has a C^1 extension on Σ such that $*\sigma_k = 0$ along Σ, namely $*\sigma_k \in H_{20}(D^-)$. If we put $\tau := \sigma_1 - \sigma_2$ on $\overline{D^-}$, then we see from (4.1) under (2) for $J^{(k)}$ that τ is an exact 1-form in D^-. Hence, $\tau \in H_{1e}(D^-) \cap *H_{20}(D^-) = \{0\}$, so that $n_x \times (J^{(1)} - J^{(2)}) = 0$ on Σ. Since $J^{(k)}$ itself is tangential on Σ, we have $J^{(1)} = J^{(2)}$ on Σ.

We divide the proof of the existence of $\mathcal{J}_i$ into two steps:

Step 1: *Fixed i ($i = 1, \dots, q$), there exists a unique harmonic 2-form $\Omega = \mathcal{A} \cdot *dX$ in a domain E with $E \supset \overline{D^-}$ in $\mathbb{R}^3$, such that*

$$\text{(i) } \Omega = 0 \ \text{ along } \ \Sigma; \qquad \text{(ii) } \int_{\gamma_j} *\Omega = \delta_{ij},\ j = 1, \dots, q.$$

In fact, for $k = 1, \dots, q$, we have the reproducing form $*\Omega_k$ for γ_k in D^- so that $*\Omega_k$ is a harmonic 1-form in a domain E with $E \supset \overline{D^-}$ in $\mathbb{R}^3$ and $\Omega_k = 0$ along Σ. For $k, l = 1, \dots q$, we can find $\xi_k \in Z_1^\infty(D^-)$ such that $\int_{\gamma_l} \xi_k = \delta_{kl},\ k, l = 1, \dots, q$. We thus see, from the definition of $*\Omega_l$, that $*\Omega_l,\ l = 1, \dots, q$ are linearly independent in $H_1(D^-)$, so that the uniqueness in Step 1 follows. Further, if we put $a_{kl} := \int_{\gamma_l} *\Omega_k\ (k, l = 1, \dots, q)$, then we see that $\det(a_{kl}) \neq 0$. For, if not, we could find $(\alpha_1, \dots, \alpha_q) \neq \mathbf{0}$, such that $\omega := \sum_{k=1}^q \alpha_k \Omega_k$ satisfies

$$\int_{\gamma_l} *\omega = 0 \ \ (l = 1, \dots, q).$$

We thus have $*\omega \neq 0$ and $*\omega \in H_{1e}(D^-) \cap *H_{20}(D^-) = \{0\}$, which is a contradiction. By some linear combination of $*\Omega_k$ $(k = 1, \ldots, q)$ we can find a harmonic 1-form $*\Omega = \mathcal{A} \cdot dX$, which is harmonic in a domain $E \supset \overline{D^-}$ such that $\Omega = 0$ along Σ and $\int_{\gamma_j} *\Omega = \delta_{ij}, j = 1, \ldots, q$. Thus Step 1 is proved.

Step 2: *If we put $J := n_x \times \mathcal{A}$ on Σ, then J is a surface current density on Σ, which satisfies (1) and (2) in Theorem 4.1.*

In fact, we have

$$\sigma_J := (n_x \times J) \cdot dX = -\mathcal{A} \cdot dX = - * \Omega \quad \text{on } \Sigma, \tag{4.3}$$

so that σ_J is a C^ω closed 1-form on Σ. Since J is tangential on Σ, J is a C^ω surface current density on Σ. Thus, J induces the magnetic field B_J in $\mathbb{R}^3 \setminus \Sigma$. We usually set $B_J = B_J^\pm$ in $D^\pm$. By Lemma 3.1, σ_J has a unique harmonic extension $\widehat{\sigma}_J$ in a neighborhood of Σ in $\mathbb{R}^3$. Further, we uniquely have a harmonic 1-form $\mathcal{G}^\pm = G^\pm \cdot dX$ in $D^\pm \cup V$, where $V \Supset \Sigma$ in $\mathbb{R}^3$, which satisfies (i) and (ii) in (3.1).

On the other hand, by Step 1, $*\Omega$ is harmonic 1-form in $E \supset \overline{D^-}$, so that (4.3) implies $\widehat{\sigma}_J = - * \Omega$ in $V' := V \cap E \Supset \Sigma$. Thus,

$$\mathcal{G}^+ - \mathcal{G}^- = \widehat{\sigma}_J = 0 - *\Omega \quad \text{in } V'.$$

Since $*\Omega(x) = O(1/\|x\|)$ at $x = \infty$, it follows from the uniqueness for (3.1) that $\mathcal{G}^+ = 0$ in D^- and $\mathcal{G}^- = *\Omega$ in D^-, namely $G^+ = 0$ in D^+ and $G^- = \mathcal{A}$ in D^-. By Theorem 3.2, $B_J^+ = 0$ in D^+ and $B_J^- = \mathcal{A}$ in D^-, so that J satisfies condition (1) in Theorem 4.1 for $G = D^-$.

By (4.1) and Step 1, we have

$$J[\gamma_j] = \int_{\gamma_j} B_J^- \cdot dX = \int_{\gamma_j} *\Omega = \delta_{ij},$$

so that J also satisfies (2). Thus, Step 2 and, hence, Theorem 4.1 are proved. ■

ACKNOWLEDGEMENT

We would like to thank Professor J. Siciak for his useful comments about jump theorem for harmonic functions.

REFERENCES

[1] Lars V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton University Press, Princeton, New Jersey, 1960.

[2] S. Axer, P. Bourdan and W. Ramey, *Harmonic function theory*, Graduate Texts in Mathematics, Springer, New York, 2001.

[3] U. Cegrell and H. Yamaguchi, Construction of equilibrium magnetic vector potentials, *Potential Analysis*, **5** (2001), 302–31.

[4] D. J. Griffiths, *Introduction to electrodynamics*, 3rd edn, Prentice-Hall, New Jersey, 1999.

[5] S. Hamano, Hartogs–Osgood theorem for separately harmonic functions, *Proceedings of the Japan Academy*, **83** (A), (2007), 16–18.

[6] T. Harada, Study of surface current densities and magnetic fields, unpublished PhD thesis (in Japanese), Nara Women's University, Nara, 2006.

[7] H. Weyl, The method of orthogonal projection in potential theory, *Duke Mathematical Journal*, **7** (1940), 411–44.

[8] H. Yamaguchi, Equilibrium vector potentials in $\mathbb{R}^3$, *Hokkaido Mathematical Journal*, **25** (1996), 1–53.