ATKINSON–WILCOX EXPANSION THEOREM FOR INHOMOGENEOUS MEDIA

By George Venkov
Department of Differential Equations,
Faculty of Applied Mathematics and Informatics, Technical University of Sofia,
8 Kliment Ohridski Str., 1756 Sofia, Bulgaria

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Abstract

The classical Atkinson–Wilcox theorem claims that in the exterior of the sphere that optimally circumscribes the scatterer, any radiating solution to the Helmholtz equation can be expanded into a uniformly and absolutely convergent series in inverse powers of the radial distance. Moreover, the full series can be recovered uniquely through a second-order recurrence relation via the leading coefficient, known as the far-field pattern. In this work we consider the medium scattering problem and prove an analogue of the expansion theorem for inhomogeneity of compact support.

1. Introduction

The direct and inverse medium problems are fundamental to the scattering theory, although they are by far less studied than the scattering problems for homogeneous media in the exterior of impenetrable obstacles. In general, the problem of scattering of wave fields by inhomogeneous media appears in several situations, among them medical imaging, analysis of biological studies at the cell level, exploration geophysics and some industrial applications. For background material, see Colton *et al.* [10], Colton and Kress [11], Gutman and Klibanov [15], Kirsch [17].

The far-field pattern (or the scattering amplitude) u_∞ plays a central role in the direct and inverse scattering theory. It was Atkinson [6] who in 1949 showed that any solution of the Helmholtz equation that satisfies the Sommerfeld radiation condition [21] is an analytic function of inverse power of the radial variable in the exterior of the sphere that optimally circumscribes the scatterer. The Sommerfeld radiation condition completely characterises the behavior of the scattered field at infinity. In fact, Atkinson presented a wave analogue of Maxwell's multipole expansion in potential theory [19], i.e. he proved that any radiation solution has an absolutely and uniformly convergent series representation in inverse powers of the radial distance in all space, exterior to the circumscribing sphere. This expansion separates the radial from the angular dependence of the solution and it can be

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seen as the wave analogue in electromagnetic and elastic scattering theory [5; 9; 12]. The radial dependence characterises the radiative nature of the scattered field, while the angular dependence incorporates the geometrical and physical characteristics of each particular obstacle.

The one-to-one correspondence between the scattered fields and their radiation patterns for the exterior Helmholtz problem was established by Rellich in [20]. This correspondence was made explicit by Wilcox, who made a generalisation of the Atkinson expansion theorem, both for the acoustic [22], as well as the electromagnetic case [23]. His work contains the important fact that all the angular dependent coefficients $\{F_n : n = 1, 2, ...\}$ in the Atkinson expansion can be recovered through a second order recurrence relation via the leading coefficient F_0, known as scattering amplitude or far-field (radiation) pattern u_∞ (see for example [11; 18]). The expansion theorem brought up the questions of recovering the radiating solution of the Helmholtz equation from a knowledge of their far-field patterns in the direct scattering problem, as well as the determination of the shape and the nature of the scatterer from a given scattering amplitude in the inverse obstacle problem.

During the last decade there were many results applying the so-called *expansion theorem* for arbitrary convex domain (see Arnaoudov *et al.* [3; 4]) for obstacles such as the prolate and the oblate spheroid (see Burnett and Holford [7]), the ellipsoid (see Arnaoudov and Venkov [1], Burnett and Holford [8] and Dassios [13; 14]) and the bi-spheres (see Arnaoudov and Venkov [2]), because it provides a very efficient way to construct methods (the infinite element method, the method of mirror images, etc.) for solving direct scattering problems. In [4] the Atkinson–Wilcox expansion was implemented for solving the inverse scattering problem. More precisely, it is proved that the order of the recurrence relation between the series coefficients determines the shape of the scattering surface. Finally, the Atkinson–Wilcox theorem was extended to scattering of electromagnetic chiral waves (Athanasiadis and Giotopoulos [5]), to elasticity (Cakoni and Dassios [9] and Dassios [12]) and to the scattering problems in $\mathbb{R}^2$ (Karp [16]).

The present work provides a proof of the Atkinson–Wilcox expansion theorem in the theory of scattering of a time-harmonic acoustic wave by an inhomogeneous medium of compact support. The usual way to prove the series expansion theorem consists in reformulating the scattering problem as an integral version of the corresponding elliptic equation, i.e. the Lippmann–Schwinger equation, and to use the analytic properties of the integral kernel (the fundamental solution to the Helmholtz equation) outside the support of the inhomogeneity. Finally, two important corollaries are obtained via the appropriate applications of the expansion theorem.

2. The expansion theorem

We restrict our discussion to the propagation of a monochromatic, time-harmonic acoustic spherical wave of small amplitude in an isotropic inhomogeneous medium. The inhomogeneity of the scattering media is characterized by the refractive index

$$n(\mathbf{r}) = n_1(\mathbf{r}) + i\frac{n_2(\mathbf{r})}{k}, \tag{2.1}$$

$$n_1(\mathbf{r}) = \frac{c_0^2}{c^2(\mathbf{r})}, \quad n_2(\mathbf{r}) = \sigma(\mathbf{r}), \tag{2.2}$$

where $c : \mathbb{R}^3 \to \mathbb{R}_+ \backslash \{0\}$ is the sound speed, $\sigma : \mathbb{R}^3 \to \mathbb{R}_+$ is a function that models the influence of absorption and c_0 denotes the sound speed in the homogeneous background medium. We shall consider the case when the inhomogeneity is of compact support, the region under consideration is all of $\mathbb{R}^3$ and $c(\mathbf{r}) \leq c_0$. Assuming the inhomogeneous region is contained inside a ball B, i.e. $n(\mathbf{r}) = 1$ for $\mathbf{r} \in \mathbb{R}^3 \backslash B$, then the acoustic scattering problem in the inhomogeneous medium is modelled by

$$u(\mathbf{r}) = u^i(\mathbf{r}) + u^s(\mathbf{r}), \tag{2.3}$$

$$\Delta u + k^2 n(\mathbf{r}) u = 0, \quad \mathbf{r} \in \mathbb{R}^3, \tag{2.4}$$

$$\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad r = |\mathbf{r}|, \tag{2.5}$$

where u^i is an entire solution of the Helmholtz equation $\Delta u + k^2 u = 0$ and u^s is the scattered field, which satisfies the Sommerfeld radiation condition (2.5) uniformly in all directions.

We begin our analysis by introducing an integral equation that is equivalent to the scattering problem (2.3)–(2.5). Let $n \in C^1(\mathbb{R}^3)$ have the general form (2.1) such that $m := 1 - n$ has compact support, $m : \mathbb{R}^3 \to (-\infty, 1) + i(-\infty, 0]$ and let $D := \{\mathbf{r} \in \mathbb{R}^3 : m(\mathbf{r}) \neq 0\}$.

Since the velocity potential has no discontinuities across the boundary of the inhomogeneous medium, i.e. the boundary conditions are absent, we shall make use of volume potentials instead of surface potentials, as in the case of impenetrable obstacles. From Green's second identity the solution of the above scattering problem solves the Lippmann–Schwinger integral equation

$$u(\mathbf{r}) = u^i(\mathbf{r}) - k^2 \int_{\mathbf{R}^3} \Phi(\mathbf{r}, \mathbf{r}') m(\mathbf{r}') u(\mathbf{r}') \, d\mathbf{r}', \quad \mathbf{r} \in \mathbb{R}^3, \tag{2.6}$$

where $\Phi(\cdot, \mathbf{r}') : \mathbb{R}^3 \backslash \{\mathbf{r}'\} \to \mathbb{C}$ is the fundamental solution to the Helmholtz equation of the form $\Phi(\mathbf{r}, \mathbf{r}') = \exp(ik|\mathbf{r} - \mathbf{r}'|)/4\pi|\mathbf{r} - \mathbf{r}'|$. We can state the following representation theorem (see, for instance, theorem 8.3 in [11]).

Theorem 2.1. *If $u \in C^2(\mathbb{R}^3)$ is a solution of* (2.3)–(2.5)*, then u is a solution of the integral equation (2.6). Conversely, if $u \in C(\mathbb{R}^3)$ is a solution of (2.6) then $u \in C^2(\mathbb{R}^3)$ and u is a solution of (2.3)–(2.5).*

We note that in (2.6) we can replace the region of integration by the ball B such that the support of m is contained and look for solutions in $C\left(\bar{B}\right)$. Then for $\mathbf{r} \in \mathbb{R}^3 \backslash \bar{B}$ we define $u(\mathbf{r})$ by the right-hand side of (2.6) and obviously obtain a continuous solution of the Lippmann–Schwinger equation in all of $\mathbb{R}^3$.

From (2.6) we have

$$u^s(\mathbf{r}) = -k^2 \int_B \Phi(\mathbf{r}, \mathbf{r}') m(\mathbf{r}') u(\mathbf{r}') \, d\mathbf{r}'. \tag{2.7}$$

Hence, letting $r = |\mathbf{r}|$ tend to infinity, with the help of the asymptotic behavior

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{e^{ikr}}{r} \left\{ e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} + O\left(\frac{1}{r}\right) \right\}, \quad \hat{\mathbf{r}} = \mathbf{r}/r, \tag{2.8}$$

we see that

$$u^s(\mathbf{r}) = \frac{e^{ikr}}{r} u_\infty(\hat{\mathbf{r}}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty. \tag{2.9}$$

The function u_∞ defined on the unit sphere is known as the far-field pattern of u and is given by

$$u_\infty(\hat{\mathbf{r}}) = -\frac{k^2}{4\pi} \int_B e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} m(\mathbf{r}') u(\mathbf{r}') \, d\mathbf{r}'. \tag{2.10}$$

In other words, as the observation point approaches infinity the radial and the angular dependence of the scattered field separate. The radial dependence is established into the scatterer-independent part e^{ikr}/r, while the angular dependence is reflected on the far-field pattern (or the scattering amplitude) $u_\infty(\hat{\mathbf{r}})$.

From the representation theorem the following expansion theorem, analogous to the Atkinson–Wilcox expansion theorem for impenetrable obstacles, follows immediately.

Theorem 2.2. *Let $u \in C^2\left(\mathbb{R}^3\right)$ be a solution of (2.3)–(2.5) and let R_0 be large enough so that D is contained in the ball $B_{R_0} := \left\{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r}| \leq R_0\right\}$. Then u has an expansion*

$$u(\mathbf{r}) = u(r, \theta, \varphi) = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{F_n(\theta, \varphi)}{r^n}, \tag{2.11}$$

that converges absolutely and uniformly on compact subsets of $\mathbb{R}^3 \backslash B_{R_0}$. Furthermore, the series in (2.11) may be differentiated term by term with respect to r, θ and φ any number of times and the resulting series all converge absolutely and uniformly.

Proof. From the representation theorem applied in the exterior of the ball B_R with $R < R_0$, we have

$$u^s(\mathbf{r}) = -k^2 \int_{B_R} \Phi(\mathbf{r}, \mathbf{r}') m(\mathbf{r}') u(\mathbf{r}') \, d\mathbf{r}', \quad |\mathbf{r}| \geq R_0. \tag{2.12}$$

Let (r, θ, φ) and (r', θ', φ') be the spherical coordinates of $\mathbf{r}$ and $\mathbf{r}'$, respectively. Thus, we may rewrite (2.12) in the form

$$u^s(\mathbf{r}) = -\frac{k^2}{4\pi} \int_0^R \left(\int_\Omega \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} m(\mathbf{r}') u(\mathbf{r}') \, d\omega' \right) r'^2 dr', \tag{2.13}$$

where Ω is the unit sphere and $d\omega' = \sin\theta' d\theta' d\varphi'$. In spherical coordinates the distance function $|\mathbf{r}-\mathbf{r}'|$ that appears in the integrand of (2.13) is given by

$$|\mathbf{r}-\mathbf{r}'| = \left(r^2 - 2rr'\cos\gamma + r'^2\right)^{1/2}, \tag{2.14}$$

with $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$. Now, by putting $\rho := r'/r$, we have

$$\frac{e^{ik(|\mathbf{r}-\mathbf{r}'|-r)}}{|\mathbf{r}-\mathbf{r}'|} = \frac{\rho e^{ikr'\left[\left(1-2\rho\cos\gamma+\rho^2\right)^{1/2}-1\right]/\rho}}{r'\left(1-2\rho\cos\gamma+\rho^2\right)^{1/2}}, \tag{2.15}$$

where the branch of the square root is chosen having the value $+1$ at $\rho = 0$. Since $\left(1-2\rho\cos\gamma+\rho^2\right)^{1/2}$ is analytic function in ρ for $\rho < 1$, then the following expansion holds

$$\frac{e^{ik(|\mathbf{r}-\mathbf{r}'|-r)}}{|\mathbf{r}-\mathbf{r}'|} = \sum_{n=1}^{\infty} f_n(\gamma)\rho^n, \quad \rho < 1. \tag{2.16}$$

The above series converges absolutely and uniformly for $r \geq R_0$ and $\gamma \in [0, 2\pi)$. Furthermore, it may be differentiated with respect to r and γ any number of times, and the resulting series all converge absolutely and uniformly. Multiplying by $m(\mathbf{r}')u(\mathbf{r}')$ and integrating term by term we arrive at the expansion (2.11) and the proof is completed. ■

The following two corollaries are immediate consequences of Theorem 2.2.

Corollary 2.1. *The coefficients $F_n(\theta, \varphi)$ of series (2.11) can be determined from the far-field pattern $u_\infty(\hat{\mathbf{r}})$ by the second-order recurrence relation of the form*

$$2iknF_n = n(n-1)F_{n-1} + \mathbf{B}_{(\theta,\varphi)}F_{n-1}, \quad n \geq 1, \tag{2.17}$$

where $F_0(\theta, \varphi) = u_\infty(\hat{\mathbf{r}})$ and

$$\mathbf{B}_{(\theta,\varphi)} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{2.18}$$

is the Laplace–Beltrami operator on the unit sphere.

Proof. From the representation theorem it follows that the expansion (2.11) satisfies equation (2.4) in $\mathbb{R}^3 \backslash B_{R_0}$. Differentiating the series in terms of spherical coordinates and equating like powers of r we obtain

$$\frac{e^{ikr}}{r}\sum_{n=1}^{\infty}\frac{1}{r^{n+1}}\left\{-2iknF_n + n\left(n-1\right)F_{n-1} + \mathbf{B}_{(\theta,\varphi)}F_{n-1}\right\} = k^2 m\left(\mathbf{r}\right)u\left(\mathbf{r}\right), \quad (2.19)$$

which holds for $r \geq R_0$. Since $D \subset B_{R_0}$, then the right-hand side of (2.19) vanishes and the corollary is proved. ∎

As in the classical acoustic and electromagnetic scattering theory [5; 22; 23], the second-order recurrence relation (2.17) establishes the one-to-one correspondence between radiating waves and their far-field patterns.

Corollary 2.2. *Let $u^s \in C^2\left(\mathbb{R}^3 \backslash \bar{D}\right)$ be the radiating solution to equation 2.4 for which the far-field pattern $u_\infty\left(\hat{\mathbf{r}}\right)$ vanishes identically. Then $u^s = 0$ in $\mathbb{R}^3 \backslash \bar{D}$.*

Proof. Since $F_0\left(\theta,\varphi\right) = u_\infty\left(\hat{\mathbf{r}}\right) = 0$, then from Corollary 2.1 and expansion (2.11) it follows that $u^s = 0$ in the exterior of any ball that contains the support of m. Then, from the analyticity of u^s it follows that $u^s = 0$ in $\mathbb{R}^3 \backslash \bar{D}$. ∎

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