BLOCH FUNCTIONS ON COMPLEX BANACH MANIFOLDS

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Abstract

Let X be a complex Banach manifold. A holomorphic function $f : X \to C$ is called a Bloch function if the family $\mathcal{F}_f = \{f \circ \varphi - f(\varphi(0)) : \varphi : \Delta \to X$ is holomorphic$\}$, $\Delta = \{z \in C : |z| < 1\}$, is a normal family in the sense of Montel. In this paper Bloch functions on complex Banach manifolds are studied. The main result shows that many of the equivalent definitions of Bloch functions on the unit disk are also equivalent in the general setting.

1. Introduction

The concept of Bloch functions was introduced by Pommerenke [7]. Pommerenke's definition was restricted to the unit disk Δ in the complex plane C. Bloch functions of several complex variables was studied by Hahn [4], Timoney [8] and Krantz and Ma [6], among others. In the infinite dimensional case Kim and Krantz [5] define Bloch functions on the unit ball in a complex Banach space.

In this note we extend the notion of Bloch functions of one complex variable to holomorphic functions defined on a complex Banach manifold. The main result shows that many of the characterisations of Bloch functions on the unit disk extend also to the case of complex Banach manifolds.

2. Definition and characterisations of Bloch functions

We refer the reader to the paper by Earle *et al.* [2] and the books by Dineen [1] and Franzoni and Vesentini [3] for background on complex analysis in infinite dimensions.

Let X be a complex Banach manifold modelled on a complex Banach space of positive, possibly infinite, dimension; X assumed to be a connected Hausdorff space. Let $O(X, C)$ be the set of all holomorphic functions on X, and let $O(\Delta, X)$ be the space of all holomorphic maps from Δ into X.

Definition 1. A function $f \in O(X, C)$ is called a Bloch function if the family

$$\mathcal{F}_f = \{f \circ \varphi - f(\varphi(0)) : \varphi \in O(\Delta, X)\}$$

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is a normal family on Δ in the sense of Montel.

Recall that a family $F \subseteq O(\Delta, C)$ is called a normal family in the sense of Montel if each sequence in F contains a subsequence that converges uniformly on each compact subset in Δ. (Here, we allow the subsequence to converge uniformly on each compact subset to the function that is identically ∞ in the sense that the subsequence of reciprocals converges uniformly on each compact subset to the function that is identically zero.)

For each $x \in X$ the tangent space to X at x will be denoted by $T_x(X)$. The tangent bundle $T(X)$ of X consists of the ordered pairs (x, v) such that $x \in X$ and $v \in T_x(X)$.

Let x be a point of the complex Banach manifold X and let v be a tangent vector to X at x. The infinitesimal Kobayashi pseudometric on the complex Banach manifold X is the function k_X on $T(X)$ defined by the formula

$$k_X(x, v) = \inf\{|\lambda| : \exists\, \varphi \in O(\Delta, X),\ \varphi(0) = x,\ \varphi_*(0)\lambda = v\},$$

where $\varphi_*(0)$ is the linear map induced by φ from $T_0(\Delta)$ to $T_{\varphi(0)}(X)$.

The Kobayashi length of a piecewise C^1 curve $\gamma : [0, 1] \to X$ in X is to be the upper Riemann integral

$$L_k(\gamma) = \overline{\int_0^1} k_X(\gamma(t), \gamma'(t))\, dt$$

and the distance $K_X(x, y)$ is the infimum of the lengths of all piecewise C^1 curves joining x to y in X. Here and in the sequel the symbol $\gamma'(t)$ denotes the tangent vector $\gamma_*(t)1$ to X at $\gamma(t)$ for those points t, where curve γ is differentiable.

Theorem 2. *Let X be a complex Banach manifold. If $k_X(x, v) > 0$ for all $(x, v) \in T(X)$, $v \neq 0$, then the following statements are mutually equivalent for $f \in O(X, C)$.*

(1) *f is a Bloch function.*

(2) *The quantity*

$$\sup\{|(f \circ \varphi)'(0)| : \varphi \in O(\Delta, X)\} \tag{2.1}$$

is finite.

(3) *There exists a constant $L > 0$ such that*

$$|f_*(x)v| \leq L \cdot k_X(x, v) \textit{for all } (x, v) \textit{ in } T(X), \tag{2.2}$$

where $f_*(x)$ is the linear map induced by f from $T_x(X)$ to $T_{f(x)}(C)$.

(4) *There exists a constant $L > 0$ such that*

$$|f(x) - f(y)| \leq L \cdot K_X(x, y) \textit{for all } x \textit{ and } y \textit{ in } X. \tag{2.3}$$

PROOF.

$(a) \Rightarrow (b)$. Suppose $f \in O(X, C)$ is a Bloch function. By Definition 2.1, $\mathcal{F}_f$ is a normal family and so by Marty's criterion (see, for example, [9]) with $K = \{0\}$, there exists a constant $M > 0$ such that

$$\frac{|g'(0)|}{1 + |g(0)|^2} < M \quad \text{for all } g \in \mathcal{F}_f. \tag{2.4}$$

Any function $g \in \mathcal{F}_f$ satisfies $g(0) = 0$ and $g'(0) = (f \circ \varphi)'(0)$, hence (2.4) implies that the quantity $\sup\{|(f \circ \varphi)'(0)| : \varphi \in O(\Delta, X)\}$ is finite.

$(b) \Rightarrow (c)$. Suppose (2.1) holds. By the definition of k_X at $(x, v) \in T(X)$, $v \neq 0$, there exists $\psi \in O(\Delta, X)$ such that $\psi(0) = x$, $\psi_*(0)a = v$ for $a > 0$ and $a/2 < k_X(x, v) < a$. Since $(f \circ \psi)'(0) = f_*(\psi(0))\psi_*(0)1$, we obtain from (2.1)

$$|f_*(x)v| < L \cdot k_X(x, v) \quad \text{for all } (x, v) \in T(X),$$

where $L = 2 \sup\{|(f \circ \varphi)'(0)| : \varphi \in O(\Delta, X)\} < \infty$. This proves (c).

$(c) \Rightarrow (d)$. Suppose (2.2) holds. Let x and y be distinct points in X. Let $\gamma : [0, 1] \to X$ be a piecewise C^1 curve joining x to y in X. Then $f \circ \gamma : [0, 1] \to C$ is a piecewise C^1 curve joining $f(x)$ to $f(y)$ in C and

$$\begin{aligned} |f(x) - f(y)| &\leq \overline{\int_0^1} |\tfrac{d}{dt}(f \circ \gamma)(t)| dt = \overline{\int_0^1} |f_*(\gamma(t))\gamma'(t)|\, dt \\ &\leq \overline{\int_0^1} L \cdot |k_X(\gamma(t), \gamma_*(t)1)| dt = L \cdot L_k(\gamma). \end{aligned}$$

Hence

$$|f(x) - f(y)| \leq L \cdot L_k(\gamma).$$

Taking the infimum of the right-hand side over all piecewise C^1 curves γ satisfying $\gamma(0) = x$, $\gamma(1) = y$, we obtain

$$|f(x) - f(y)| \leq L \cdot K_X(x, y).$$

Therefore the function f is uniformly continuous in the considered metrics.

$(d) \Rightarrow (c)$. Let $(x, v) \in T(X)$ be given and let $t \to \gamma(t)$ be a C^1 map of an open interval $(-\varepsilon, \varepsilon)$ into X with $\gamma(0) = x$ and $\gamma'(0) = v$. Then from inequality (2.3) we obtain

$$\limsup_{|t| \to 0} \frac{|f(\gamma(t)) - f(x)|}{t} \leq L \cdot \limsup_{|t| \to 0} \frac{K_X(\gamma(0), \gamma(t))}{|t|}.$$

Because

$$\lim_{t \to 0} \frac{f(\gamma(t)) - f(x)}{t} = f_*(x)v$$

we have

$$|f_*(x)v| \le L \cdot \limsup_{|t|\to 0} \frac{K_X(\gamma(0), \gamma(t))}{|t|}.$$

By Lemma 3 of [2]

$$\limsup_{|t|\to 0} \frac{K_X(\gamma(0), \gamma(t))}{|t|} \le k_X(x, v) \quad \text{for all } (x, v) \in T(X).$$

Putting the above together we get

$$|f_*(x)v| \le L \cdot k_X(x, v) \quad \text{for all } (x, v) \in T(X).$$

$(c) \Rightarrow (b)$. Suppose (2.2) holds. Given any $\varphi \in O(\Delta, X)$. Using (2.2) with $x = \varphi(0)$ and $v = \varphi_*(0)1$, we have

$$|f_*(\varphi(0))\varphi_*(0)1| \le L \cdot k_X(\varphi(0), \varphi_*(0)1).$$

Taking into account $f_*(\varphi(0))\varphi_*(0)1 = (f \circ \varphi)'(0)$ we get

$$|(f \circ \varphi)'(0)| \le L \cdot k_X(\varphi(0), \varphi_*(0)1). \tag{2.5}$$

The distance-decreasing property of the Kobayashi metric implies

$$k_X(\varphi(0), \varphi_*(0)1) \le k_\Delta(0, 1) = 1. \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$|(f \circ \varphi)'(0)| \le L.$$

Taking the supremum of the right-hand side over all $\varphi \in O(\Delta, X)$ we obtain

$$\sup\{|(f \circ \varphi)'(0)| : \varphi \in O(\Delta, X)\} \le L.$$

This is to say that (2.1) holds.

$(b) \Rightarrow (a)$. Suppose (2.1) holds. Denote by $Aut(\Delta)$ the group of all automorphisms of Δ (one-to-one holomorphic functions of Δ onto itself). If $\varphi \in O(\Delta, X)$ and $\gamma \in Aut(\Delta)$ then $\varphi \circ \gamma \in O(\Delta, X)$. Hence,

$$|(f \circ \varphi \circ \gamma)'(0)| \le L$$

for all $\varphi \in O(\Delta, X)$ and all $\gamma \in Aut(\Delta)$. If $\gamma(0) = a$, then $|\gamma'(0)| = 1 - |a|^2$, and

$$|(f \circ \varphi)'(a)| \cdot (1 - |a|^2) \le L \text{ for all } a \in \Delta.$$

Therefore

$$\frac{|(f(\varphi) - f(\varphi(0))'(a)|}{1 + |f(\varphi(a)) - f(\varphi(0))|^2} \le |(f \circ \varphi)'(a)| \le \frac{L}{1 - |a|^2}$$

for all $a \in \Delta$. Since the right-hand side is bounded on each compact set $K \subset \Delta$, by Marty's criterion $\mathcal{F}_f$ is a normal family. This concludes the proof. ■

Remark 3. By (c) of Theorem 2 it is not difficult to see that in the case $X = \Delta$, the class of all Bloch functions coincides with the Bloch space on Δ that consists of those functions f holomorphic on Δ for which

$$\sup_{z\in\Delta} |f'(z)|(1-|z|^2) < \infty.$$

Corollary 4. *The class of all bounded holomorphic functions on X is a subspace of all Bloch functions on X.*

Proof. Let $f : X \to C$ be a bounded holomorphic function. Proposition 1 and corollary 1 of [2] give

$$|f_*(x)v| \leq A \cdot k_X(x,v) \text{for all } (x,v) \in T(X), \tag{2.7}$$

where A is an upper bound of $|f|$ on X. By Theorem 2, f is a Bloch function. ■

Remark 5. Kim and Krantz introduced the definition of normal functions on bounded domains in complex Banach spaces [5, definition 4.3]. This definition can be generalised naturally to the case of complex Banach manifolds by replacing a domain in the complex Banach space by an arbitrary complex Banach manifold. Thus in the infinite dimension case we give the following definition.

A function $f \in O(X,C)$ is said to be normal on X if there exists a constant $L > 0$ such that

$$\frac{|f_*(x)v|}{1+|f(x)|^2} \leq L \cdot k_X(x,v) \text{for all } (x,v) \text{ in } T(X), \tag{2.8}$$

where $f_*(x)$ is the linear map induced by f from $T_x(X)$ to $T_{f(x)}(C)$.

The above definition is equivalent to the following one:

A function $f \in O(X,C)$ is said to be normal on X if the family $\{f\circ\varphi : \varphi \in O(\Delta, X)\}$ is a normal family on Δ in the sense of Montel.

Corollary 6. *The class of all Bloch functions on X is a subspace of all normal functions on X. If f is a Bloch function on X, then exp(f) is a normal function on X.*

Proof. Let f be a Bloch function. By (c) of Theorem 2 the inequality (2.2) holds. Because

$$\frac{|f_*(x)v|}{1+|f(x)|^2} \leq |f_*(x)v| \text{ for all } (x,v) \text{ in } T(X),$$

(2.8) follows immediately from (2.2). Therefore f is a normal function.

Since

$$\frac{|exp(f(x))| \cdot |f_*(x)v|}{1 + |exp(f(x))|^2} \leq \frac{|f_*(x)v|}{2} \quad \text{for all } (x, v) \text{ in } T(X)$$

it is easy to check that $exp(f)$ is a normal function on X. ■

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