REVERSIBLE MAPS IN ISOMETRY GROUPS OF SPHERICAL, EUCLIDEAN AND HYPERBOLIC SPACE

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Abstract

An element of a group is *reversible* if it is conjugate to its own inverse, and it is *strongly reversible* if it is conjugate to its inverse by an involution. A group element is strongly reversible if and only if it can be expressed as a composite of two involutions. In this paper the reversible maps, the strongly reversible maps and those maps that are expressible as a composite of three involutions are determined in the isometry groups of spherical, Euclidean and hyperbolic space in several dimensions.

1. Introduction

1.1. Background on reversibility

An element g of a group G is *reversible* in G if it is conjugate to its own inverse, that is, if there is an element h in G with $hgh^{-1} = g^{-1}$. The element g is *strongly reversible* in G if it is conjugate to its own inverse by an involution, that is, if there is an involution σ in G with $\sigma g \sigma = g^{-1}$. Notice that g is strongly reversible if and only if there are involutions σ and τ in G with $g = \sigma\tau$. We omit the words 'in G' from the phrase 'reversible in G' if the group is clear from the context. The property of being reversible and, for a natural number n, the property of being the composite of n involutions are invariants of conjugation.

Reversible and strongly reversible elements have been studied in many contexts. For finite groups, the terms *real* and *strongly real* are used instead of *reversible* and *strongly reversible* [7, section 9.1] because of the connections with real characters. Questions of reversibility for classical groups have been addressed in works such as [6; 8; 9; 10]. The authors of these papers use the term *bireflectional* to describe a group comprised entirely of strongly reversible maps. There are numerous other papers on reversible maps; a brief selection is [1; 11; 13; 15].

The purpose of this paper is to determine the reversible and strongly reversible maps in the isometry groups of spherical, Euclidean and hyperbolic space in each finite dimension. For $N \geqslant 1$, let S^N denote N-dimensional spherical space, let E^N denote N-dimensional Euclidean space and let H^N denote N-dimensional

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hyperbolic space. These are the only simply connected, complete Riemannian N-manifolds with constant curvature. Denote the corresponding isometry groups by $\mathrm{Isom}(S^N)$, $\mathrm{Isom}(E^N)$ and $\mathrm{Isom}(H^N)$. These groups are each generated by reflections. Denote the three subgroups of these three isometry groups, comprised of orientation-preserving isometries, by $\mathrm{Isom}^+(S^N)$, $\mathrm{Isom}^+(E^N)$ and $\mathrm{Isom}^+(H^N)$. A map in $\mathrm{Isom}(S^N)$ lies in $\mathrm{Isom}^+(S^N)$ if and only if it can be expressed as a composite of an even number of reflections. Similar comments apply to the groups $\mathrm{Isom}^+(E^N)$ and $\mathrm{Isom}^+(H^N)$. This paper is divided into three sections: Section 2 is about S^N, Section 3 is about E^N and Section 4 is about H^N.

1.2. Statement of results

The reversible maps, strongly reversible maps and those maps that are the composite of three involutions are classified according to the six theorems below. A discussion of which results are known already is given in Section 1.3.

Theorem 1.1. *Each map in* $\mathrm{Isom}(S^N)$, $\mathrm{Isom}(E^N)$ *and* $\mathrm{Isom}(H^N)$ *is strongly reversible.*

Theorem 1.2. *Each map in* $\mathrm{Isom}^+(S^N)$, $\mathrm{Isom}^+(E^N)$ *and* $\mathrm{Isom}^+(H^N)$ *is strongly reversible if and only if it is reversible.*

Theorem 1.3. *For* $N \geqslant 2$, *each map in* $\mathrm{Isom}^+(S^N)$, $\mathrm{Isom}^+(E^{N+1})$ *and* $\mathrm{Isom}^+(H^N)$ *is the composite of three involutions.*

The cases omitted from Theorem 1.3—namely, the four groups $\mathrm{Isom}^+(S^1)$, $\mathrm{Isom}^+(E^1)$, $\mathrm{Isom}^+(E^2)$ and $\mathrm{Isom}^+(H^1)$—are considered separately in Sections 2.4, 3.4 and 4.2. It remains to describe the strongly reversible maps in $\mathrm{Isom}^+(S^N)$, $\mathrm{Isom}^+(E^N)$ and $\mathrm{Isom}^+(H^N)$. The description differs for each group, so we state three results.

Theorem 1.4. *For* $f \in \mathrm{Isom}^+(S^N)$, *either*

(i) $N \neq 1 \pmod 4$, *in which case* f *is strongly reversible or*
(ii) $N = 1 \pmod 4$, *in which case* f *is strongly reversible if and only if there is a pair of antipodal points in* S^N *that is fixed, as a set, by* f.

Theorem 1.5. *For* $f \in \mathrm{Isom}^+(E^N)$, *either*

(i) $N = 0, 3 \pmod 4$, *in which case* f *is strongly reversible;*
(ii) $N = 1 \pmod 4$, *in which case* f *is strongly reversible if and only if either* f *has a fixed point or there is a two-dimensional* f*-invariant plane; or*
(iii) $N = 2 \pmod 4$, *in which case* f *is strongly reversible if and only if either* f *is fixed point free or there is an* f*-invariant line.*

Theorem 1.6. *For* $f \in \mathrm{Isom}^+(H^N)$, *either*

(i) $N = 0, 3 \pmod 4$, *in which case* f *is strongly reversible;*
(ii) $N = 1 \pmod 4$, *in which case* f *is strongly reversible if and only if either* f *is elliptic, parabolic or there is an* f*-invariant two-dimensional plane; or*

(iii) $N = 2 \pmod 4$*, in which case* f *is strongly reversible if and only if either* f *is loxodromic, or* f *is elliptic and there is an* f*-invariant line or* f *is parabolic and there is an* f*-invariant three-dimensional plane.*

The results on spherical geometry are proven in Section 2, the results on Euclidean geometry are proven in Section 3 and the results on hyperbolic geometry are proven in Section 4.

1.3. Literature review

Most existing results on composites of involutions in isometry groups of spherical, Euclidean and hyperbolic space involve reflections. Many such results can be found, for example, in the work of Coxeter ([3] or [4, p. 99]). Since we allow ourselves the freedom of working with more general involutions than just reflections, and since there are no reflections in the orientation-preserving isometry groups, our results tend to differ from the classic results on composites of reflections.

The isometry group of S^N is the orthogonal group of $\mathbb{R}^{N+1}$ with the usual inner product, and questions of reversibility are known for orthogonal groups with general inner products. In particular, the part of Theorem 1.1 involving spherical geometry follows from results in [8]. Likewise, Theorem 1.4 and the part of Theorem 1.3 involving spherical geometry both follow from [9, theorem B]. We supply only brief proofs of results on spherical geometry; these results are used later in the sections on Euclidean and hyperbolic geometry.

It seems likely that others have considered the Euclidean and hyperbolic parts of Theorem 1.1 before, because they are straightforward to prove. Nevertheless, I am unable to find a reference to a comparable result that is valid in all dimensions. The part of Theorem 1.3 involving hyperbolic geometry was proven in a different context by Knüppel and Thomsen [10, theorem 8.8]. Knüppel and Thomsen were working with orthogonal groups with respect to general symmetric bilinear forms—not necessarily Euclidean inner products—and they were examining composites of involutions in commutator subgroups of orthogonal groups. Equip $\mathbb{R}^{N+1}$ with the Lorentz inner product $\langle x, y\rangle = x_1y_1+\ldots+x_Ny_N-x_{N+1}y_{N+1}$, then the commutator subgroup of the corresponding orthogonal group is isomorphic to $\text{Isom}^+(H^N)$. All our results on reversibility in hyperbolic space can be re-proven in a more algebraic fashion using Lorentz transformations. Finally, Theorems 1.1, 1.2, 1.3, and 1.6 have all been proven for H^5 in [12] using quaternionic Möbius transformations.

2. Spherical geometry

2.1. Spherical isometries

The standard model of N-dimensional spherical space is the unit sphere $\mathbb{S}^N$ embedded in $\mathbb{R}^{N+1}$ ([14, section 2]). The group of spherical isometries of $\mathbb{S}^N$ is the restriction of the group O_{N+1} of orthogonal maps of $\mathbb{R}^{N+1}$ to $\mathbb{S}^N$. It is convenient to identify $\mathbb{R}^{N+1}$ with the vector space of real column vectors of length $N+1$, and to identify O_{N+1} with the group of $(N+1)$-by-$(N+1)$ orthogonal matrices. The subgroup O^+_{N+1} of O_{N+1} that consists of orientation-preserving orthogonal maps is the special orthogonal group. Let $\mathrm{O}^-_{N+1} = \mathrm{O}_{N+1} \setminus \mathrm{O}^+_{N+1}$. The results of this section rest on the following well-known principal axis theorem, which is [5, theorem 30.5]. Let

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for $\theta \in \mathbb{R}$, and let I_k denote the k-by-k identity matrix.

Theorem 2.1. *In either of the groups* O_M *or* O_M^+, *each matrix is conjugate to another matrix of the form*

$$\begin{pmatrix} R_{\theta_1} & & & & \\ & \ddots & & & \\ & & R_{\theta_r} & & \\ & & & -I_s & \\ & & & & I_t \end{pmatrix}, \tag{2.1}$$

where $\theta_1, \ldots, \theta_r \in (-\pi, \pi)\backslash\{0\}$.

An orthogonal matrix is in *standard form* if it is of the same form as the matrix of (2.1). For an orthogonal matrix A, let r_A, s_A and t_A denote the integers r, s and t of a matrix in standard form that is conjugate to A. There is no ambiguity in this definition as s_A and t_A are the dimensions of the eigenspaces of -1 and 1, and $r_A = (M - s_A - t_A)/2$. The involutions in O_M are those orthogonal maps A with $r_A = 0$.

PROOF OF THE SPHERICAL PART OF THEOREM 1.1.

Given an orthogonal map A in standard form, define

$$B = \begin{pmatrix} J & & & & & \\ & \ddots & & & & \\ & & J & & & \\ & & & \epsilon_1 & & \\ & & & & \ddots & \\ & & & & & \epsilon_{s_A+t_A} \end{pmatrix}, \tag{2.2}$$

where there are r_A blocks

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the ϵ_i are either -1 or 1. Then B is both orthogonal and an involution, and $BAB = A^{-1}$, no matter how the ϵ_i are chosen from $\{-1, 1\}$. Since each orthogonal map is conjugate to an orthogonal map in standard form, Theorem 1.1 is proven for spherical isometries. ■

2.2. Proof of Theorem 1.4 and the spherical part of Theorem 1.2

The next proposition is used in the proofs of Theorems 1.4, 1.5, and 1.6.

Proposition 2.2. *Let A be an element of O_M^+ with $s_A + t_A = 0$, and let S be an element of O_M such that $SAS^{-1} = A^{-1}$. Then* $\det S = 1$ *if* $M = 0 \pmod 4$ *and* $\det S = -1$ *if* $M = 2 \pmod 4$.

PROOF. Assume, by conjugation, that A is in standard form (2.1). For $i = 1, \dots, r_A$, let V_i be the subspace spanned by the standard basis vectors e_{2i-1} and e_{2i}. Then $AS(V_i) = S(V_i)$. For $i, j \in \{1, \dots, r_A\}$, it cannot be that $S(V_i) \cap V_j$ is one-dimensional because A does not fix a one-dimensional subspace of V_j. Therefore S permutes the collection $\{V_1, \dots, V_{r_A}\}$. Thus the matrix S is an r_A-by-r_A array of two-by-two blocks such that all but one two-by-two block in each row and column is the 0 block. Suppose that X is a non-zero orthogonal two-by-two block in S, in the (i,j)th position. Then $XR_{\theta_j}X^{-1} = R_{-\theta_i}$. If $\det X = 1$ this means that $\theta_i = -\theta_j$ and if $\det X = -1$ this means that $\theta_i = \theta_j$. By studying the orbits of the V_i under S we deduce that there are an even number of blocks X with determinant 1. After interchanging an even number of columns, all non-zero two-by-two blocks of S can be brought to the leading diagonal. Hence $\det S = (-1)^{r_A}$. ■

In contrast to Proposition 2.2, if $s_A + t_A \geqslant 1$ (for example, if M is odd) then we can choose whether $\det S = 1$ or $\det S = -1$ by using the matrix B from (2.2) and adjusting the entry ϵ_1.

Proposition 2.3. *For A in O_M^+, the following are equivalent:*
(i) A is strongly reversible in O_M^+;
(ii) A is reversible in O_M^+;
(iii) either $M = 0 \pmod 4$ *or* $s_A + t_A \geqslant 1$.

PROOF. That (i) implies (ii) is true in all groups. That (ii) implies (iii) follows from Proposition 2.2 and the remarks following that proposition. Assume condition (iii). This condition ensures that the involution B from (2.2) can be chosen to have positive determinant. Hence, by conjugating A to standard form, we see that A is strongly reversible in O_M^+. ■

An orthogonal map A fixes, as a set, a line through the origin if and only if it fixes, as a set, a pair of antipodal points on the unit sphere. These circumstances occur if and only if either -1 or 1 is an eigenvalue of A, that is, if and only if $s_A + t_A \geqslant 1$. Thus we immediately deduce Theorem 1.4 and the spherical part of Theorem 1.2 from Proposition 2.3.

Proof of the spherical part of Theorem 1.3

The abelian group $\mathrm{Isom}^+(S^1)$ of rotations of the unit circle is equal to the group O_2^+. In an abelian group, the set of reversible maps, the set of strongly reversible maps and the set of involutions coincide and form a group. Hence there are elements in O_2^+—rotations by angles that are not integer multiples of π—that are not expressible as composites of involutions. For all other special orthogonal groups we have the following proposition (from which the spherical part of Theorem 1.3 follows immediately).

Proposition 2.4. *Each member of* O_M^+*,* $M \geqslant 3$*, can be expressed as the composite of three involutions in* O_M^+*. Also, each member of* O_M^+ *can be expressed as the composite of three involutions in* O_M*, two of which lie in* O_M^-*.*

Proof. Consider an element A of O_M^+, which, by conjugation, we may assume is in standard form. In Section 2.2 we saw that there is an orthogonal involution B given by (2.2) and another orthogonal involution C such that $A = I_M BC$. Either all three of these matrices have determinant 1 or else only I_M has determinant 1. We can switch between these two possibilities either by swapping the sign of ϵ_1 in B and swapping the sign of the $(2r_A + 1)$th diagonal entry in I_M, if $s_A + t_A \geqslant 1$, or else by adjusting a pair of two-by-two blocks in I_M and B using the following identity:

$$\begin{pmatrix} I_2 & \\ & I_2 \end{pmatrix}\begin{pmatrix} J & \\ & J \end{pmatrix} = \begin{pmatrix} I_2 & \\ & J \end{pmatrix}\begin{pmatrix} J & \\ & I_2 \end{pmatrix}.$$

Both parts of Proposition 2.4 have thereby been accounted for. ■

3. Euclidean geometry

3.1. Euclidean isometries

The standard model of N-dimensional Euclidean space is $\mathbb{R}^N$. The full isometry group of $\mathbb{R}^N$, denoted $\mathcal{E}_N$, consists of maps of the form $x \mapsto A(x) + v$, where $A \in \mathrm{O}_N$ and $v \in \mathbb{R}^N$. Such maps preserve orientation if and only if $A \in \mathrm{O}_N^+$. Denote the orientation-preserving isometry group of $\mathbb{R}^N$ by $\mathcal{E}_N^+$.

Each member of $\mathcal{E}^N$ either has a fixed point, in which case it is conjugate to an orthogonal map, or it does not have a fixed point. An isometry f from the latter category may be expressed in the form $f(x) = A(x) + v$, where A is an orthogonal map, $v \neq 0$, and $A(v) = v$. Such isometries have infinite order, therefore the only involutions in $\mathcal{E}^N$ are those maps that are conjugate to orthogonal involutions.

3.2. Proof of the Euclidean part of Theorem 1.1

Proof. Select a member f of $\mathcal{E}^N$. If f has a fixed point then it is conjugate to an orthogonal map, which means that it is strongly reversible, by the spherical part of Theorem 1.1. Otherwise $f(x) = A(x) + v$, where A is an orthogonal map, $v \neq 0$, and $A(v) = v$. From the spherical part of Theorem 1.1, we may choose orthogonal involutions S_1 and S_2 that both fix v such that $A = S_1 S_2$. Define Euclidean isometries σ_1 and σ_2 by the formulae $\sigma_1(x) = -S_1(x) + v$ and $\sigma_2(x) = -S_2(x)$. Both maps are involutions, and they satisfy $f = \sigma_1\sigma_2$. ■

3.3. Proof of Theorem 1.5 and the Euclidean part of Theorem 1.2.

We consider isometries with fixed points and isometries without fixed points separately.

Proposition 3.1. *Let* A *be an orthogonal map in* $\mathcal{E}_N^+$*. The following are equivalent:*

(i) *A is reversible in* O_N^+*;*
(ii) *A is strongly reversible in* $\mathcal{E}_N^+$*;*

(iii) *A is reversible in $\mathcal{E}_N^+$.*

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are straightforward. Assume (iii), that is, assume there is a Euclidean isometry g given by $g(x) = B(x) + w$, for $B \in \mathrm{O}_N^+$ and $w \in \mathbb{R}^N$, such that $gAg^{-1} = A^{-1}$. By expanding out this equation we see that $BAB^{-1} = A^{-1}$, so A is reversible in O_N^+. ■

Proposition 3.2. *Let f be an element of $\mathcal{E}_N^+$ such that $f(x) = A(x) + v$, where A is an orthogonal map, $v \neq 0$ and $A(v) = v$. The following are equivalent:*

(i) *there is a map B in O_N^+ such that $BAB^{-1} = A^{-1}$ and $B(v) = -v$;*
(ii) *f is strongly reversible in $\mathcal{E}_N^+$;*
(iii) *f is reversible in $\mathcal{E}_N^+$.*

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are straightforward. Assume (iii), that is, assume there is a Euclidean isometry g given by $g(x) = B(x) + w$, for $B \in \mathrm{O}_N^+$ and $w \in \mathbb{R}^N$, such that $gfg^{-1} = f^{-1}$. By expanding out this equation we see that $BAB^{-1} = A^{-1}$ and $B(v) + w - A^{-1}(w) = -v$. Since A is orthogonal and fixes v, if we take the scalar product of both sides of the second equation with v then we obtain $\langle B(v), v\rangle = -\langle v, v\rangle$. Since B is orthogonal, we see that $B(v) = -v$. ■

PROOF OF THE EUCLIDEAN PART OF THEOREM 1.2. Immediate from the equivalence of (ii) and (iii) in Propositions 3.1 and 3.2. ■

Before we prove Theorem 1.5 we need a lemma that explains how the geometric conditions of Theorem 1.5 relate to our results on orthogonal maps. We supply only the key steps in the proof of the lemma.

Lemma 3.3. *Let f be an element of $\mathcal{E}_N^+$.*

(i) *If f is orthogonal then there is an f-invariant line in $\mathbb{R}^N$ if and only if there is an f-invariant line through the origin.*
(ii) *If $f(x) = A(x) + v$, where $A \in \mathrm{O}_N^+$, $v \neq 0$ and $A(v) = v$, then there is an f-invariant two-dimensional plane in $\mathbb{R}^N$ if and only if there is an A-invariant line through the origin that is orthogonal to v.*

PROOF. To prove (i), suppose that f fixes a line ℓ that does not pass through the origin. Then f fixes the unique point w on ℓ that is closest to 0. Therefore the line spanned by w is fixed by f.

To prove (ii), suppose that f fixes a two-dimensional plane Π that does not pass through the origin. The line spanned by v does not intersect Π because otherwise f would fix the intersection point. This means that v is parallel to Π, which means that the unique point w on Π that is closest to 0 is fixed by A. This point w satisfies $\langle w, v\rangle = 0$, and the line spanned by w is fixed by A. ■

PROOF OF THEOREM 1.5. Let f be an element of $\mathcal{E}_N^+$. If f has a fixed point then we may assume, by conjugation, that f is an orthogonal map. In this case, by

Proposition 3.1, f is strongly reversible in $\mathcal{E}_N^+$ if and only if it is reversible in O_N^+. By Theorems 1.2 and 1.4, f is reversible in O_N^+ unless $N-1=1 \pmod 4$ and there is no f-invariant line through the origin. By Lemma 3.3, there is an f-invariant line through the origin if and only if there is an f-invariant line. Theorem 1.5 has now been established for maps with fixed points.

Now suppose that $f(x)=A(x)+v$, where A is an orthogonal map, $v\neq 0$ and $A(v)=v$. By Proposition 3.2, f is strongly reversible if and only if there is a map B in O_N^+ such that $BAB^{-1}=A^{-1}$ and $B(v)=-v$. Let A_0 and B_0 be the restrictions of A and B to the orthogonal complement of v. Since B preserves orientation if and only if B_0 reverses orientation, we see from Proposition 2.2 (and the remarks following Proposition 2.2) that such a map B exists unless $N-1=0 \pmod 4$ and $s_{A_0}+t_{A_0}=0$. Since $s_{A_0}+t_{A_0}\geqslant 1$ if and only if there is an A_0-invariant line through the origin, we can apply Lemma 3.3 to deduce Theorem 1.5 for maps without fixed points. ■

3.4. Proof of the Euclidean part of Theorem 1.3

Theorem 1.3 does not hold for $\mathrm{Isom}^+(E^1)$ or $\mathrm{Isom}^+(E^2)$. The group $\mathcal{E}_1^+$ consists only of translations $x\mapsto x+v$, where $v\in\mathbb{R}$. It is an abelian group and the only involution is the identity. In $\mathcal{E}_2^+$, the only maps of order two are of the form $x\mapsto -x+v$, for $v\in\mathbb{R}^2$. The collection of strongly reversible maps is equal to the collection of involutions and translations, and this collection is a group. The remaining maps in $\mathcal{E}_2^+$ are rotations by angles that are not integer multiples of π, and these maps are not expressible as composites of involutions.

Proof. Let f be an element of $\mathcal{E}_N^+$. If f has a fixed point then it is conjugate to an orthogonal map and we can apply Theorem 1.3 for spherical isometries. Otherwise, $f(x)=A(x)+v$, where A is an orthogonal map, $v\neq 0$ and $A(v)=v$. By Theorem 1.5, the map f is strongly reversible if N is even, so we assume that N is odd. By Proposition 2.4 we can find orthogonal involutions S_1, S_2 and S_3—two from O_N^- and one from O_N^+—that each fix v and satisfy $A=S_1S_2S_3$. We can assume that $S_1,S_2\in\mathrm{O}_N^-$ by composing suitably with the map $x\mapsto -x$. Define $\sigma_1(x)=-S_1(x)+v$, $\sigma_2(x)=-S_2(x)$, and $\sigma_3(x)=S_3(x)$. These maps are involutions in $\mathcal{E}_N^+$ that satisfy $f=\sigma_1\sigma_2\sigma_3$. ■

4. Hyperbolic geometry

4.1. Hyperbolic isometries

The *Möbius group*, $\mathcal{M}_N$, is the group of bijections of $\mathbb{R}_\infty^N$, the one-point compactification of $\mathbb{R}^N$, generated by reflections in $(N-1)$-dimensional planes and spheres. It contains $\mathcal{E}_N$. There is a subgroup $\mathcal{M}_N^+$ of $\mathcal{M}_N$ of index 2, which consists of those members of $\mathcal{M}_N$ that can be expressed as a composite of an even number of reflections and inversions. We consider $\mathbb{R}_\infty^N$ to be the ideal boundary of the upper-half space model of $(N+1)$-dimensional hyperbolic space, and identify $\mathcal{M}_N$ with the full group of hyperbolic isometries of H^{N+1}. We identify $\mathcal{M}_N^+$ with the group of orientation-preserving hyperbolic isometries of H^{N+1}. Alternatively, the unit ball $\mathbb{B}^{N+1}=\{x\in\mathbb{R}^N : |x|<1\}$ is a model of H^{N+1}, and the subgroup $\mathcal{M}_{N+1}(\mathbb{B}^{N+1})$

of $\mathcal{M}_{N+1}$ that fixes $\mathbb{B}^{N+1}$ forms the hyperbolic isometry group of $\mathbb{B}^{N+1}$. We use both models of $\mathrm{Isom}(H^{N+1})$. See [2; 14] for information on hyperbolic space. One-dimensional hyperbolic space is isometric to one-dimensional Euclidean space, and Theorems 1.1, 1.2, and 1.6 all hold for H^1, so it suffices to work with $\mathcal{M}_N$, $N \geqslant 1$, henceforth.

Möbius transformations other than the identity map can be classified according to their conjugacy type as *elliptic*, *parabolic* or *loxodromic* ([14, p. 142]). When dealing with elliptic Möbius transformations, typically one works with the ball model of hyperbolic space, as all elliptic Möbius transformations fixing the unit ball are conjugate to orthogonal maps. For the other two types of map, the upper-half space model of hyperbolic space is used. Parabolic maps are conjugate in $\mathcal{M}_N$ to Euclidean isometries and loxodromic maps are conjugate to maps of the form $x \mapsto \lambda A(x)$, where A is an orthogonal map and $\lambda > 1$. The comments of this paragraph on conjugacy apply even if we restrict to orientation-preserving Möbius transformations ([2, theorem 3.5.1]). Since parabolic and loxodromic maps are of infinite order, the only Möbius transformation involutions are those elliptic maps that, in the ball model, are conjugate to orthogonal involutions.

Let γ denote the inversion $x \mapsto x/|x|^2$, which is an orientation-reversing involution. For $\lambda > 0$ we let λ also denote the map $x \mapsto \lambda x$. The proof of the next lemma is straightforward and omitted.

Lemma 4.1. *Let A be an orthogonal map and let $\lambda > 0$. Then*
(i) $\gamma A = A\gamma$,
(ii) $\gamma\lambda = \lambda^{-1}\gamma$.

PROOF. Elliptic and loxodromic maps are conjugate to Euclidean isometries, therefore both types of map are strongly reversible by the Euclidean part of Theorem 1.1. Suppose then that f is a loxodromic member of $\mathcal{M}_N$. By conjugation we may assume that $f(x) = \lambda A(x)$, where $\lambda > 1$ and $A \in \mathrm{O}_N$. Choose orthogonal involutions S and T such that $A = ST$ and define involutions $\sigma = \lambda\gamma S$ and $\tau = \gamma T$. Then $\sigma, \tau \in \mathcal{M}_N$ and $g = \sigma\tau$. ■

4.2. Proof of Theorem 1.6 and the hyperbolic part of Theorem 1.2

To determine whether a map in $\mathcal{M}_N^+$ is strongly reversible, it is useful to consider elliptic, parabolic and loxodromic maps separately.

Proposition 4.2. *Let A be an orthogonal map in $\mathcal{M}_{N+1}^+$. The following are equivalent:*
(i) *A is reversible in O_{N+1}^+;*
(ii) *A is strongly reversible in $\mathcal{M}_{N+1}^+(\mathbb{B}^{N+1})$;*
(iii) *A is reversible in $\mathcal{M}_{N+1}^+(\mathbb{B}^{N+1})$.*

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are straightforward. Suppose that (i) is false. From Theorem 1.4 we know that the only fixed point of A is 0.

If g is an element of $\mathcal{M}_{N+1}(\mathbb{B}^{N+1})$ and $gAg^{-1} = A^{-1}$ then $Ag(0) = 0$. Therefore $g(0) = 0$. This means that g is an element of O^{-}_{N+1}. Hence (iii) is false. ■

Proposition 4.3. *Let f be a member of $\mathcal{E}^{+}_{N}$ without a fixed point. The following are equivalent:*

(i) *f is reversible in $\mathcal{E}^{+}_{N}$;*
(ii) *f is strongly reversible in $\mathcal{M}^{+}_{N}$;*
(iii) *f is reversible in $\mathcal{M}^{+}_{N}$;*

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are straightforward. Assume (iii), that is, assume there is an orientation-preserving Möbius transformation g such that $gfg^{-1} = f^{-1}$. As usual, there is an orthogonal map A that fixes a non-zero vector v such that $f(x) = A(x)+v$. Since $fg(\infty) = \infty$, and ∞ is the only fixed point of f, we see that $g(\infty) = \infty$. Therefore there is an orthogonal map W, a positive number λ and an element w of $\mathbb{R}^N$ such that $g(x) = \lambda B(x) + w$. By expanding out the equation $gfg^{-1} = f^{-1}$ we see that $w - A^{-1}(w) + \lambda B(v) = -v$. Take the scalar product of each side of this equation with v to see that $\lambda\langle B(v), v\rangle = -\langle v, v\rangle$. We deduce that $\lambda = 1$. Hence $g \in \mathcal{E}^{+}_{N}$. ■

Proposition 4.4. *Suppose that an element f of $\mathcal{M}^{+}_{N}$ satisfies the equation $f(x) = \lambda A(x)$, for $A \in \mathrm{O}^{+}_{N}$ and $\lambda > 1$. The following are equivalent:*

(i) *there exists an element B of O^{-}_{N} such that $BAB^{-1} = A^{-1}$;*
(ii) *f is strongly reversible in $\mathcal{M}^{+}_{N}$;*
(iii) *f is reversible in $\mathcal{M}^{+}_{N}$.*

PROOF. The implication (i) $\Rightarrow$ (ii) holds because the B can be chosen to be an involution, and the map $g = B\gamma$, where $\gamma(x) = x/|x|^2$, satisfies $gfg^{-1} = f^{-1}$. The implication (ii) $\Rightarrow$ (iii) is straightforward. Assume (iii), that is, assume there is an element g of $\mathcal{M}^{+}_{N}$ such that $gfg^{-1} = f^{-1}$. The map g must either fix each of the two fixed points of f, 0 and ∞, or else swap them. The only Möbius transformations fixing 0 and ∞ are orthogonal maps followed by dilations. Such maps commute with f. We conclude that $g(0) = \infty$ and $g(\infty) = 0$. This means that we can express g in the form $g(x) = \mu B\gamma(x)$, where $\mu > 1$ and B is an element of O^{-}_{N}. Condition (i) now follows from expanding out the equation $gfg^{-1} = f^{-1}$, using Lemma 4.1. ■

PROOF OF THE HYPERBOLIC PART OF THEOREM 1.2. Immediate, as (ii) and (iii) are equivalent in Propositions 4.2, 4.3 and 4.4. ■

Before we prove Theorem 1.6 we need a lemma that explains how the geometric conditions of Theorem 1.6 relate to our results on orthogonal maps. We supply only the key steps in the proof of the lemma.

Lemma 4.5. *Let f be an element of* $\mathrm{Isom}^{+}(H^{N+1})$.

(i) *If, when considered as an element of $\mathcal{M}^{+}_{N+1}(\mathbb{B}^{N+1})$, f is orthogonal, then* *there is an f-invariant hyperbolic line if and only if there is an f-invariant Euclidean line through the origin.*

(ii) *If, when considered as an element of $\mathcal{M}_N^+$, f is a Euclidean isometry, then there is an f-invariant three-dimensional hyperbolic plane if and only if there is an f-invariant two-dimensional Euclidean plane.*

(iii) *If, when considered as an element of $\mathcal{M}_N^+$, $f(x) = \lambda A(x)$, for $A \in \mathrm{O}_N^+$ and $\lambda > 0$, then there is an f-invariant two-dimensional hyperbolic plane if and only if there is an A-invariant Euclidean line through the origin.*

PROOF. To prove (i), suppose that f fixes a hyperbolic line ℓ in $\mathbb{B}^{N+1}$ that does not pass through the origin. The pair of end-points of ℓ on $\mathbb{S}^N$ are either each fixed by f, or else they are swapped. In the former case the Euclidean line through 0 and one of the fixed points is f-invariant. In the latter case, the Euclidean line through 0 that intersects ℓ orthogonally is f-invariant.

To prove (ii), suppose that f fixes a three-dimensional hyperbolic plane Π. Then f fixes the two-dimensional boundary of Π, which must be a Euclidean plane, since f does not fix any compact sets in $\mathbb{R}^N$.

To prove (iii), suppose that f fixes a two-dimensional hyperbolic plane Π. Then f fixes the one-dimensional boundary of Π, which must be a Euclidean line through 0 and ∞, since these points are the attracting and repelling fixed points of f. ■

PROOF OF THEOREM 1.6. Let f be an element of $\mathrm{Isom}^+(H^N)$, $N \geqslant 2$. If f is elliptic then, by conjugation, we consider it to be an orthogonal map in $\mathcal{M}_N(\mathbb{B}^N)$. Using Theorem 1.4 and Proposition 4.2 we see that f is strongly reversible unless $N-1 = 1 \pmod 4$ and there is no f-invariant line through the origin. We deduce Theorem 1.6 for elliptic maps from Lemma 4.5 (i).

If f is a parabolic map then, by conjugation, we consider it to be a fixed point free element of $\mathcal{E}_{N-1}^+$ within $\mathcal{M}_{N-1}^+$. Using Theorem 1.5 and Proposition 4.3 we see that f is strongly reversible unless $N-1 = 1 (\mathrm{pmod} 4)$ and there is no f-invariant two-dimensional Euclidean plane in $\mathbb{R}^{N-1}$. We deduce Theorem 1.6 for parabolic maps from Lemma 4.5 (ii).

If f is loxodromic then, by conjugation, we may assume that f is an element of $\mathcal{M}_{N-1}^+$ of the form $f(x) = \lambda A(x)$, for $A \in \mathrm{O}_{N-1}^+$ and $\lambda > 1$. Using Proposition 2.2 and Proposition 4.4 we see that f is strongly reversible unless $N-1 = 0 \pmod 4$ and there is no A-invariant Euclidean line through the origin in $\mathbb{R}^{N-1}$. We deduce Theorem 1.6 for loxodromic maps from Lemma 4.5 (iii). ■

4.3. Proof of the hyperbolic part of Theorem 1.3

PROOF. Choose f in $\mathcal{M}_N^+$, $N \geqslant 1$. If f is elliptic or parabolic, then f can be expressed as a composite of three involutions, by the part of Theorem 1.3 concerning Euclidean geometry. If f is loxodromic then, by conjugation, we can assume that $f(x) = \lambda A(x)$, where A is orthogonal and $\lambda > 1$. By Proposition 2.4 there are involutions $S_1, S_2 \in \mathrm{O}_N^-$ and $S_3 \in \mathrm{O}_N^+$ such that $A = S_1S_2S_3$. Define involutions $\sigma_1 = \lambda\gamma S_1$, $\sigma_2 = \gamma S_2$ and $\sigma_3 = S_3$ in $\mathcal{M}_N^+$. Then $f = \sigma_1\sigma_2\sigma_3$. ■

4.4. Examples

In this section, the results on strongly reversible orientation-preserving hyperbolic isometries are reproved in two and three dimensions using alternative representations of the hyperbolic isometry groups.

First consider the group of orientation-preserving Möbius transformations fixing the unit disc. Each non-trivial map f from this group is the composite of two distinct reflections. That is, $f = \tau_1\tau_2$, where τ_1 is a reflection in a hyperbolic line ℓ_1 and τ_2 is a reflection in a different hyperbolic line ℓ_2. The map f is classified as elliptic, parabolic, or loxodromic, depending on whether ℓ_1 and ℓ_2 intersect, are disjoint and meet on the ideal boundary point, or are disjoint and do not meet on the ideal boundary (see Figure 1).

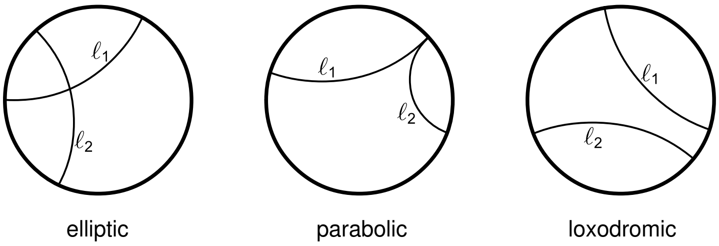

FIG. 1—The dynamic type of a Möbius map that fixes the unit disc can be classified geometrically according to whether ℓ_1 and ℓ_2 intersect, are parallel or are disjoint and not parallel.

Only elliptic maps and the identity map are ever of finite order, and an elliptic map is of order two if and only if its angle of rotation is π. The angle of rotation is π if and only if the lines ℓ_1 and ℓ_2 intersect orthogonally. The next theorem is Theorem 1.6 in the special case $N = 2$.

Theorem 4.6. *A Möbius transformation fixing the unit disc that is not an involution is strongly reversible if and only if it is loxodromic.*

PROOF. Suppose that f is a loxodromic map, say $f = \tau_1\tau_2$ for reflections τ_1 and τ_2 in disjoint and not parallel lines ℓ_1 and ℓ_2. Define ℓ_3 to be the unique hyperbolic line orthogonal to both ℓ_1 and ℓ_2, and define τ_3 to be the reflection in ℓ_3. The maps $\sigma_1 = \tau_1\tau_3$ and $\sigma_2 = \tau_3\tau_2$ are both elliptic maps of order two, and $f = \sigma_1\sigma_2$. Conversely, define f to be the composite $\sigma_1\sigma_2$ for elliptic rotations σ_1 and σ_2 of order two about distinct points p_1 and p_2. Define ℓ_3 to be the unique hyperbolic line containing p_1 and p_2. Define ℓ_1 to be the unique hyperbolic line containing p_1 that is orthogonal to ℓ_3 and define ℓ_2 to be the unique hyperbolic line containing p_2 that is orthogonal to ℓ_3. Note that ℓ_1 and ℓ_2 are disjoint and do not meet on the ideal boundary. Let τ_1, τ_2 and τ_3 be the corresponding reflections in ℓ_1, ℓ_2 and ℓ_3. Then $\sigma_1 = \tau_1\tau_3$ and $\sigma_2 = \tau_3\tau_2$. Therefore $f = \tau_1\tau_2$, so f is loxodromic. ■

It is straightforward to prove the other theorems of Section 1 for two-dimensional hyperbolic space with techniques similar to those used in Theorem 4.6. The strongly reversible maps in $\mathcal{E}_2^+$ can also be classified in a similar fashion.

We finish with an alternative proof of Theorem 1.6 in the case $N = 3$, using the well known representation of $\mathcal{M}_2^+$ in terms of fractional linear transformations of $\mathbb{C}_\infty$. (Maps of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$.) After conjugating suitably we can assume that a map f in $\mathcal{M}_2^+$ either fixes only the point ∞, or else fixes both 0 and ∞. In the first case $f(z) = z + v$, $v \neq 0$, and in the second case $f(z) = \lambda z$, $\lambda \neq 0$. The identities $z + v = -(-z) + v$ and $\lambda z = \lambda(z^{-1})^{-1}$ suggest how to decompose f as a composite of two involutions. Thus, in agreement with Theorem 1.6, all members of $\mathrm{Isom}^+(H^3)$ are strongly reversible.

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REFERENCES

[1] L.V. Ahlfors, *Möbius transformations in several dimensions*, University of Minnesota lecture notes, Minnesota, 1981.

[2] M. Baake and J.A.G. Roberts, Reversing symmetry groups of GL(2, Z) and PGL(2, Z) matrices with connections to cat and trace maps, *Journal of Physics A: Mathematical and Theoretical* **30** (1997), 1549–73.

[3] A.F. Beardon, *The geometry of discrete groups*, Springer-Verlag, Berlin and New York, 1983.

[4] C.W. Curtis, *Linear algebra: an introductory approach*, Undergraduate texts in mathematics, Springer, New York, 1984.

[5] D.Z. Djokovic, Products of two involutions, *Archiv der Mathematik* **18** (1967), 582–4.

[6] E.W. Ellers and O. Villa, The special orthogonal group is trireflectional, *Archiv der Mathematik* **82** (2004), 122–7.

[7] A. Gomez and J.D. Meiss, Reversible polynomial automorphisms of the plane: the involutary case, *Physics Letters A* **312** (2003), 49–58.

[8] D. Gorenstein, *Finite groups*, Harper's series in modern mathematics, Harper and Row, New York, 1968.

[9] F. Knuppel and K. Nielsen, On products of two involutions in the orthogonal group of a vector space, *Linear Algebra and its Applications* **94** (1987), 209–16.

[10] F. Knuppel and K. Nielsen, Products of involutions in O+(V), *Linear algebra and its applications* **94** (1987), 217–22.

[11] F. Knuppel and G. Thomsen, Involutions and commutators in orthogonal groups, *Australian Mathematical Society Journal Series A. Pure Mathematics and Statistics* **65** (1) (1998), 1–36.

[12] J.S.W. Lamb and J.A.G. Roberts, Time-reversal symmetry in dynamical systems: a survey, *Physica D. Nonlinear Phenomena* **112** (1998), 1–49.

[13] R. Lávička, A.G. O'Farrell and I. Short, Reversible maps in the group of quaternionic mobius transformations, *Mathematical Proceedings of the Cambridge Philosophical Society* **143** (2007), 57–69

[14] A.G. O'Farrell, Conjugacy, involutions and reversibility for real homeomorphisms, *Irish Mathematical Society Bulletin* **54** (2004), 41–52.

[15] J.G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts, No. 149, Springer-Verlag, New York, 1994.