OSCILLATORY INTEGRALS FOR BOCHNER–RIESZ OPERATORS WITH CRITICAL ORDER

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Abstract

We establish the L^p boundedness for oscillatory integrals for Bochner–Riesz operators with critical order provided that $1 < p < \infty$.

1. Introduction

Let $n \geq 2$ denote the dimension of the Euclidean space. For f defined on $\mathbb{R}^n$, $\hat{f}$ denotes the Fourier transform of f, and $f^\vee$ denotes the inverse Fourier transform of f.

We define the Bochner–Riesz operator

$$(B_\tau^\delta f)\hat{}(\xi) = (1 - \tau^2|\xi|^2)_+^\delta \hat{f}(\xi).$$

Let

$$\varphi(x) = [(1 - |\xi|^2)_+^\delta]^\vee(x),\ \delta > 0,\ (a)_+ = \max(a, 0).$$

Then, the Bochner–Riesz operator $B_\tau^\delta f$ can be rewritten as

$$B_\tau^\delta f(x) = (f * \varphi_\tau)(x),$$

where $\varphi_\tau(x) = \tau^{-n}\varphi(\frac{x}{\tau})$. In particular, we say that $B_\tau^\delta f$ is a Bochner–Riesz operator with critical order when $\delta = \frac{n-1}{2}$, that is,

$$B_\tau^{\frac{n-1}{2}} f(x) = (f * \phi_\tau)(x), \tag{1.1}$$

where $\phi(x) = [(1 - |\xi|^2)_+^{\frac{n-1}{2}}]^\vee(x)$.

The Bochner–Riesz operators have been studied by many [see 4; 5 and 6, for example]. In particular, E.M. Stein [6] proved L^p boundedness for Bochner–Riesz operators with critical order for $1 < p < \infty$. Then, X. Shi and Q. Sun [5] obtained weighted L^p boundedness for Bochner–Riesz operators with critical order for $1 < p < \infty$.

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It is well known that in 1987, Ricci and Stein [4] proved that a class of oscillatory singular integral operator with polynomial phases and smooth kernel is bounded on $L^p(\mathbb{R}^n)$. In 1992, Lu and Zhang [3] extended the above results to the case of rough kernel; see also [2]. Naturally, it is an interesting question whether the oscillatory integrals for Bochner–Riesz operators with critical order are L^p bounded for $1 < p < \infty$.

In this note, we will answer the question. More precisely, we have the following results.

Theorem 1.1. *Let $p(x, y)$ be a real valued polynomial, and let ϕ_τ be the same as (1.1), the oscillatory integral for Bochner–Riesz operators with critical order defined by*

$$B_\tau f(x) = \int_{\mathbb{R}^n} e^{ip(x,y)} \phi_\tau(x - y) f(y) dy.$$

Then for $1 < p < \infty$

$$\|B_\tau f\|_p \le C\|f\|_p,$$

where C is independent of the coefficients of $p(x, y)$, τ and f, but depends on the degree of $p(x, y)$.

Furthermore, we have the following weighted result for the oscillatory integral of Bochner–Riesz operators with critical order.

Theorem 1.2. *Let $B_\tau f$ be the same as in Theorem 1.1, and $\omega \in A_p(\mathbb{R}^n)$(the Mouckenhoupt weight class). Then for $1 < p < \infty$*

$$\|B_\tau f\|_{p,\omega} \le C\|f\|_{p,\omega},$$

where C is independent of the coefficients of $p(x, y)$ and f, but depends on the degree of $p(x, y)$.

Throughout this paper, C is a positive constant that is independent of the main parameters and not necessarily the same at each occurrence. For a measurable set E, denote by χ_E the characteristic function of E. For f defined on $\mathbb{R}^n$, $\hat{f}$ denotes the Fourier transform of f.

2. Proof of the Theorems

First, we state several lemmas.

Lemma 2.1. [see 7, p. 169] *Let ϕ be the same as in (1.1). Then*

(i) $\phi(x) = C\dfrac{J_{n-\frac{1}{2}}(|x|)}{|x|^{n-\frac{1}{2}}}$, *$J_v(x)$ denotes the Bessel function;*

(ii) $|\phi(x)| \le C(1 + |x|)^{-n}$*;*

(iii) $|\nabla\phi(x)| \le C(1+|x|)^{-n}$.

Lemma 2.2. (see [4]) *Suppose* $p(x)=\sum_{|\alpha|\le d} a_\alpha x^\alpha$ *is a polynomial of degree* d *in* $\mathbb{R}^n$*, with* $\epsilon < 1/d$*. Then*

$$\sup_{y\in\mathbb{R}^n}\int_{|x|\le 1} |p(x-y)|^{-\epsilon} d\sigma(x) \le A_\epsilon \left(\sum_{|\alpha|=d} |a_\alpha|\right)^{-\epsilon}.$$

Now, let us turn to prove the theorems.

PROOF OF THEOREM 1.1. Write

$$p(x,y)=\sum_{|\alpha|\le k,|\beta|\le l} a_{\alpha,\beta}x^\alpha y^\beta.$$

By the dilation invariance, we may assume that $\sum_{|\alpha|=k,|\beta|=l}|a_{\alpha\beta}|=1$.

Decompose

$$\begin{aligned} B_\tau f(x) &= \int_{|x-y|\le\tau} e^{ip(x,y)}\phi_\tau(x-y)f(y)dy \\ &\quad + \int_{|x-y|>\tau} e^{ip(x,y)}\phi_\tau(x-y)f(y)dy \\ &= T_0 f(x) + T_\infty f(x). \end{aligned}$$

For T_0, it is easy to see that

$$\left|\int_{|x-y|<\tau} e^{ip(x,y)}\phi_\tau(x-y)f(y)dy\right| \le C\tau^{-n}\int_{|x-y|<\tau}|f(y)|dy \le CMf(x),$$

where M denotes the standard Hardy–Littlewood function.

Then, we have

$$\|T_0 f\|_p \le C\|f\|_p. \tag{2.1}$$

Now we return to consider T_∞.We write

$$\begin{aligned} T_\infty f(x) &= \sum_{j=1}^{\infty}\int_{2^{j-1}\tau\le|x-y|<2^j\tau} e^{ip(x,y)}\phi_\tau(x-y)f(y)dy \\ &= \sum_{j=1}^{\infty} T_j f(x). \end{aligned}$$

We next consider two cases for τ.

Case 1. $0<\tau\le 1$. We claim that there exists a positive number δ such that

$$\|T_j f\|_p \le C2^{-j\delta}\|f\|_p \tag{2.2}$$

holds for $j \geq 1$. From (2.2), we obtain

$$\|T_\infty f\|_p \leq C\|f\|_p \left(1 + \sum_{j=1}^{\infty} 2^{-j\delta}\right) \leq C\|f\|_p,$$

which together with (2.1) will prove the theorem in the case $0 < \tau \leq 1$. We now prove (2.2). To do this we consider $T_j^* T_j$. This operator has as its kernel $\bar{L}_j(y,z)$ given by

$$\int_{2^{j-1}\tau<|x-z|,|x-y|<2^j\tau} e^{i(p(x,z)-p(x,y))}\phi_\tau(x-z)\phi_\tau(x-y)dx.$$

By rescaling we would obtain the same norm if we replace $\bar{L}_j(y,z)$ by $L_j(y,z)$ defined by

$$L_j(y,z) = 2^{jn}\bar{L}_j(2^j y, 2^j z).$$

Then

$$[L_j(y,z) = \int_{\tau/2<|x-z|,|x-y|<\tau} e^{i(p(2^j x,2^j z)-p(2^j x,2^j y))}\phi_{2^{-j}\tau}(x-z)\phi_{2^{-j}\tau}(x-y)dx. \tag{2.3}$$

Now we make the changes of variables $x \to x+y$ in (2.3), and then write x-integration in polar coordinates with $x = rx', r = |x|,\ |x'| = 1,\ dx = r^{n-1}drd\sigma(x)$. We now write $p(x,y)$ as follows

$$p(x,y) = \sum_{|\alpha|=k} x^\alpha Q_\alpha(y) + R(x,y),$$

where $R(x,y)$ is a polynomial with x-degree less than k, and $Q_\alpha(y)$ is a polynomial with degree l. So we can write

$$L_j(y,z) = \int_{|x'|=1}\int_{\frac{\tau}{2}<r<\tau,\frac{\tau}{2}<|rx'-z+y|<\tau} e^{i(E+F)}\Phi_{j,\tau}(r,x')drd\sigma(x'),$$

where

$$E = (2^j r)^k \sum_{|\alpha|=k} {x'}^\alpha[Q_\alpha(2^j z) - Q_\alpha(2^j y)],$$

and F with r-degree less than k, and

$$\Phi_{j,\tau}(r,x') = \phi_{2^{-j}\tau}(rx'-z+y)\phi_{2^{-j}\tau}(r)r^{n-1}.$$

By Van der Corput's lemma (see [6, p. 332]), for any $\frac{1}{2} \leq t \leq 1$, we have

$$\left|\int_{\frac{\tau}{2}}^{t\tau} e^{i(E+F)}dr\right| \leq C\left|(2^j\tau)^k \sum_{|\alpha|=k} {x'}^\alpha[Q_\alpha(2^j\ z) - Q_\alpha(2^j y)]\right|^{-\frac{1}{k}}.$$

From integration by parts, and using Lemma 2.1, we have

$$\begin{aligned}
&\left|\int_{\frac{\tau}{2}<r<\tau,\frac{\tau}{2}<|rx'-z+y|<\tau} e^{i(E+F)}\Phi_{j,\tau}(r,x')dr)\right| \\
&\qquad \le C\left|(2^j\tau)^k\sum_{|\alpha|=k} {x'}^{\alpha}[Q_\alpha(2^jz)-Q_\alpha(2^jy)]\right|^{-\frac{1}{k}} \\
&\qquad\quad \times\left(1+\int_{\frac{\tau}{2}<r<\tau,\frac{\tau}{2}<|rx'-z+y|<\tau}|d\Phi_{j,\tau}(r,x')|\right) \\
&\qquad \le C\tau^{-1}\left|\sum_{|\alpha|=k} {x'}^{\alpha}[Q_\alpha(2^jz)-Q_\alpha(2^jy)]\right|^{-\frac{1}{k}}.
\end{aligned} \tag{2.4}$$

Using Lemma 2.1, it is easy to see that

$$\left|\int_{\frac{\tau}{2}<r<\tau,\frac{\tau}{2}<|rx'-z+y|<\tau} e^{i(E+F)}\Phi_{j,\tau}(r,x')dr\right| \le C\tau^n\chi_{B_{2\tau}}(y-z),$$

where $\chi_{B_{2\tau}}$ is the characteristic function of the ball radius 2τ centered at zero.

Since $\tau<1$, from (2.4) and the above inequality, we get the estimate

$$\begin{aligned}
&\left|\int_{\frac{\tau}{2}<r<\tau,\frac{\tau}{2}<|rx'-z+y|<\tau} e^{i(E+F)}\Phi_{j,\tau}(r,x')dr\right| \\
&\qquad \le C\left|\sum_{|\alpha|=k} {x'}^{\alpha}[Q_\alpha(2^jz)-Q_\alpha(2^jy)]\right|^{-\frac{\delta_1}{k}}\chi_{B_{2\tau}}(y-z),
\end{aligned}$$

which holds uniformly in $\delta_1\in(0,1]$. Thus, by Lemma 2.2, we get

$$|L_j(y,z)|\le C\left|\sum_{|\alpha|=k}[Q_\alpha(2^jz)-Q_\alpha(2^jy)]\right|^{-\frac{\delta_1}{k}}\chi_{B_{2\tau}}(y-z). \tag{2.5}$$

Now, we take $\delta_1\in(0,1]$ such that $\delta_1/k<1/l$, then from Lemma 2.2, it follows

$$\begin{aligned}
\int|L_j(y,z)|dz &\le C2^{-\delta_1 j}\int_{|z-y|\le 2}\left|\sum_{|\alpha|=k}[Q_\alpha(2^jz)-Q_\alpha(2^jy)]\right|^{-\frac{\delta_1}{k}}dz \\
&\le C2^{-lj\delta_1/k}=C2^{-2j\epsilon}.
\end{aligned}$$

That is,

$$\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}|L_j(y,z)|dy\le C2^{-2j\epsilon}.$$

Similarly,

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|L_j(y,z)|dz \le C2^{-2j\epsilon}.$$

Hence, we have

$$\|T_jf\|_2 \le C2^{-j\epsilon}\|f\|_2. \tag{2.6}$$

Using (ii) of Lemma 2.1, we have

$$\begin{aligned}|T_jf(x)| &\le \int_{2^{j-1}\tau\le|x-y|<2^j\tau}|\phi_\tau(x-y)f(y)|dy\\ &\le C\int_{2^{j-1}\tau\le|x-y|<2^j\tau}\frac{|f(y)|}{|x-y|^n}dy\\ &\le CMf(x).\end{aligned}$$

Thus

$$\|T_jf\|_{p_0} \le C_{p_0}\|f\|_{p_0}, \quad for\ 1<p_0<\infty. \tag{2.7}$$

By interpolation between (2.6) and (2.7), we obtain (2.2).

Case 2. $\tau\ge 1$. We claim that there exists a positive δ such that

$$\|T_jf\|_p \le C(2^j\tau)^{-\delta}\|f\|_p \tag{2.8}$$

holds for $j\ge 1$.

The proof is parallel to the proof in case 1. We now prove (2.8). To do this we consider $T_j^*T_j$. This operator has as its kernel $\bar{L}_j(y,z)$ given by

$$\int_{2^{j-1}\tau<|x-z|,|x-y|<2^j\tau} e^{i(p(x,z)-p(x,y))}\phi_\tau(x-z)\phi_\tau(x-y)dx.$$

By rescaling we would obtain the same norm if we replace $\bar{L}_j(y,z)$ by $L_j(y,z)$ defined by

$$L_j(y,z) = (2^j\tau)^n\bar{L}_j(2^j\tau y, 2^j\tau z).$$

Then

$$L_j(y,z) = \int_{1/2<|x-z|,|x-y|<1} e^{i(p(2^j\tau x,2^j\tau z)-p(2^j\tau x,2^j\tau y))}\phi_{2^{-j}}(x-z)\phi_{2^{-j}}(x-y)dx. \tag{2.9}$$

It is easy to see that

$$|L_j(y,z)| \le C\chi_{B_2}(y-z), \tag{2.10}$$

where χ_{B_2} is the characteristic function of the ball radius 2 center at zero.

Now we make the changes of variables $x\to x+y$ in (2.9), and then write x-integration in polar coordinates with $x=rx', r=|x|,\ |x'|=1,\ dx=r^{n-1}drd\sigma(x)$.

We now write $p(x, y)$ as follows

$$p(x, y) = \sum_{|\alpha|=k} x^\alpha Q_\alpha(y) + R(x, y),$$

where $R(x, y)$ is a polynomial with x-degree less than k, and $Q_\alpha(y)$ is a polynomial with degree l. So we can write

$$L_j(y, z) = \int_{|x'|=1} \int_{\frac{1}{2}<r<1, \frac{1}{2}<|rx'-z+y|<1} e^{i(E+F)} \Phi_j(r, x') dr d\sigma(x'),$$

where

$$E = (2^j \tau r)^k \sum_{|\alpha|=k} x'^\alpha [Q_\alpha(2^j \tau z) - Q_\alpha(2^j \tau y)],$$

and F with r-degree less than k, and

$$\Phi_j(r, x') = \phi_{2^{-j}}(rx' - z + y) \phi_{2^{-j}}(r) r^{n-1}.$$

By Van der Corput's lemma (see [6]), for any $1/2 \leq t \leq 1$, we have

$$\left| \int_{\frac{1}{2}}^{t} e^{i(E+F)} dr \right| \leq C \left| (2^j \tau)^k \sum_{|\alpha|=k} x'^\alpha [Q_\alpha(2^j \tau z) - Q_\alpha(2^j \tau y)] \right|^{-\frac{1}{k}}.$$

From integration by parts, and using Lemma 1, we have

$$\begin{aligned} &\left| \int_{\frac{1}{2}<r<1, \frac{1}{2}<|rx'-z+y|<1} e^{i(E+F)} \Phi_j(r, x') dr \right| \\ &\leq C \left| (2^j \tau)^k \sum_{|\alpha|=k} x'^\alpha [Q_\alpha(2^j \tau z) - Q_\alpha(2^j \tau y)] \right|^{-\frac{1}{k}} \\ &\quad \times \left(1 + \int_{\frac{1}{2}<r<1, \frac{1}{2}<|rx'-z+y|<1} |d\Phi_j(r, x')| \right) \\ &\leq C \left| (2^j \tau)^k \sum_{|\alpha|=k} x'^\alpha [Q_\alpha(2^j \tau z) - Q_\alpha(2^j \tau y)] \right|^{-\frac{1}{k}}. \end{aligned}$$

From (2.10) and the above inequality, we get the estimate

$$\begin{aligned} &\left| \int_{\frac{1}{2}<r<1, \frac{1}{2}<|rx'-z+y|<1} e^{i(E+F)} \Phi_j(r, x') dr \right| \\ &\leq C \left| (2^j \tau)^k \sum_{|\alpha|=k} x'^\alpha [Q_\alpha(2^j \tau z) - Q_\alpha(2^j \tau y)] \right|^{-\frac{\delta_1}{k}} \chi_{B_2}(y - z), \end{aligned}$$

which holds uniformly in $\delta_1 \in (0,1]$. Thus, by Lemma 2.2, we get

$$|L_j(y,z)| \leq C(2^j\tau)^{-\delta_1} \left| \sum_{|\alpha|=k} [Q_\alpha(2^j\tau z) - Q_\alpha(2^j\tau y)] \right|^{-\frac{\delta_1}{k}} \chi_{B_2}(y-z).$$

Now, we take $\delta_1 \in (0,1]$ such that $\delta_1/k < 1/l$, then from Lemma 2.2, it follows

$$\begin{aligned} \int |L_j(y,z)|dz &\leq C(2^j\tau)^{-\delta_1} \int_{|z-y|\leq 2} \left| \sum_{|\alpha|=k} [Q_\alpha(2^j\tau z) - Q_\alpha(2^j\tau y)] \right|^{-\frac{\delta_1}{k}} dz \\ &\leq C(2^j\tau)^{-\delta_1}(2^j\tau)^{-l\delta_1/k} = C(2^j\tau)^{-2\epsilon}. \end{aligned}$$

That is,

$$\sup_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} |L_j(y,z)|dy \leq C(2^j\tau)^{-2\epsilon}.$$

Similarly,

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |L_j(y,z)|dz \leq C(2^j\tau)^{-2\epsilon}.$$

Hence, we have

$$\|T_j f\|_2 \leq C(2^j\tau)^{-\epsilon}\|f\|_2.$$

Adapting the arguments in the proof of (2.7), there exists a positive number δ such that

$$\|T_j f\|_p \leq C(2^j\tau)^{-\delta}\|f\|_p$$

holds for $j \geq 1$. Thus, (2.8) is proved.

Since $\tau > 1$, from (2.8), we obtain

$$\|T_\infty f\|_p \leq C\|f\|_p \left(1 + \sum_{j=1}^{\infty} (2^j\tau)^{-\delta}\right) \leq C\|f\|_p.$$

Thus, the proof of Theorem 1.1 is complete. ■

Proof of Theorem 1.2. Adapting the idea of [2], similar to the proof Theorem 1.1, we can obtain the desired result. ■

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