ON THE SUM OF THE CHARACTER DEGREES OF A FINITE GROUP

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Abstract

Let $T(G)$ be the sum of the degrees of the irreducible complex representations of a finite group G, and suppose that p^n divides $|G|$ where p is a prime number. We produce a lower bound for $T(G)$ in general. For n at most 6 we produce best possible bounds for $T(G)$.

1. Introduction

In [8] and [9] Héthelyi and Külshammer proved the following remarkable results: Let G be a finite soluble group with exactly $k(G)$ conjugacy classes, and let p be a prime number. Then

(1) if p divides $|G|$ then $k(G) \geq 2\sqrt{p-1}$; and
(2) if p^2 divides $|G|$ then $k(G) \geq (49p+1)/60$.

Let $T(G) = \sum_{i=1}^{k(G)} d_i$, where the d_i are the degrees of the irreducible complex representations of G. We note that the complex group algebra $\mathbb{C}G$ is naturally a module for G and $\mathbb{C}G$ is a direct sum of exactly $T(G)$ irreducible modules. In this paper we derive results for $T(G)$ that are analogous to those of [8] and [9] but without the restriction of solubility. Specifically, we investigate lower bounds for $T(G)$ in terms of the prime powers dividing $|G|$.

Clearly $T(G) = |G|$ if and only if G is abelian.

We make use of the following result of Berkovich and Mann [3, p. 647]: if H is a proper subgroup of G then $T(H) < T(G)$. We note the corresponding result for conjugacy classes does not hold — consider D_5 the dihedral group of order 10. This has an abelian subgroup H of order 5, so $k(H) = 5$; however, $k(D_5) = 4$.

Let p denote a prime number throughout. Other notation is standard with the following addition — $\mathcal{G}_p$ denotes the set of all finite groups with order divisible by the prime p but by no smaller prime.

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2. Main Results

Theorem 1. *If* p^n *divides* $|G|$ *then* $T(G) \geq p^r$ *where* $r = \frac{1}{2}[\sqrt{8n+1}-1]$, *where the square brackets indicate the greatest integer function. (For example, if* p^{10} *divides* $|G|$ *then* $T(G) \geq p^4$*).*

Proof. Since p^n divides $|G|$, G has a subgroup P of order p^n. Now, by a result of Miller *et al.* [15, p.120] P has an abelian subgroup A of order p^r where $r(r+1) \geq 2n$. Solving the equation $r^2 + r - 2n = 0$ we obtain $r = \frac{1}{2}[\sqrt{8n+1} - 1]$ so that in the inequality we have $r \geq \frac{1}{2}[\sqrt{8n+1} - 1]$. Now, A is abelian so $p^r = |A| = T(A) \leq T(P) \leq T(G)$, which gives $T(G) \geq p^{\frac{1}{2}[\sqrt{8n+1}-1]}$. ■

While Theorem 1 holds for all finite groups, it is by no means the best possible result. By examining specific values of n we can achieve better results in general.

Theorem 2. *If* p *divides* $|G|$ *then* $T(G) \geq p$.

Proof. If p divides $|G|$, then G has a subgroup P of order p and $T(G) \geq T(P) = p$. This result is best possible since $T(C_p) = p$, where C_p is the cyclic group of order p. ■

Theorem 3. *If* $T(G) = p$, *where* p *is a prime, then* $G \cong C_p$.

This result is originally due to Isaacs [11]. For a proof see [1].

Theorem 4. *If* p^2 *divides* $|G|$ *then* $T(G) \geq p^2$.

Proof. If p^2 divides $|G|$ then G has a subgroup P of order p^2, which is abelian, and so $T(G) \geq T(P) = p^2$. ■

We note than if $T(G) = p^2$ then G is not necessarily abelian, e.g. $T(S_3) = 4$ and $T(G_{21}) = 9$ where G_{21} is the non-abelian group of order 21. However, if G is a p-group then clearly it must be abelian of order p^2.

Lemma 1. *If* H *is a non-abelian group,* $H \in \mathcal{G}_p$ *then* $T(H) \leq \frac{2p-1}{p^2}|H|$ *with equality if and only if* $(H : Z(H)) = p^2$, *where* p *is a prime.*

For a proof see [1].

Theorem 5. *If* p^3 *divides* $|G|$ *then* $T(G) \geq 2p^2 - p$. *This is the best possible result.*

Proof. G has a subgroup H of order p^3. If H is abelian then $T(H) = p^3$ and $T(G) \geq T(H) = p^3 \geq 2p^2 - p$, since $p(p-1)^2 > 0$. If H is non-abelian $(H : Z(H)) = p^2$, so $T(H) = \frac{2p-1}{p^2}|H| = 2p^2 - p$, so $T(G) \geq T(H) = 2p^2 - p$, and the result is established. ■

The non-abelian groups of order p^3 show that this result is best possible for both $p = 2$ and p is odd.

We note that Theorem 1 in this case gives merely $T(G) \geq p^2$.

Theorem 6. *If p^4 divides $|G|$ then $T(G) \geq p^3 + p^2 - p$. This is the best possible result.*

PROOF. If p^4 divides $|G|$, then G has a subgroup P of order p^4 and $T(G) \geq T(P)$. So, we must consider possible values of $T(P)$ for $|P| = p^4$. If P is abelian, then $T(P) = p^4$ and $T(G) \geq T(P) = p^4 \geq p^3 + p^2 - p$. By Burnside [5, p. 146], if P is non-abelian and $|P| = p^4$, then $k(P)$ is either $p^3 + p^2 - p$ or $2p^2 - 1$.

Now $|P| = p^4 = \alpha + \beta p^2$, where $\alpha = (P : P')$ and β is a non-negative integer. Furthermore $k(P) = \alpha + \beta$.

If $k(P) = p^3 + p^2 - p$ then we have

$$p^4 - p^3 - p^2 + p = \beta(p^2 - 1),$$

so

$$\beta = p^2 - p$$

and

$$\alpha = p^3.$$

Thus $T(P) = 2p^3 - p^2$ and $T(G) \geq T(P) = \alpha + \beta p = 2p^3 - p^2 \geq p^3 + p^2 - p$ since $p(p-1)^2 \geq 0$.

Finally, if $k(P) = 2p^2 - 1$ then

$$p^4 - 2p^2 + 1 = \beta(p^2 - 1)$$

so

$$\beta = p^2 - 1$$

and

$$\alpha = p^2.$$

Thus $T(P) = p^3 + p^2 - p$ and $T(G) \geq T(P) = p^3 + p^2 - p$, as required. ■

Theorem 7. *If p^5 divides $|G|$, then $T(G) \geq 2p^3 - p$. For $p \geq 3$ this is the best possible result.*

PROOF. If p^5 divides $|G|$ then G has a subgroup P of order p^5 and $T(G) \geq T(P)$. If P is abelian, then $T(G) \geq T(P) = p^5 \geq 2p^3 - p$, as required. If P is non-abelian, then by Boston and Isaacs [4] we have that the possible values for $k(P)$ are

(1) $p^4 + p^3 - p^2$;
(2) $p^4 + p - 1$;
(3) $2p^3 - p$;

(4) $p^3 + p^2 - 1$; or
(5) $2p^2 + p - 2$.

For $p = 2$ case 5, $k(P) = 2p^2 + p - 2 = 8$ does not occur — there is no group P of order 2^5 with 8 conjugacy classes. However, there are groups P of order 2^5 with $k(P)$ corresponding to the first four cases.

Now, $|P| = p^5 = \alpha + \beta p^2 + \gamma p^4$, where $\alpha = (P : P')$ is the number of linear characters, β is the number of characters of degree p and γ is the number of characters of degree p^2. Furthermore, $k(P) = \alpha + \beta + \gamma$ and $T(P) = \alpha + \beta p + \gamma p^2$. Computing $T(P)$ involves determining α, β and γ. We note that α, β and γ must be non-negative integers. By way of illustration we will do this in one instance.

Let $k(P) = 2p^3 - p$. Now, possible values for α are p^2, p^3 and p^4. Writing γ in terms of $k = k(P)$ and α we have

$$\gamma = \frac{p^5 - kp^2 + \alpha(p^2 - 1)}{p^4 - p^2},$$

and writing β in terms of k, α and γ:

$$\beta = k - \alpha - \gamma.$$

If $\alpha = p^2$, we have $\gamma = 1 - p$ and $\beta = 2p^3 - p^2 - 1$, giving $T(P) = p^4 - p^3 + 2p^2 - p$. If $\alpha = p^3$ we get $\gamma = 0$, $\beta = p^3 - p$ and $T(P) = p^4 - p^3 - p$. Finally, if $\alpha = p^4$ then $\gamma = p^2 - p$, $\beta = -p^4 + 2p^3 - p^2$ and $T(P) = -p^5 + 4p^4 - 2p^3$. Both the first and final cases may be discounted immediately, as α and β must be non-negative integers. This leaves $\alpha = p^3$ and $T(P) = p^4 + p^3 - p^2$.

Working in a similar manner to the case above we get the following: if $|P| = p^5$ and $p > 2$ then one of the following occurs:
(1) $k(P) = p^4 + p^3 - p^2$, $T(P) = 2p^4 - p^3$;
(2) $k(P) = p^4 + p - 1$, $T(P) = p^4 + p^3 - p^2$;
(3) $k(P) = 2p^3 - p$, $T(P) = p^4 + p^3 - p^2$;
(4) $k(P) = p^3 + p^2 - 1$, $T(P) = p^4 + p^2 - p$ or $3p^3 - 2p^2$;
(5) $k(P) = 2p^2 + p - 2$, $T(P) = 2p^3 - p$.

Since $T(G) \geq T(P)$ in each of the above cases $T(G) \geq 2p^3 - p$, and the result is established for odd primes. If $p = 2$ then one of the following occurs:

- $k(P) = 20$, $T(P) = 24$;
- $k(P) = 17$, $T(P) = 20$;
- $k(P) = 14$, $T(P) = 20$; or
- $k(P) = 11$, $T(P) = 18$ or 16.

In each of these cases $T(P) \geq 2p^3 - p = 14$, so $T(G) \geq 2p^3 - p$, and the result is established for all primes p. ■

We note that, for $n = 5$, Marefat [14] considers a problem closely related to ours. Our method is quite different but our results are consistent with his.

Theorem 8. *If* p^6 *divides* $|G|$ *then* $T(G) \geq p^4 + p^3 - p$. *For* $p \geq 5$ *this is the best possible result.*

PROOF. Again, if p^6 divides $|G|$, then G has a subgroup P of order p^6 and $T(G) \geq T(P)$. In [12] James lists some pertinent invariants of the 43 isoclinism classes of groups of order p^6. From this list we get the following: if P has order p^6 and $p > 3$ then one of the following occurs:
(1) $k(P) = p^6$, $T(P) = p^6$;
(2) $k(P) = p^5 + p^4 - p^3$, $T(P) = 2p^5 - p^4$;
(3) $k(P) = 2p^4 - p^2$, $T(P) = p^5 + p^4 - p^3$;
(4) $k(P) = p^5 + p^2 - p$, $T(P) = p^5 + p^4 - p^3$;
(5) $k(P) = p^4 + p^3 - p$, $T(P) = p^5 + p^3 - p^2$ or $3p^4 - 2p^3$;
(6) $k(P) = 2p^3 + p^2 - 2p$, $T(P) = 2p^4 - p^2$ or $3p^4 - 3p^3 + 2p^2 - p$;
(7) $k(P) = p^4 + 2p^3 - p^2 - 2p + 1$, $T(P) = 4p^4 - 4p^3 + p^2$;
(8) $k(P) = p^4 + p^2 - 1$, $T(P) = 2p^4 - p^2$ or $p^5 + p^2 - p$;
(9) $k(P) = 3p^3 - 3p + 1$, $T(P) = 3p^4 - 2p^3$;
(10) $k(P) = p^3 + 2p^2 - p - 1$, $T(P) = p^4 + 2p^3 - 2p^2$ or $2p^4 - p^3 + p^2 - p$;
(11) $k(P) = 2p^2 - 2$, $T(P) = p^4 + p^3 - p$;

If P has order p^6 and $p = 2$ then one of the following occurs:
(1) $k(P) = 64$, $T(P) = 64$;
(2) $k(P) = 40$, $T(P) = 48$;
(3) $k(P) = 28$, $T(P) = 40$;
(4) $k(P) = 34$, $T(P) = 40$;
(5) $k(P) = 22$, $T(P) = 36$ or 32;
(6) $k(P) = 16$, $T(P) = 28$;
(7) $k(P) = 25$, $T(P) = 36$;
(8) $k(P) = 19$, $T(P) = 28$, 34 or 32;
(9) $k(P) = 13$, $T(P) = 24$.

In each of these cases $T(G) \geq T(P) \geq p^4 + p^3 - p = 22$. If P has order p^6 and $p = 3$ then one of the following occurs:
(1) $k(P) = 729$, $T(P) = 729$;
(2) $k(P) = 297$, $T(P) = 405$;
(3) $k(P) = 153$, $T(P) = 297$;
(4) $k(P) = 249$, $T(P) = 297$;
(5) $k(P) = 105$, $T(P) = 261$ or 189;
(6) $k(P) = 57$, $T(P) = 153$ or 177;
(7) $k(P) = 121$, $T(P) = 225$;
(8) $k(P) = 89$, $T(P) = 153$ or 249;
(9) $k(P) = 73$, $T(P) = 189$;
(10) $k(P) = 41$, $T(P) = 117$ or 141.

In each of these cases $T(G) \geq T(P) \geq p^4 + p^3 - p = 105$.

So, for any prime p, if P has order p^6 then $T(P) \geq p^4 + p^3 - p$ and the result is established.

The polynomial expressions for $T(P)$ may either be easily deduced from James' paper or derived in the manner of the example in the proof of Theorem 7. Since $|P| = p^6$ we may assume P has no representation of degree p^3 or greater. ■

We conjecture (admittedly on flimsy evidence) that, for all n:

- if p^{2n+1} divides $|G|$ then $T(G) \geq 2p^{n+1} - p$;
- if p^{2n} divides $|G|$ then $T(G) \geq p^{n+1} + p^n - p$.

In [13] Kovács and Leedham-Green produce for each odd prime p a group G of order p^p with exactly $k(G) = \frac{1}{2}(p^3 - p^2 + p + 1)$. In this case we estimate that $T(G) = 2p^{\left(\frac{p+1}{2}\right)} - p$ and as these groups are likely to be the worst possible cases, this provides evidence for our conjecture.

What can be said about $k(G)$ in the case where G is insoluble? In theorem 8 of [3], Berkovich and Mann prove another result, again striking in its simplicity: let G be an insoluble group, then $T(G) > 2(G : G')$.

We combine this result with lemma 1(d) of [2], which states that for a non-abelian group G, $(T(G))^2 < k(G)|G|$. This at once gives the following:

Theorem 9. *Let G be a finite insoluble group. Then $k(G) > \frac{4|G|}{|G'|^2}$.*

Finally, suppose that G is insoluble and p divides $|G|$. Then G has an abelian subgroup of order p. By a well-known result (see [10, problem 2.9, p. 30]), $d_i \leq \frac{|G|}{p}$, so $T(G) < \frac{k(G)|G|}{p}$. It follows that

$$\frac{2|G|}{|G'|} < T(G) < \frac{k(G)|G|}{p},$$

from which we get at once that

$$k(G) > \frac{2p}{|G'|}.$$

So we have

Theorem 10. *If G is insoluble and p divides $|G|$, then*

$$k(G) > \frac{2p}{|G'|}.$$

If p^2 divides $|G|$, then G has an abelian subgroup of order p^2, so $d_i \leq \frac{|G|}{p^2}$. As in the proof of Theorem 10, we at once get

Theorem 11. *If G is insoluble and p^2 divides $|G|$ for some prime p, then*

$$k(G) > \frac{2p^2}{|G'|}.$$

We note that if $|G|$ is squarefree, then G must be soluble (in fact metacyclic) so if G is insoluble p^2 must divide $|G|$ for some prime p.

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