A NOTE ON THE NILPOTENCY CLASS OF THE UNIT GROUP OF A MODULAR GROUP ALGEBRA

By

Francesco Catino,

Salvatore Siciliano

and

Ernesto Spinelli*
Dipartimento di Matematica "E. De Giorgi", Università del Salento, Via Provinciale Lecce-Arnesano, 73100 Lecce, Italy.

[Accepted 4 October 2007. Published 27 October 2008.]

Abstract

It is well known that a modular group algebra KG is Lie nilpotent if, and only if, its unit group $\mathcal{U}(KG)$ is nilpotent. In this note we prove that if G is a torsion group, then the equality $cl_L(KG) = \mathrm{cl}(\mathcal{U}(KG))$ occurs.

1. Introduction

Let KG be the group algebra of a group G over a field K. We regard it as a Lie algebra under the Lie multiplication $[a, b] = ab - ba$, for all $a, b \in KG$. Denote by $\gamma_k(KG)$ the k-th term of the lower Lie central series of KG. We say that KG is *Lie nilpotent* if there exists an integer n such that $\gamma_n(KG) = 0$. In this case, the minimal integer m such that $\gamma_{m+1}(KG) = 0$ is called the *Lie nilpotency class* of KG and denoted by $cl_L(KG)$.

Assume that K has positive characteristic p and G contains at least one element of order p. According to the well-known result by Passi, Passman and Sehgal (theorem V.4.4 of [10]) and that by Khripta (theorem VI.3.1 of [10]), the group algebra KG is Lie nilpotent if and only if its unit group $\mathcal{U}(KG)$ is nilpotent and this occurs if and only if G is a nilpotent group and its commutator subgroup G' is a finite p-group.

At the end of 1980s, Shalev ([11]) proposed a systematical study of the nilpotency class of the unit group of a group algebra of a finite p-group G over the field with p elements. By resuming the original idea of Coleman and Passman ([4]), some results were proved principally by Shalev himself and Mann ([8] and [12]). A fundamental contribution was successively given by Du's Theorem

*Corresponding author, e-mail: ernesto.spinelli@unile.it
doi:10.3318/PRIA.2008.108.1.65

Cite as follows: Francesco Catino, Salvatore Siciliano and Ernesto Spinelli, A note on the nilpotency class of the unit group of a modular group algebra, *Mathematical Proceedings of the Royal Irish Academy* **108**A (2008), 65–68; doi:10.3318/PRIA.2008.108.1.65.

([5]), which allowed the conclusion that $cl_L(KG) = \mathrm{cl}(\mathcal{U}(KG))$. In this way, group commutator calculations were replaced by ones involving Lie commutators, which are considerably easier. In confirmation of this, the most prominent results in this framework, presented in [13] were just deduced on the basis of the breakthrough of Du.

The above equality is easily seen to be satisfied when G is a p-group, not necessarily a finite one, but, to the best of the authors' knowledge, very little is known in literature in the case in which G is arbitrary. In this note we set the torsion case by proving the following:

Theorem. *Let K be a field of positive characteristic p and G a torsion group containing an element of order p such that $\mathcal{U}(KG)$ is nilpotent. Then $cl_L(KG) = cl(\mathcal{U}(KG))$.*

Related results were proved by Jespers, Riley and the second author for algebraic algebras over perfect fields with more than two elements ([7]).

We stress that one cannot expect that our Theorem is valid for arbitrary modular group algebras. In fact, theorem 4.3, theorem 4.4 and theorem 5.2 of [2], as well as the lemma by Khripta and Kurdics cited in p. 45 and p. 50 of [2], offer examples in which the equality does not hold. We remark that the arguments of [2] seem to be different according to whether $G' \neq Syl_p(G)$ or $G' = Syl_p(G)$. For this purpose, it should be noted that when G is torsion, by virtue of McLain's Theorem (lemma 2.22 of [9]), the latter case does not occur.

2. Proof of the Theorem and concluding remarks

Proof. By [6], we know that $\mathrm{cl}(\mathcal{U}(KG)) \leq cl_L(KG)$. Thus, it remains to show that $\mathrm{cl}(\mathcal{U}(KG)) \geq cl_L(KG)$.

To this end, assume first that K is a perfect field of positive characteristic p. Let $\mathcal{N}(KG)$ denote the set of all nilpotent elements of KG. We claim that $\mathcal{N}(KG)$ is an ideal of KG. Let $x, y \in \mathcal{N}(KG)$ and denote by $\Delta(G)$ the augmentation ideal of KG. Since $KG/\Delta(G')KG \cong K[G/G']$, for every positive integer t one has that $(x+y)^{p^t} \equiv x^{p^t} + y^{p^t} (\mathrm{mod}\, \Delta(G')KG)$. Consequently, for a sufficiently large t we have $(x+y)^{p^t} \in \Delta(G')KG$. From lemma I.2.21 of [10] it follows that $\Delta(G')KG$ is nilpotent, hence we can conclude that $x + y \in \mathcal{N}(KG)$. Moreover, if a is an arbitrary element of KG, from the fact that $(ax)^{p^t} \equiv a^{p^t} x^{p^t} (\mathrm{mod}\, \Delta(G')KG)$, we deduce that $ax \in \mathcal{N}(KG)$ and, in an analogous manner, $xa \in \mathcal{N}(KG)$, by proving the claim.

Consider now an arbitrary element x of KG and denote by H the unitary associative subalgebra generated by x. As G is a nilpotent torsion group, G is actually locally finite, hence KG is a locally finite-dimensional associative algebra. In particular, H is finite-dimensional and, since K is perfect, by the Wedderburn–Malcev Theorem H decomposes as $H = \mathrm{Rad}(H) \oplus F$, where $\mathrm{Rad}(H)$ is the Jacobson radical of H and the complement F is isomorphic to a direct sum of separable extension fields of K. As a consequence, there exist two elements $x_n, x_s \in H$ such that $x = x_n + x_s$, x_n is nilpotent, and the minimal polynomial of x_s has no multiple roots in any extension field of K. Now let y be an arbitrary element of KG and consider the associative subalgebra B generated by x_s and y. Then B is a finite-dimensional algebra and, for the property of the minimal polynomial of x_s quoted above, the adjoint map $\mathrm{ad}_B x_s$ turns out to be a semisimple linear transformation of B. Since B is Lie nilpotent, Engel's Theorem forces $\mathrm{ad}_B x_s = 0$ and so, in particular, x_s and y commute. Hence, $x_s \in Z(KG)$, the centre of KG, and $KG = Z(KG) + \mathcal{N}(KG)$.

By combining the above deductions with Du's Theorem applied to the radical ring $\mathcal{N}(KG)$, one has that

$$\mathrm{cl}(\mathcal{U}(KG)) \geq \mathrm{cl}(1 + \mathcal{N}(KG)) = cl_L(\mathcal{N}(KG)) = cl_L(KG),$$

by proving the statement under the assumption that K is perfect.

Finally, suppose that K is arbitrary. Let F_p denote the prime subfield of K. Obviously, F_p is perfect and $KG \cong K \otimes_{F_p} F_pG$. Since the Lie nilpotency class of an associative algebra is preserved by extensions of the ground field, the proof is complete. ■

Set $KG^{(1)} = KG$ and, by induction, let $KG^{(k)}$ be the associative ideal generated by all the Lie products $[x, r]$, with $x \in KG^{(k-1)}$ and $r \in KG$. The group algebra KG is said to be *strongly Lie nilpotent* if there exists an integer n such that $KG^{(n)} = 0$; in this case the minimal integer m such that $KG^{(m+1)} = 0$ is called the *strong Lie nilpotency class* of KG and denoted by $cl^L(KG)$. According to (V.6.16) of [10], KG is strongly Lie nilpotent if, and only if, KG is Lie nilpotent.

Under the assumption that the characteristic of the ground field exceeds 3, Bhandari and Passi proved in [1] that $cl_L(KG) = cl^L(KG)$. A direct consequence of this fact is the following:

Corollary. *Let K be a field of characteristic $p > 3$ and G be a torsion group containing an element of order p such that $\mathcal{U}(KG)$ is nilpotent. Then $cl_L(KG) = cl(\mathcal{U}(KG)) = cl^L(KG)$.*

As stressed in the Introduction, replacing group commutators computation by Lie commutators computation (which is easier, but surely non-trivial) is a first advantage of our Theorem. Furthermore, by virtue of the Corollary, the determination of the nilpotency class of the unit group is reduced to that of the strong Lie nilpotency class of the group algebra, which can be easily computed by looking at the structure of the underlying group. In fact, an extension of Jennings's theory provides a rather satisfactory formula for the strong Lie nilpotency class of a group algebra, which involves only the central series of the *upper Lie dimension subgroups* (for details, we refer the reader to [13]).

Finally, we mention that it is an *open question* whether the equality $cl_L(KG) = cl^L(KG)$ holds in the special cases $p \leq 3$ and only few results are known in this topic (see the papers [2] and [3]).

Acknowledgement

The first author was partially supported by the National Research Project "Group Theory and Applications".

References

[1] A.K. Bhandari and I.B.S. Passi, Lie nilpotency indices of group algebras, *Bulletin of the London Mathematical Society* **24**, 68–70. 1992.

[2] A.A. Bovdi and J. Kurdics, Lie properties of the group algebra and the nilpotency class of the group of units, *Journal of Algebra* **212**, 28–64, 1999.

[3] F. Catino and E. Spinelli, Lie nilpotent group algebras and upper Lie codimension subgroups, *Communications in Algebra* **34**, 3859–73, 2006.

[4] D.B. Coleman and D.S. Passman, Units in modular group rings, *Proceedings of the American Mathematical Society* **25**, 510–12, 1970.

[5] X. Du, The centers of a radical ring, *Canadian Mathematical Bulletin* **35**, 174–9, 1992.

[6] N. Gupta and F. Levin, On the Lie ideals of a ring, *Journal of Algebra* **81**, 225–31, 1983.

[7] E. Jespers, D.M. Riley and S. Siciliano, Group identities on algebraic algebras with applications to restricted enveloping algebras, *Journal of Algebra* **319** (2008), 4008–17.

[8] A. Mann and A. Shalev, The nilpotency class of the unit group of a modular group algebra II, *Israel Journal of Mathematics* **70**, 267–77, 1990.

[9] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups, Part 1*, Springer-Verlag, New York-Berlin, 1972.

[10] S.K. Sehgal, *Topics in group rings*, Dekker, New York, 1978.

[11] A. Shalev, On some conjectures concerning units in p-group algebras, *Proceedings of the 2nd Bressanone Conference in Group Theory, Palermo*, 279–88, 1989.

[12] A. Shalev, The nilpotency class of the unit group of a modular group algebra I, *Israel Journal of Mathematics* **70**, 257–66, 1990.

[13] A. Shalev, The nilpotency class of the unit group of a modular group algebra III, *Archiv der Mathematik (Basel)* **60**, 136–45, 1993.