WEYL TYPE THEOREMS FOR POSINORMAL OPERATORS

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Abstract

Let A be a bounded linear operator acting on infinite dimensional separable Hilbert space H. The study of operators satisfying Weyl's theorem, Browder's theorem, the SVEP and Bishop's property is of significant interest and is currently being done by a number of mathematicians around the world. It is known that Weyl's theorem holds for M-hyponormal operators, but does not hold for dominant operators. Hence it is an interesting problem to seek a condition that implies Weyl's theorem for dominant operators. Ho Jeon *et al.* proved that if A is dominant and satisfies $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I)|_M$ for every $M \in Lat(A)$, then Weyl's theorem holds for A. Recently Cao showed that the generalized a-Weyl's theorem holds for $f(A)$, where f is an analytic function defined in an open neighbourhood of $\sigma(A)$ in the case where A^* is p-hyponormal or M-hyponormal. Also Aiena showed that a-Weyl's theorem holds for some classes of operators. In this paper we prove that if A^* is conditionally totally posinormal (with certain condition) or totally posinormal, then the generalized a-Weyl's theorem holds for A and for $f(A)$, where f is an analytic function defined in an open neighbourhood of $\sigma(A)$.

1. Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on infinite dimensional separable Hilbert space H. If $A \in B(H)$ we shall write $N(A)$ and $R(T)$ for the null space and the range of A, respectively. Also, let $\alpha(A) := \dim N(A)$, $\beta(A) := \dim(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of A, respectively.

An operator $A \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite co-dimension.

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The index of a Fredholm operator is given by

$$\text{ind}\,(A) = \alpha(A) - \beta(A).$$

An operator $A \in B(H)$ is called Weyl if it is a Fredholm of index zero, and Browder if it is Fredholm of finite ascent and descent; equivalently [12, theorem 7.9.3] if A is Fredholm and $A - \lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in \mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by [12; 13]

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda \,\text{is not Weyl}\},$$

$$\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda \,\text{is not Fredholm}\},$$

$$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda \,\text{is not Browder}\},$$

respectively. Evidently

$$\sigma_e A \subseteq \sigma_w(A) \subseteq \sigma_b A = \sigma_e(A) \cup acc\sigma(A),$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$. If we write $isoK = K \setminus accK$, then we let

$$\pi_{00}(A) := \{\lambda \in iso\sigma(A) : 0 < \alpha(A - \lambda) < \infty\},$$

$$p_{00}(A) := \sigma(A) \setminus \sigma_b(A).$$

Definition 1.1. *We say that Weyl's theorem holds for A if*

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A).$$

Definition 1.2. *We say that the generalized Weyl's theorem holds for A if*

$$\sigma(A) \setminus \sigma_{Bw}(A) = E(A),$$

where $E(A)$ and $\sigma_{Bw}(A)$ denote the isolated points of the spectrum that are eigenvalues (no restriction multiplicity) and the set of all complex numbers λ for which $A - \lambda I$ is not B-Weyl, respectively.

Let $A \in B(H)$, n be a nonnegative integer and define $A_{[n]}$ to be the restriction A to $R(A^n)$ viewed as a map from $R(A^n)$ to $R(A^n)$ (in particular $A_{[0]} = A$). If for some integer n, the range space $R(A^n)$ is closed and $A_{[n]}$ is upper (resp. a lower) semi-Fredholm operator, then A is called an upper (resp. a lower) semi-B-Fredholm operator. Moreover if $A_{[n]}$ is a Fredholm (Weyl or Browder) operator, then A is called a B-Fredholm (B-Weyl or B-Browder) operator. Similarly, we can define the B-Fredholm's spectrum $\sigma_{BF}(A)$, B-Weyl's spectrum $\sigma_{Bw}(A)$ and B-Browder's spectrum $\sigma_{BB}(A)$. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator.

Note that if the generalized Weyl's theorem holds for A, then so does Weyl's theorem [2]. We say that Browder's theorem holds for A if

$$\sigma(A) \setminus \sigma_w(A) = p_{00}(A).$$

An operator $A \in B(H)$ is said to be posinormal (the word posinormal stands for positive normal), if there exists a $P \geq 0$ in $B(H)$ such that $AA^* = A^*PA$. Equivalently, $A \in B(H)$ is posinormal if there exists a co-isometry $V^* \in B(H)$ and a positive operator $P \in B(H)$ such that $A = A^*PV^*$.

Rhaly [20] introduced posinormal operators and proved many interesting properties of posinormal operators that have since been considered by Jeon *at al.* [14].

The class of posinormal operators contains, in particular, the classes consisting of hyponormal operators ($A \in B(H) : AA^* \leq A^*A$), M-hyponormal ($A \in B(H) : |(A-\lambda I)^*|^2 \leq M|(A-\lambda I)|^2$ for some real number $M > 0$) and dominant operators ($A \in B(H) : |(A-\lambda I)^*|^2 \leq M_\lambda |A-\lambda I|^2$ for some real number $M_\lambda > 0$ and all complex number λ). A posinormal operator A is said to be conditionally totally posinormal (resp. totally posinormal), shortened to $A \in CTP$ (resp. $A \in TP$), if to each complex number λ there corresponds a positive P_λ such that $|(A-\lambda I)^*|^2 = |P_\lambda^{\frac{1}{2}}(A-\lambda I)|^2$ (resp. if there exists a positive operator P such that $|(A-\lambda I)^*|^2 = |P^{\frac{1}{2}}(A-\lambda I)|^2$ for all λ).

In [26], Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators [4; 5; 7; 16], and to several classes of operators including semi-normal operators [6]. Berkani [2] showed that if A is hyponormal, then A satisfies the generalized Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the B-Weyl spectrum $\sigma_{Bw}(A)$ of A satisfies the spectral mapping theorem. In [18] the author showed that the generalized Weyl's theorem holds for (p,k)-quasihyponormal operators. Recently X. Cao [8] showed that the generalized a- Weyl's theorem holds for $f(A)$, where f is an analytic function defined in an open neighbourhood of $\sigma(A)$ in the case where A^* is p-hyponormal or M-hyponormal, and in [19] the author proved that the generalized a-Weyl's theorem holds for some classes of operators. Also Aiena [1] showed that a-Weyl's theorem holds for some classes of operators. In this paper we prove that if A^* is conditionally totally posinormal (with certain condition) or totally posinormal, then the generalized a-Weyl's theorem holds for A and for $f(A)$, where f is an analytic function defined in an open neighbourhood of $\sigma(A)$.

2. Main results

An operator $A \in B(H)$ is said to have Bishop's property (β) if $(A-z)f_n(z) \to 0$ uniformly on every compact subset of D for analytic functions $f_n(z)$ on D, then $f_n(z) \to 0$ uniformly on every compact subset of D. A is said to have the single valued extension property if there exists no nonzero analytic function f such that $(A-z)f(z) \equiv 0$. It is clear that if A has Bishop's property (β), then A has the single valued extension property. In this case, the local resolvent $\rho_A(x)$ of $x \in H$ denotes the maximal open set in which there exists a unique analytic function $f(z)$ satisfying $(A-z)f(z) \equiv x$. The local spectrum $\sigma_A(x)$ of $x \in H$ is defined by

$\sigma_A(x) = \mathbb{C} \setminus \rho_A(x)$ and $X_A(F) = \{x \in H : \sigma_A(x) \subset H\}$ for a subset $F \subset \mathbb{C}$. A is said to have finite ascent if $\ker A^m = \ker A^{m+1}$ for some positive integer m, and finite descent if $R(A^n) = R(A^{n+1})$ for some positive integer n. Laursen [15, proposition 1.8] proved that if $A - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, then A has the single valued extension property (SVEP).

We will say in the following Lemma that an operator $A \in B(H)$ is conditionally totally posinormal (CTP), if to each $\lambda \in \mathbb{C}$ there corresponds an operator $P_\lambda \geq 0$, such that $|(A-\lambda)^*|^2 \leq |P_\lambda^2(A-\lambda I)|^2$; A will be said to be totally posinormal (TP), if A is CTP and the positive operator P_λ can be chosen independent of λ.

Lemma 2.1. *Let $A \in B(H)$ be a CTP operator. Then $A - \lambda I$ has finite ascent for all $\lambda \in \mathbb{C}$. In particular A has the single valued extension property.*

Proof. It is easy to see that if $A \in CTP$, then $\ker(A-\lambda I) \subseteq \ker(A-\lambda I)^*$. Hence $ascent(A - \lambda I) \leq 1$ for all $\lambda \in \mathbb{C}$. Thus A has SVEP. ■

Since an operator $A \in TP$ has Bishop's property (β) [10], the proof of the following lemma is immediate.

Lemma 2.2. *Let $A \in TP$. Then A has the single valued extension property.*

Theorem 2.1. *Let $A \in B(H)$ have SVEP and let $\lambda \in \sigma(A)$ be an isolated point of $\sigma(A)$. Then*

$$X_A(\{\lambda\}) = \{x \in H : ||(A-\lambda)^n x||^{\frac{1}{n}} \to 0\} = E_\lambda H,$$

where E_λ denotes the Riesz idempotent for λ. In particular the above equalities hold for TP or CTP operator.

Proof. Since A has the single valued extension property, the first equality follows from Corollary 2.4 of [15] and the second equality follows from [21, p. 424]. ■

Proposition 2.1. *Let $A \in B(H)$ be a CTP or a TP operator and $\mathcal{M} \subset H$ be an invariant subspace of A. Then the restriction $A|_{\mathcal{M}}$ is also CTP or TP.*

Proof. Let P be the orthogonal projection on $\mathcal{M}$. Then for all $z \in \mathbb{C}$ and for all $x \in H$,

$$||(A - zI|_{\mathcal{M}})^* x|| = ||P(A-z)^* x|| = ||(A-zI)^* x|| = \mathcal{M}_z ||(A|_{\mathcal{M}} - zI)x||.$$

■

Lemma 2.3. [25, lemma 10] *Let $A \in B(H)$ be a TP operator. If $\sigma(A - \lambda I) = 0$, then $A - \lambda I = 0$.*

Lemma 2.4. *Let A be a quasinilpotent algebraically posinormal operator. Then A is nilpotent.*

PROOF. Assume that $p(A)$ is totally posinormal for some nonconstant polynomial p. Since $\sigma(p(A)) = p(\sigma(A))$, the operator $p(A)-p(0)$ is quasinilpotent. Thus Lemma 2.3 implies that

$$cA^m(A-\lambda_1)(A-\lambda_2)\cdots(A-\lambda_n) \equiv p(A)-p(0) = 0,$$

where $m \geq 1$. Since $A-\lambda_i$ is invertible for every $\lambda \neq 0$, we must have $A^m = 0$. ∎

It is known that the SVEP is stable under the functional calculus, i.e. if $A \in B(H)$ has SVEP, then so does $f(A)$ for each function f analytic in a neighbourhood of $\sigma(A)$.

Lemma 2.5. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then $f(A)$ has SVEP for each function f analytic in a neighbourhood $\sigma(A)$.*

Theorem 2.2. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then $f(A)$ satisfies Browder's theorem for each function f analytic in a neighbourhood $\sigma(A)$.*

PROOF. It is known that operators with SVEP satisfy Browder's theorem [9]. Then $f(A)$ satisfies Browder's theorem. This completes the proof. ∎

We recall the following well-known theorems that will be used for the sequel.

Theorem 2.3. [17] *Let $A \in B(H)$ be totally posinormal. Then the generalized Weyl's theorem holds for A.*

Theorem 2.4. [17] *Let $A \in B(H)$ be totally posinormal. Then $f(A)$ satisfies the generalized Weyl's theorem for every function f analytic in a neighbourhood of $\sigma(A)$. In particular, Weyl's theorem holds for $f(A)$.*

Restricting themselves to only those $A \in CTP$ for which the spectrum $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I)|_M = 0$ for every $M \in Lat(A)$, Jeon *et al.* [14, proposition 3.5] have shown that A satisfies Weyl's theorem. In the following theorems we can give more.

Theorem 2.5. *[17] Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I)|_M = 0$ for every $M \in Lat(A)$. Then the generalized Weyl's theorem holds for A.*

Remark 2.1. Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I)|_M = 0$ for every $M \in Lat(A)$. If A is quasinilpotent, then Lemma 2.4 implies that A is nilpotent. By using the same techniques used in the proof of [7, lemma 2.3], it is easy to see that A is isoloid.

Theorem 2.6. [17] *Let $A \in B(H)$ be a conditionally totally posinormal operator such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A-\lambda I)|_M = 0$ for every $M \in Lat(A)$. Then $f(A)$ satisfies the generalized Weyl's theorem for each function f analytic in a neighbourhood of $\sigma(A)$. In particular, Weyl's theorem holds for $f(A)$.*

The essential approximate point spectrum $\sigma_{ea}(A)$ is defined by

$$\sigma_{ea}(A) = \bigcap\{\sigma_{ap}(A+K) : K \text{ is a compact operator}\}$$

and the essential approximate point Browder's spectrum σ_{ab} is defined by

$$\sigma_{ab}(A) = \bigcap\{\sigma_{ap}(A+K) : AK = KA, K \text{ is a compact operator}\},$$

where $\sigma_{ap}(A)$ is the approximate point spectrum of A. We consider the set

$$\Phi_+^-(H) = \{A \in B(H) : A \text{ is left semi-Fredholm and ind}\,(A \le 0\}.$$

Rakocevic [22] proved that

$$\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+^-\}$$

and the inclusion $\sigma_{ea}(f(A)) \subset f(\sigma_{ea}(A))$ holds for all function $f(z)$ that are analytic in some open neighbourhood of $\sigma(A)$ with no restriction on A. The next theorem shows the spectral mapping theorem on the essential approximate point spectrum of conditionally totally posinormal or totally posinormal operator.

Lemma 2.6. *Let $A \in B(H)$ and let $\lambda \in \mathbb{C}$. If $A-\lambda$ is semi-Fredholm and it has finite ascent, then* $\text{ind}\,(A-\lambda) \le 0$*.*

Proof. If $A-\lambda$ has finite descent, then $\text{ind}\,((A-\lambda) = 0$ by [24, theorem V 6.2]. If $A-\lambda$ does not have finite descent, then

$$n\text{ind}\,((A-\lambda) = \dim N(A-\lambda)^n - \dim R((A-\lambda)^n))^\perp \to -\infty.$$

Hence $\text{ind}\,((A-\lambda) < 0$. ■

Note that the proof of Lemma 2.6 follows also from the stability of the index.

Corollary 2.1. *Let $A \in B(H)$ be a conditionally totally posinormal operator or totally posinormal. If $A-\lambda I$ is semi-Fredholm for some $\lambda \in \mathbb{C}$, then* $\text{ind}(A-\lambda I) \le 0$*.*

Theorem 2.7. [1] *Let $A \in B(H)$ have SVEP. Then*

$$\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for every function $f(z)$ that is analytic in some open neighbourhood G of $\sigma(A)$.

Corollary 2.2. *Let $A \in B(H)$ be conditionally totally posinormal or totally posinormal. Then*

$$\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for every function $f(z)$ that is analytic on some open neighbourhood G of $\sigma(A)$.

We say that a-Browder's theorem holds for A if $\sigma_{ea}(A) = \sigma_{ab}(A)$. It is well known that

$$a - \text{Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

In general [3] Weyl's theorem does not hold for operators having SVEP only, but a-Browder's theorem holds for operators having the single valued extension property only as we will show in Theorem 2.8.

Theorem 2.8. *Let $A \in B(H)$ has SVEP. Then a-Browder's theorem holds for A.*

PROOF. It is well known that $\sigma_{ea}(A) \subseteq \sigma_{ab}(A)$. Conversely, assume that $\lambda \in \sigma_{ap} \setminus \sigma_{ea}(A)$. Then $A - \lambda I \in \Phi_+^-(H)$ and $A - \lambda I$ is not bounded below. Since A has SVEP and $A - \lambda I \in \Phi_+^-$, [1, theorem 2.6] implies that $A - \lambda I$ has finite ascent. Hence [23, theorem 2.1] implies that $\lambda \in \sigma_{ap}(A) \setminus \sigma_{ab}(A)$. This implies that a-Browder's theorem holds for A. ■

Corollary 2.3. *Let $A \in B(H)$ be a CTP or a TP operator. Then a-Browder's theorem holds for $f(A)$ for each function f analytic in a neighbourhood of $\sigma(A)$.*

PROOF. By applying Theorem 2.7 we get

$$\sigma_{ab}(f(A)) = f(\sigma_{ab}(A)) = f(\sigma_{ea}(A)) = \sigma_{ea}(f(A)).$$

Therefore a-Browder's theorem holds for $f(A)$. ■

Let SBF_+ be the class of all upper semi-B-Fredholm operators, SBF_+^- the class of $A \in SBF_+$ such that $\operatorname{ind}((A) \leq 0$, and let

$$\sigma_{SBF_+^-}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin SBF_+^-\}$$

be called the semi-B-essential approximate point spectrum.

Definition 2.1. *We say that A obeys the generalized a-Weyl's theorem if*

$$\sigma_{SBF_+^-}(A) = \sigma_{ap}(A) \setminus E^a(A),$$

where E^a is the set of all eigenvalues of A that are isolated in $\sigma_{ab}(A)$.

Definition 2.2. *An operator $A \in B(H)$ is said to obey Weyl's theorem if*

$$\sigma_{ab}(A) \setminus \sigma_{SF_+^-}(A) = E_0^a,$$

where E_0^a is the set of all isolated points of $\sigma_{ap}(A)$ that are eigenvalues of finite multiplicity and $\sigma_{SF_+^-}(A)$ is the set of $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not an upper semi-Fredholm operators with $\operatorname{ind}(A - \lambda I) \leq 0$.

Recall in [3] that

Generalized a-Weyl's theorem $\Rightarrow$ Generalized Weyl's theorem $\Rightarrow$ Weyl's theorem

$\Rightarrow$ Browder's theorem.

Generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem $\Rightarrow$ Weyl's theorem

$\Rightarrow$ Browder's theorem.

Generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem $\Rightarrow$ Generalized Browder's theorem

$\Rightarrow$ Browder's theorem.

The converse of the previous implications are false (see [3, example 3.12]).

Theorem 2.9. *Let A^* be totally posinormal. Then the generalized a-Weyl's theorem holds for A.*

Proof. We have to prove that $\sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A) = E^a(A)$. For this, assume that $\lambda \in \sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A)$. Then $A - \lambda I$ is an upper semi-B-Fredholm operator and $\operatorname{ind}((A - \lambda I) \leq 0$. Hence for n large enough, $A - (\lambda + \frac{1}{n})I$ is an upper semi-Fredholm operator and $\operatorname{ind}((A - (\lambda + \frac{1}{n})I) = \operatorname{ind}((A - \lambda I)$ [3]. Therefore $\operatorname{ind}((A - (\lambda + \frac{1}{n})I) \leq 0$. Since A^* has SVEP, [1] implies that $\operatorname{ind}((A - (\lambda + \frac{1}{n})I) \geq 0$. Thus $\operatorname{ind}((A - (\lambda + \frac{1}{n})I) = 0$. It follows that $\operatorname{ind}((A - \lambda I) = 0$. This implies that $A - \lambda I$ is a B-Fredholm operator of index zero. Since A^* has SVEP, we have $\sigma(A) = \sigma_{ap}(A)$ and we have $\lambda \in \sigma(A) \setminus \sigma_{Bw}(A)$. Then it follows from Theorem 2.3 that $\lambda \in E(A)$. Hence $\lambda \in E^a(A)$. Conversely, let $\lambda \in E^a(A)$. Then λ is an isolated point of $\sigma_{ap}(A)$. Therefore $\overline{\lambda}$ is an isolated point of $\sigma(A^*)$. Let P be the spectral projection defined by

$$P = \int_{\partial B_0} (\lambda_0 I - A^*)^{-1} d\lambda_0,$$

where B_0 is an open disk centered at $\overline{\lambda}$ that contains no other points of $\sigma(A^*)$. Then A^* can be represented as the direct sum

$$A^* = A_1 \oplus A_2, \text{where } \sigma(A_1) = \overline{\lambda} \text{ and } \sigma(A_2) = \sigma(A^*) \setminus \{\overline{\lambda}\}.$$

Then $\overline{\lambda} I - A_2$ is invertible. We have to consider the two following cases:

Case 2.1, where $\lambda = 0$. Assume that $\lambda = 0$. Then $\sigma(A_1) = \{0\}$. Since A_1 is a TP operator, it follows that $A_1 = 0$ by Lemma 2.1. Therefore $\overline{\lambda}I - A^* = 0 \oplus \overline{\lambda}I - A_2$.

Case 2.2, where $\lambda \neq 0$. Since A_1 is an invertible TP operator, it follows that A_1^{-1} is an invertible TP operator. Then $||A_1|| = |\lambda|$ and $||A_1^{-1}|| = \frac{1}{\lambda}$. Therefore, for any $x \in R(P)$, we have

$$||x|| \leq ||A_1^{-1}||||A_1 x|| = \frac{1}{|\lambda|}||A_1 x|| \leq \frac{1}{|\lambda|}|\lambda|||x|| = ||x||.$$

Hence $\frac{1}{\lambda}A_1$ is unitary. Therefore A_1 is normal and $\overline{\lambda}I - A_1$ is also normal. Since $\overline{\lambda}I - A_1$ is quasinilpotent and the only normal quasinilpotent operator is zero, it follows that $\overline{\lambda} - A^* = 0 \oplus \overline{\lambda}I - A_2$. Now since $\overline{\lambda}I - A_2$ is invertible, it is known that $\overline{\lambda}I - A^*$ has finite ascent and descent. Therefore $\lambda I - A$ has finite ascent and descent. This implies that $\lambda \in \sigma_{ap}(A) \setminus \sigma_{SBF_+^-}(A)$. This completes the proof. ■

By the same arguments as in the proof of the previous theorem we prove the following theorem.

Theorem 2.10. *Let A^* be conditionally totally posinormal such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A - \lambda I)|_M = 0$ for every $M \in Lat(A)$. Then A satisfies the generalized a-Weyl's theorem.*

Let

$$A_2(H) = \{A \in B(H) : \text{ind}\,((A - \lambda I)\text{ind}\,((A - \mu I) \geq 0 \text{ for all}\, \lambda, \mu \in \mathbb{C} \setminus \sigma_{SF^+}\}.$$

An operator $A \in B(H)$ is said to be approximate-isoloid (abbrev., a-isoloid) if every isolated point of $\sigma_{ap}(A)$ is an eigenvalue of A, and isoloid if every isolated point of $\sigma(A)$ is an eigenvalue of A. Clearly, if A is a-isoloid, then it is isoloid. However, the converse is not true.

Lemma 2.7. *Let A^* be a TP operator. Then A is a-isoloid.*

PROOF. Since A^* is a TP operator, Theorem 2.9 implies that a-Weyl's theorem holds for A and $\sigma(A) = \sigma_{ap}(A)$. If we assume that $\lambda \in iso\sigma_{ap}(A) = iso\sigma(A)$, then $\overline{\lambda} \in iso\sigma(A^*)$. Since A^* is a TP operator and it is isoloid, $N(\overline{\lambda}I - A^*) \neq \{0\}$. Since $N(\overline{\lambda}I - A^*) \subseteq N(\lambda I - A)$, we have $N(\lambda I - A) \neq \{0\}$. Thus A is a-isoloid. ■

Using the same arguments as above we prove the following lemma.

Lemma 2.8. *Let A^* be a conditionally totally posinormal operator such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A - \lambda I)|_M = 0$ for every $M \in Lat(A)$. Then A is a-isoloid.*

Lemma 2.9. *Let A^* be conditionally totally posinormal or totally posinormal. Then $A \in A_2(H)$.*

PROOF. Let $\lambda \in \mathbb{C} \setminus \sigma_{SF^+}(A)$. Since $N(\overline{\lambda} I - A^*) \subseteq N(\lambda I - A)$, we have ind $((A - \lambda I) \geq 0$. This implies that $A \in A_2(H)$. ■

Theorem 2.11. *Let A^* be totally posinormal. Then $f(A)$ obeys the generalized a-Weyl's theorem for every function f analytic in a neighbourhood of $\sigma(A)$.*

PROOF. Since A is a-isoloid, $A \in A_2(H)$ and A obeys the generalized a-Weyl's theorem, [8, theorem 2.2] implies that $f(A)$ obeys the generalized a-Weyl's theorem. ■

Again by the same arguments as above we prove the following theorem.

Theorem 2.12. *Let A^* be conditionally totally posinormal such that $\sigma((A-\lambda I)|_M) = \{0\} \Rightarrow (A - \lambda I)|_M = 0$ for every $M \in Lat(A)$. Then $f(A)$ obeys the generalized a-Weyl's theorem for every function f analytic in a neighbourhood of $\sigma(A)$.*

As a consequence of the previous theorem we obtain

Corollary 2.4. [8] *Let A^* be M-hyponormal. Then $f(A)$ obeys the generalized a-Weyl's theorem for every function f analytic in a neighbourhood of $\sigma(A)$.*

Corollary 2.5. *Let A^* be a dominant operator such that $\sigma((A - \lambda I)|_M) = \{0\} \Rightarrow (A - \lambda I)|_M = 0$ for every $M \in Lat(A)$. Then $f(A)$ obeys the generalized a-Weyl's theorem for every function f analytic in a neighbourhood of $\sigma(A)$.*

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REFERENCES

[1] P. Aiena and F. Villafane, Weyl's theorem for some classes of operators, *Integral equations Operator Theory* **53** (2005), 453–66.

[2] M. Berkani and A. Arroud, Generalized Weyl's theorem and hyponormal operators, *Journal of the Australian Mathematical Society* **76** (2004), 291–302.

[3] M. Berkani and J. Koliha, Weyl type theorems for bounded linear operators, *Acta Scientiarum Mathematicarum (Szeged)* **69** (2003), 359–76.

[4] S.K. Berberian, An extension of Weyl's theorem to a class of not necessarily normal operators, *Michigan Mathematical Journal* **16** (1969), 273–9.

[5] S.K. Berberian, The Weyl spectrum of an operator, *Indiana University Mathematics Journal* **20** (1970), 529–44.

[6] L.A. Coburn, Weyl's theorem for non normal operators, *Michigan Mathematical Journal* **13** (1966), 285–8.

[7] R. Curto and Y.M. Han, Weyl's heorem for algebraically paranormal operators, *Integral Equations and operator theory* **47** (2003), 307–14.

[8] C. Xiaohong, G. Maozhng and M. Bin, Weyl type theorems for p-hyponormal and M-hyponormal operators, *Studia Mathematica* **163**(2004), 177–88.

[9] S.V. Djordjevic and Y.M. Han, Browder's theorem and spectral theory, *Glasgow Mathematical Journal* **42** (2000), 479–86.

[10] J. Eschmeir and M. Putinar, Bishop's property (β) and rich extensions of linear operators, *Indiana University Mathematics Journal* **37** (1988), 325–47.

[11] Y.M. Han and W.Y. Lee, Weyl's theorem in the class of algebraically p-hyponormal operators, *Proceedings of the American Mathematical Society* **128** (2000), 2291–6.

[12] R. Harte, Fredholm, Weyl and Browder theory, *Mathematical Proceedings of the Royal Irish Academy* **85**A (1985), 151–76.

[13] R. Harte, *Invertibility and singularity for bounded linear operators*, Dekker, New York, 1988.

[14] I.H. Jeon, S.H. Kim, E. Ko and J.E. Park, On positive normal operators, *Bulletin of the Korean Mathematical Society* **39** (2002), 33–41.

[15] K.B. Laursen, Operators with finite ascent, *Pacific Journal of Mathematics* **152**(1992), 323–36.

[16] S.H. Lee and W.Y. Lee, A spectral mapping theorm for the Weyl spectrum, *Glasgow Mathematical Journal* **38** (1996), 61–4.

[17] S. Mecheri, Generalized Weyl's theorem for posinormal operators, *Mathematical Proceedings of the Royal Irish Academy* **107**A (2007), 20–8.

[18] S. Mecheri, Generalized Weyl's theorem for algebraically (p,k)-quasihyponormal operators, *Kyungpook Mathematical Journal* **46** (2006), 553–63.

[19] S. Mecheri, Generalized a-Weyl's theorem for some classes of operators, *Nihonkai Mathematical Journal* **17** (2006), 155–65.

[20] H.C. Rhaly Jr, Posinormal operators, *Journal of the Mathematical Society of Japan* **46** (1994), 587–607.

[21] F. Riesz and B.Sz. Nagy, *Functional analysis*, Frederick Ungar, New York, 1955.

[22] V. Rakocevic, On the essential point spectrum II, *Matematicki Vesnik* **36** (1984), 89–97.

[23] V. Rakocevic, Approximate point spectrum and commuting compact perturbations, *Glasgow Mathematical Journal* **28** (1986), 193–8.

[24] A.E. Taylor and D.C. Lay, *Introduction to functional analysis*, 2nd edn, Krieger, Florida, 1980.

[25] A. Uchiyama and T. Yoshino, On the class $\mathcal{Y}$ operators, *Nihonkai Mathematical Journal* **8** (1997), 179–94.

[26] H. Weyl, Uber beschrankte quadratishe formen, deren differenz vollsteig ist, *Rendiconti del Circolo Matematico di Palermo* **27** (1909), 373–92.