

# DRAZIN INVERSES IN JÖRGENS ALGEBRAS OF BOUNDED LINEAR OPERATORS

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## ABSTRACT

Let  $X$  be a Banach space and  $T$  be a bounded linear operator from  $X$  to itself ( $T \in B(X)$ .) An operator  $D \in B(X)$  is a Drazin inverse of  $T$  if  $TD = DT$ ,  $D = TD^2$  and  $T^k = T^{k+1}D$  for some nonnegative integer  $k$ . In this paper we look at the Jörgens algebra, an algebra of operators on a dual system, and characterise when an operator in that algebra has a Drazin inverse that is also in the algebra. This result is then applied to bounded inner product spaces and \*-algebras.

## 1. Introduction

Let  $T \in B(X)$ , the Banach algebra of bounded linear operators from a Banach space  $X$  to itself. We shall denote the null space of  $T$  as  $\mathcal{N}(T)$  and the range of  $T$  as  $\mathcal{R}(T)$ . An operator  $D \in B(X)$  is a Drazin inverse of  $T$  if  $TD = DT$ ,  $D = TD^2$  and  $T^k = T^{k+1}D$  for some nonnegative integer  $k$ . The smallest such  $k$  is called the index of  $T$  and shall be denoted by  $k = \text{ind}_D(T)$ .

In Section 2 we summarise some known results about Drazin inverses. In Section 3 we look at a Banach algebra called the Jörgens Algebra. This algebra is so-named because K. Jörgens presented this algebra in [7] as a way to study integral operators. The algebra and its spectral theory were also studied by B. Barnes in [1]. Generalised inverses in this algebra were characterised in [11]. Examples of these algebras can be found in [7; 10].

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be Banach spaces in normed duality. That is, suppose there is a nondegenerate bilinear form  $\langle \cdot, \cdot \rangle$  on  $X \times Y$  such that for some  $M > 0$ ,

$$|\langle x, y \rangle| \leq M \|x\|_X \|y\|_Y \text{ for all } x \in X \text{ and } y \in Y. \quad (1.1)$$

Suppose  $T \in B(X)$  has an adjoint with respect to this bilinear form denoted by  $T^\dagger$ ; i.e.  $\langle Tx, y \rangle = \langle x, T^\dagger y \rangle$  for all  $x \in X$  and  $y \in Y$ . Define the *Jörgens algebra*  $J_Y(X) = \mathcal{A}$  to be

$$\mathcal{A} = \{T \in B(X) \mid T^\dagger \text{ exists in } B(Y)\}$$

with norm  $\|T\| = \max\{\|T\|_{op}, \|T^\dagger\|_{op}\}$ .

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With this defined norm,  $\mathcal{A}$  is a Banach algebra [7].  $\mathcal{A}$  will denote the Jörgens algebra. Because the bilinear form is nondegenerate, an operator  $T$  in  $\mathcal{A}$  is uniquely determined by  $T^\dagger$  and vice-versa. Note that a Jörgens algebra is a saturated algebra, or more specifically a  $Y$ -saturated algebra, and any saturated algebra is also a Jörgens algebra [6; 7, exercise 3.18].

In Section 3 we present the main result of this paper, which is to characterise when an operator in the Jörgens Algebra has a Drazin inverse that is also in the algebra.

In Section 4 we study Banach spaces that have a bounded inner product. We look at the algebra  $\mathcal{B}$  of operators that have an adjoint with respect to this inner product. By defining a specific norm on this algebra, it is made into a Banach  $*$ -algebra. We extend the main result to this situation.

## 2. Drazin inverses

Following the convention that for an operator  $T \in B(X)$ ,  $T^0 = I$ , the identity operator, there are two useful chains of subspaces:

$$\begin{aligned} \{0\} &= \mathcal{N}(T^0) \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2) \subseteq \dots; \text{ and} \\ X &= \mathcal{R}(T^0) \supseteq \mathcal{R}(T) \supseteq \mathcal{R}(T^2) \supseteq \dots. \end{aligned}$$

The *ascent* of an operator  $T$  is the smallest nonnegative integer  $k$  such that  $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$ , and will be denoted by  $k = \alpha(T)$ . When no such number exists, the ascent is considered infinite. The *descent* of an operator  $T$  is the smallest nonnegative  $k$  such that  $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$ , and will be denoted by  $k = \delta(T)$ . If no such number exists, the descent is infinite. Many algebraic results can be obtained with these concepts, but only those relevant to this paper will be mentioned.

**Theorem 2.1.** [12, theorem 3.7] *If  $T \in B(X)$  such that  $\alpha(T) < \infty$  and  $\delta(T) < \infty$ , then they are actually equal to the same number  $k$  and*

$$X = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k).$$

**Theorem 2.2.** [8, theorem 4] *Let  $T \in B(X)$ . Then  $T$  has a Drazin inverse if and only if  $T$  has finite ascent and descent, in which case  $\text{ind}_D(T) = \alpha(T) = \delta(T)$ .*

The following theorem and its proof can be found in [2] for the finite dimensional case and in [8] for the more general Banach space case. Again, we state it here in order to use it later.

**Theorem 2.3.** [2; 8] *Let  $T \in B(X)$  have Drazin inverse  $D$  with  $\text{ind}_D(T) = k$ . Then*

- (1)  $\mathcal{R}(D) = \mathcal{R}(T^k)$ ;
- (2)  $\mathcal{N}(D) = \mathcal{N}(T^k)$ ; and
- (3)  $TD = DT$  is the projection onto  $\mathcal{R}(T^k)$  along  $\mathcal{N}(T^k)$ .

### 3. Jörgens algebras

Before we characterise Drazin inverses in Jörgens algebras, some useful previous results from [11] will be stated. For ease of notation, for  $k \in \mathbb{N}$  we shall denote  $(T^k)^\dagger = (T^\dagger)^k$  by  $T^{k\dagger}$ .

**Lemma 1.** [11, lemma 4] *Let  $T \in \mathcal{A}$ .*

- (1)  $\mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger)$ ;
- (2)  ${}^\perp\mathcal{R}(T^\dagger) = \mathcal{N}(T)$ ;
- (3)  ${}^\perp\mathcal{N}(T^\dagger) = \text{cl}_{\mathcal{Y}}\mathcal{R}(T)$  and
- (4)  $\mathcal{N}(T)^\perp = \text{cl}_{\mathcal{X}}\mathcal{R}(T^\dagger)$ .

**Lemma 2.** [11, lemma 3] *The following are true for any projection  $P \in \mathcal{A}$ :*

- (1)  $\mathcal{N}(P) = {}^\perp\mathcal{R}(P^\dagger)$ ;
- (2)  $\mathcal{R}(P) = {}^\perp\mathcal{N}(P^\dagger)$ ;
- (3)  $\mathcal{R}(P^\dagger) = \mathcal{N}(P)^\perp$ ; and
- (4)  $\mathcal{N}(P^\dagger) = \mathcal{R}(P)^\perp$ .

*Thus  $\mathcal{R}(P)$  and  $\mathcal{N}(P)$  are both  $\mathcal{Y}$ -closed and  $\mathcal{R}(P^\dagger)$  and  $\mathcal{N}(P^\dagger)$  are both  $\mathcal{X}$ -closed.*

Using the above facts about Drazin inverses and Jörgens algebras, a useful lemma is obtained.

**Lemma 3.** *Let  $T \in \mathcal{A}$ . If  $\delta(T) = k < \infty$ , then  $\alpha(T^\dagger) \leq k$ . Similarly, if  $\delta(T^\dagger) = k < \infty$  then  $\alpha(T) \leq k$ . In particular, if  $T$  and  $T^\dagger$  both have finite index, then they must have equal index.*

PROOF. Suppose  $T \in \mathcal{A}$  with  $\delta(T) = k$ . Then by definition,  $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$ . But by Lemma 1,  $\mathcal{R}(T^k)^\perp = \mathcal{N}(T^{k\dagger})$  and  $\mathcal{R}(T^{k+1})^\perp = \mathcal{N}(T^{(k+1)\dagger})$ . Thus  $\mathcal{N}(T^{k\dagger}) = \mathcal{N}(T^{(k+1)\dagger})$ , and so  $\alpha(T^\dagger) \leq k$ . A similar argument can be shown if  $\delta(T^\dagger) = k < \infty$ .

■

Now we can characterise Drazin inverses in Jörgens algebras.

**Theorem 3.1.** *Let  $T \in \mathcal{A}$  with  $\text{ind}_D(T) = k$ . Then the following are equivalent:*

- (1)  $T$  has a Drazin inverse  $D \in \mathcal{A}$ ;
- (2)  $T^\dagger$  has a Drazin inverse;
- (3)  $\delta(T^\dagger) < \infty$ ; and
- (4)  $\mathcal{R}(T^{(k+1)\dagger})$  is  $\mathcal{X}$ -closed, i.e.,  $\mathcal{N}(T^k)^\perp = \mathcal{N}(T^{k+1})^\perp = \mathcal{R}(T^{(k+1)\dagger})$ .

PROOF. (1)  $\implies$  (2) is clear as  $D^\dagger$  must be a Drazin inverse of  $T^\dagger$  due to the properties of the bilinear form.

(2)  $\implies$  (1). Let  $B$  be the Drazin inverse of  $T^\dagger$  and  $D$  the Drazin inverse of  $T$ . We need to show that  $B = D^\dagger$ . By Lemma 3,  $\text{ind}_D(T^\dagger) = k$ . By Theorem 2.3, we also have

$$\mathcal{R}(T^\dagger B) = \mathcal{R}(B) = \mathcal{R}(T^{k\dagger}) \tag{3.1}$$

and

$$\mathcal{N}(T^\dagger B) = \mathcal{N}(B) = \mathcal{N}(T^{k\dagger}) = \mathcal{R}(T^k)^\perp = \mathcal{R}(D)^\perp. \quad (3.2)$$

By Lemma 1,  $\mathcal{N}(T^{k\dagger}) = \mathcal{R}(T^k)^\perp = \mathcal{R}(D)^\perp$ . Using Theorem 2.1 along with (3.1) above, any  $y \in Y$  can be uniquely expressed as  $y = T^\dagger B y + y_n$ , where  $y_n \in \mathcal{N}(T^\dagger B)$ . Similarly, any  $x \in X$  can be uniquely expressed as  $x = T D x + x_n$ , where  $x_n \in \mathcal{N}(D) = \mathcal{R}(D)^\perp$ . Thus

$$\begin{aligned} \langle D x, y \rangle &= \langle D x, T^\dagger B y \rangle + \langle D x, y_n \rangle \\ &= \langle D x, T^\dagger B y \rangle \\ &= \langle T D x, B y \rangle \\ &= \langle T D x, B y \rangle + \langle x_n, B y \rangle \\ &= \langle x, B y \rangle. \end{aligned}$$

Since  $x$  and  $y$  were arbitrary,  $B = D^\dagger$  and  $D \in \mathcal{A}$ .

(2)  $\implies$  (3) is clear by Theorem 2.2.

(3)  $\implies$  (2). Let  $\delta(T^\dagger) < \infty$ . Since  $\delta(T) = k$ ,  $\alpha(T^\dagger) \leq k$  by Lemma 3 and thus  $\text{ind}_D(T^\dagger) = k$  also. Thus  $T^\dagger$  has a Drazin inverse by Theorem 2.2.

(4)  $\implies$  (3). Let  $\mathcal{R}(T^{(k+1)\dagger})$  be  $\mathcal{X}$ -closed. By hypothesis,  $\delta(T) = k = \alpha(T)$  and so by Lemma 3  $\alpha(T^\dagger) \leq k$ . But by Lemma 1 we have

$$\mathcal{R}(T^{(k+1)\dagger}) = \mathcal{N}(T^{k+1})^\perp = \mathcal{N}(T^k)^\perp = \text{cl}_{\mathcal{X}} \mathcal{R}(T^{k\dagger}). \quad (3.3)$$

Hence

$$\mathcal{R}(T^{(k+1)\dagger}) \subseteq \mathcal{R}(T^{k\dagger}) \subseteq \text{cl}_{\mathcal{X}} \mathcal{R}(T^{k\dagger}) = \mathcal{R}(T^{(k+1)\dagger}) \quad (3.4)$$

and therefore  $\mathcal{R}(T^{k\dagger}) = \text{cl}_{\mathcal{X}} \mathcal{R}(T^{k\dagger}) = \mathcal{R}(T^{(k+1)\dagger})$ . Thus  $\delta(T^\dagger) \leq k < \infty$ .

(3)  $\implies$  (4). We have now proven that (1), (2) and (3) are equivalent, so  $D \in \mathcal{A}$  and from Lemma 3,  $\text{ind}_D(T^\dagger) = k$  also. By Theorem 2.3, the projection  $P$  onto  $\mathcal{R}(T^k)$  along  $\mathcal{N}(T^k)$  is  $T D$  so must also be in  $\mathcal{A}$ . Similarly,  $P^\dagger = T^\dagger D^\dagger$  is the projection onto  $\mathcal{R}(T^{k\dagger})$  along  $\mathcal{N}(T^{k\dagger})$ . By Lemma 2,  $\mathcal{R}(T^{k\dagger})$  is  $\mathcal{X}$ -closed. ■

It is indeed necessary for  $\mathcal{R}(T^{(k+1)\dagger})$ , and not just  $\mathcal{R}(T^{k\dagger})$ , to be  $\mathcal{X}$ -closed as the following example that is discussed in [7] will illustrate.

*Example.* Consider the Jörgens algebra with  $X = Y = C[0, 1]$  with the standard bilinear form  $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$ . Let  $\gamma \in \mathbb{C}$  with  $\text{Re}(\gamma) < 0$ . Define the operator  $T_\gamma \in B(C[0, 1])$  by

$$T_\gamma f(x) = x^{\gamma-1} \int_0^x t^{-\gamma} f(t) dt, \quad x \in (0, 1] \quad (3.5a)$$

$$T_\gamma f(0) = (1 - \gamma)^{-1} f(0). \quad (3.5b)$$

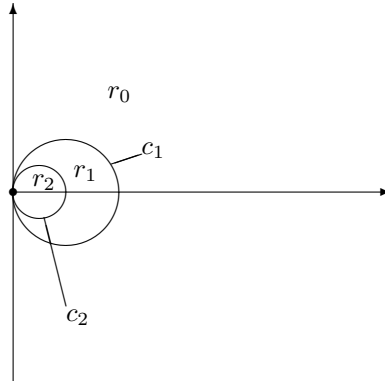


FIG. 1—Regions of the complex plane based on  $a = Re(\gamma)$ .

It can be shown that  $T_\gamma \in \mathcal{A}$  with

$$T_\gamma^\dagger f(x) = x^{-\gamma} \int_x^1 t^{\gamma-1} f(t) dt, \quad x \in (0, 1] \tag{3.6a}$$

$$T_\gamma^\dagger f(0) = -\gamma^{-1} f(0). \tag{3.6b}$$

Consider the complex plane broken up into the following regions based on  $a = Re(\gamma)$  (see Figure 1)

- $c_1 =$  circle with centre  $-\frac{1}{2a}$  and radius  $-\frac{1}{2a}$
- $c_2 =$  circle with centre  $\frac{1}{2(1-a)}$  and radius  $\frac{1}{2(1-a)}$
- $r_0 =$  region outside  $c_1$
- $r_1 =$  region inside  $c_1$  and outside  $c_2$
- $r_2 =$  region inside  $c_2$ .

We will denote the spectrum and essential spectrum of an operator  $T$  by  $\sigma(T)$  and  $\sigma_e(T)$  and the Fredholm index will be denoted by  $\iota$ . It can be shown that  $\sigma(T_\gamma) = r_2 \cup c_2$  and  $\sigma_e(T_\gamma) = c_2$ . Also it can be shown that  $\sigma(T_\gamma^\dagger)$  is the closed disc with boundary  $c_1$  and  $\sigma_e(T_\gamma^\dagger) = c_1$ . In particular Table 1 describes the operators  $\lambda - T_\gamma$  and  $\lambda - T_\gamma^\dagger$  [7, p. 113].

On the regions  $\lambda \in r_1 \cup c_1 \setminus \{0\}$ , the operator  $\lambda - T_\gamma$  is invertible, and thus has a Drazin inverse with  $\text{ind}_D(\lambda - T_\gamma) = k = 0$ . If this inverse were in  $\mathcal{A}$ , the operator  $\lambda - T_\gamma^\dagger$  would also have to be invertible but it is not. Clearly  $\mathcal{R}([\lambda - T_\gamma]^{k\dagger}) = C[0, 1]$  is  $\mathcal{X}$ -closed and thus the hypothesis of  $\mathcal{R}([\lambda - T_\gamma]^{(k+1)\dagger}) = \mathcal{R}(\lambda - T_\gamma^\dagger)$  to be  $\mathcal{X}$ -closed is needed.

TABLE 1—Summary of invertibility of  $\lambda - T_\gamma$  and  $\lambda - T_\gamma^\dagger$ .

$\lambda$	$\lambda - T_\gamma$	$\lambda - T_\gamma^\dagger$
$r_0$	Invertible	Invertible
$r_1$	Invertible	Fredholm, $\iota = -1$
$r_2$	Fredholm, $\iota = 1$	Fredholm, $\iota = -1$
$c_1 \setminus \{0\}$	Invertible	Not Fredholm
$c_2 \setminus \{0\}$	Not Fredholm	Fredholm, $\iota = -1$
0	Not Fredholm	Not Fredholm

#### 4. Banach spaces with bounded inner product

As in [11], we extend Theorem 3.1 to the case where  $X$  having a bounded inner product. Let  $X$  be a Banach space with a bounded inner product  $(\cdot, \cdot)$ . For  $T \in B(X)$ , define  $T^*$  to be the adjoint of  $T$  with respect to the inner product. That is,

$$(Tx, y) = (x, T^*y) \text{ for all } x, y \in X.$$

Define the algebra  $\mathcal{B} = \{T \in B(X) \mid \exists T^* \in B(X)\}$ . This is equivalent to the algebra of all bounded linear operators on  $X$  that have bounded extensions to the Hilbert space completion of  $X$  [9]. Define a norm on the elements of  $\mathcal{B}$  similar to the Jörgens algebra; that is, for  $T \in \mathcal{B}$ ,

$$\|T\| = \max\{\|T\|_{op}, \|T^*\|_{op}\}.$$

This makes  $\mathcal{B}$  a Banach  $*$ -algebra, and Moore-Penrose inverses in  $\mathcal{B}$  were discussed in [11].

Throughout the rest of this section,  $\mathcal{B}$  will denote the  $*$ -algebra above with the inner product space  $X$  and  $T^*$  will denote the adjoint of  $T$  in this algebra. All of the results about Drazin inverses in Jörgens algebras are analogous in this setting. In particular, we have the following result.

**Theorem 4.1.** *Let  $T \in \mathcal{B}$  with  $\text{ind}_D(T) = k$ . Then the following are equivalent:*

- (1)  $T$  has a Drazin inverse  $D \in \mathcal{B}$ ;
- (2)  $T^*$  has a Drazin inverse;
- (3)  $\delta(T^*) < \infty$ ; and
- (4)  $\mathcal{R}(T^{(k+1)*})$  is  $\mathcal{X}$ -closed, i.e.,  $\mathcal{N}(T^k)^\perp = \mathcal{N}(T^{k+1})^\perp = \mathcal{R}(T^{(k+1)*})$ .

The proof of the previous lemmas and theorem are the same as in the Jörgens algebra setting as the only difference is that there is a sesquilinear rather than bilinear form.

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