REGULAR DILATIONS OF REPRESENTATIONS OF PRODUCT SYSTEMS

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Abstract

We study completely contractive representations of product systems X of correspondences over the semigroup $\mathbb{Z}_+^k$. We present a necessary and sufficient condition for such a representation to have a regular isometric dilation. We discuss representations that doubly commute and show that these representations induce completely contractive representations of the norm closed algebra generated by the image of the Fock representation of X.

1. Introduction

A C^*-correspondence E over a C^*-algebra A is a (right) Hilbert C^*-module over A that carries also a left action of A (by adjointable operators). It is also called a Hilbert bimodule in the literature. A c.c. representation of E on a Hilbert space H is a pair (σ, T), where σ is a representation of A on H and $T : E \to B(H)$ is a completely contractive linear map that is also a bimodule map (that is, $T(a \cdot \xi \cdot b) = \sigma(a)T(\xi)\sigma(b)$ for $a, b \in A$ and $\xi \in E$). The representation is said to be isometric (or Toeplitz) if $T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle)$ for every $\xi, \eta \in E$.

In [18], Pimsner associated with such a correspondence two C^*-algebras ($\mathcal{O}(E)$ and $\mathcal{T}(E)$) with certain universal properties. In [11] we studied the operator algebra $\mathcal{T}_+(E)$ (called the tensor algebra) that is universal for c.c. representations of E.

A product system X of C^*-correspondences over a semigroup P is, roughly speaking, a family $\{X_s : s \in P\}$ of C^*-correspondences (over the same C^*-algebra A), with $X_e = A$, such that $X_s \otimes X_t$ is isomorphic to X_{st} for all $s, t \in P \backslash \{e\}$ (see Section 2 for the precise definition). A c.c. (resp. isometric) representation of X is a family $\{T_s\}$ such that, for all $s \in P \setminus \{e\}$, (T_e, T_s) is a c.c. (resp. isometric) representation of X_s and such that, whenever $x \in X_s$ and $y \in X_t$, $T_{st}(\theta_{s,t}(x \otimes y)) = T_s(x)T_t(y)$ (where $\theta_{s,t}$ is the isomorphism from $X_s \otimes X_t$ onto X_{st}).

If E is a C^*-correspondence over A then, setting $X(n) = E^{\otimes n}$ (and $X(0) = A$),

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we get a product system over $P = \mathbb{Z}_+$ and every product system over $\mathbb{Z}_+$ arises in this way.

In [4], Fowler studied product systems over more general (discrete) semigroups P. He proved the existence of a C^*-algebra $\mathcal{T}(X)$ that is universal with respect to Toeplitz representations. In [21, proposition 3.2], we proved the existence of an operator algebra $\mathcal{T}_+(X)$ (the *universal tensor algebra*), which is universal for c.c. representations of X; that is, there is a c.c. representation of X whose image generates $\mathcal{T}_+(X)$ and every c.c. representation of X gives rise to a completely contractive representation of the algebra $\mathcal{T}_+(X)$.

In [21, theorem 4.4] we also proved that every c.c. representation of a product system X over $P = \mathbb{Z}_+^2$ can be dilated to an isometric representation of X. (This was then used to dilate a pair of commuting CP maps). Specialising to the case where $A = \mathbb{C}$ and $X(\mathbf{n}) = \mathbb{C}$, $\mathbf{n} \in \mathbb{Z}_+^2$, this result recovers Ando's dilation result ([1]). Ando proved that, given a pair (T_1, T_2) of commuting contractions in $B(H)$, there is a Hilbert space K, containing H, and a pair (V_1, V_2) of commuting isometries in $B(K)$ such that, for all $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}_+^2$,

$$P_H V_1^{n_1} V_2^{n_2} | H = T_1^{n_1} T_2^{n_2}.$$

It is well known (see [16] or [17]) that such a result is false, in general, for $\mathbb{Z}_+^k$, $k \geq 3$ (that is, for k-tuples of commuting contractions with $k \geq 3$). Thus, in particular, [21, theorem 4.4], cannot be proved for product systems over $\mathbb{Z}_+^k$, $k \geq 3$.

It is known, however, that, if $(T_1, T_2, \ldots, T_k)$ is a commuting tuple of contractions in $B(H)$ satisfying an additional condition, then there are isometries $(V_1, V_2, \ldots, V_k)$ (in $B(K)$ for some Hilbert space K containing H) that dilate $(T_1, T_2, \ldots, T_k)$, (see [2] or [14, theorem 9.1]). The additional condition requires that, for every subset $v \subseteq \{1, \ldots, k\}$,

$$S(v) := \sum_{u \subseteq v} (-1)^{|u|} (T^{\mathbf{e}(u)})^* T^{\mathbf{e}(u)} \geq 0, \tag{1.1}$$

where, for $u = \{i_1, \ldots, i_m\}$, $|u| = m$ and $T^{\mathbf{e}(u)} = T_{i_1} \cdots T_{i_m}$. In fact, this condition is a necessary and sufficient condition to have an isometric dilation $(V_1, \ldots, V_k)$ with the additional property that, for every $\mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^k$ with $\mathbf{n} \wedge \mathbf{m} = \mathbf{0}$,

$$P_H V^{\mathbf{n}*} V^{\mathbf{m}} | H = T^{\mathbf{n}*} T^{\mathbf{m}}$$

where $T^{\mathbf{n}} = \prod T_i^{n_i}$ and $V^{\mathbf{n}} = \prod V_i^{n_i}$. Such a dilation is called a *regular dilation.*

In Definition 3.2 we define regular isometric dilations for c.c. representations of the product system X (over $\mathbb{Z}_+^k$) and, in Theorem 3.5, we prove that a condition similar to condition (1.1) is a necessary and sufficient condition for the existence of an isometric regular dilation. It is also possible, in this case, to find an isometric regular representation that is minimal (in an obvious sense) and, in Proposition 3.7, we show that such a dilation is unique up to unitary equivalence.

In the classical case, it is known [14, proposition 9.2] that, if the k-tuple $(T_1, \ldots, T_k)$ of contractions doubly commutes (that is, the operators commute and, in addition, $T_i T_j^* = T_j^* T_i$ for all $i \neq j$), then it satisfies condition 1.1 (and, thus, a regular, minimal isometric dilation exists). It is also known [6, theorem 1] or [22, theorem 2] that, in this case, the regular, minimal isometric dilation also doubly commutes.

In Theorem 3.10 we prove a similar result for representations of X. (See Definition 3.8 for the definition of a doubly commuting representation of a product system X). Then, in Lemma 3.11, we observe that, for an isometric representation, the doubly commuting condition is equivalent to a condition known in the literature (e.g.[4; 5; 15]) as *Nica covariance*. We then note, using results of [4], that the C^*-algebra generated by the image of the Fock representation L on the Fock space $\mathcal{F}(X) := \sum X(\mathbf{n})$ is isomorphic to the algebra $\mathcal{T}_{cov}(X)$. The algebra $\mathcal{T}_{cov}(X)$ was studied by Fowler in [4] and was shown there to be universal for Nica-covariant representations provided X is compactly aligned (Definition 3.14). Considering the Banach algebra generated by the image of the Fock representation L (and writing $\mathcal{T}_{+,c}(X)$ for it), we use Theorem 3.10 to show, in Corollary 3.17, that every doubly commuting, c.c. representation $\{T_{\mathbf{n}}\}$ of X on H gives rise to a unique completely contractive representation of $\mathcal{T}_{+,c}(X)$ mapping $L(x)$, for $x \in X(\mathbf{n})$, to $T_{\mathbf{n}}(x)$. We refer to $\mathcal{T}_{+,c}(X)$ as the *concrete tensor algebra* associated with X.

Although some of the results of this paper may be expected in view of the case of commuting tuple of operators, the examples in the last section show that they form a far-reaching generalisation.

Recently, k-graphs and the C^*-algebras associated with them have been studied extensively. (See [8] where these C^*-algebras were introduced, the survey article [19] and the references there). Note that every k-graph can be defined by a product system of graphs over $\mathbb{Z}_+^k$ [20]. The algebra $\mathcal{T}_{+,c}$ for such a product system, associated with a k-graph Λ, is the multivariable analogue of the quiver algebra of [12] and can be referred to as a k-quiver algebra and denoted $\mathcal{T}_{+,c}(\Lambda)$. These algebras (and their weak closures) were studied in [7]. In Subsection 4.4, we discuss the case of a single-vertex k-graph in more details.

The next section is devoted to recalling some preliminary results and notation. In Section 3 we present and prove the main results of the paper and in Section 4 we present some examples.

2. Preliminaries

We begin by recalling the notion of a C^*-correspondence. For the general theory of Hilbert C^*-modules which we use, we will follow [9]. In particular, a Hilbert C^*-module E over a C^*-algebra A will be a *right* Hilbert C^*-module. We write $\mathcal{L}(E)$ for the algebra of continuous, adjointable A-module maps on E. It is known to be a C^*-algebra.

Definition 2.1. *A C^*-correspondence over a C^*-algebra A is a Hilbert C^*-module E over A endowed with the structure of a left A-module via a *-homomorphism $\varphi_E : A \to \mathcal{L}(E)$.*

When dealing with a specific C^*-correspondence E it will be convenient to write φ (instead of φ_E) or even to suppress it and write $a\xi$ or $a \cdot \xi$ for $\varphi(a)\xi$.

If E and F are C^*-correspondences over A, then the balanced tensor product $E \otimes_A F$ is a C^*-correspondence over A. It is defined as the Hausdorff completion of the algebraic balanced tensor product with the internal inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \varphi_F(\langle \xi_1, \xi_2 \rangle_E)\eta_2 \rangle_F \tag{2.1}$$

for all $\xi_1, \xi_2 \in E$ and $\eta_1, \eta_2 \in F$. The left and right actions of $a \in M$ are defined by

$$\varphi_{E\otimes F}(a)(\xi \otimes \eta)b = \varphi_E(a)\xi \otimes \eta b \tag{2.2}$$

for all $a, b \in M$, $\xi \in E$ and $\eta \in F$.

Definition 2.2. *An isomorphism of C^*-correspondences E and F is a surjective, bimodule map that preserves the inner products. We write $E \cong F$ if such an isomorphism exists.*

If E is a C^*-correspondence over A and σ is a representation of A on a Hilbert space H then $E \otimes_\sigma H$ is the Hilbert space obtained as the Hausdorff completion of the algebraic tensor product with respect to $\langle \xi \otimes h, \eta \otimes k \rangle = \langle h, \sigma(\langle \xi, \eta \rangle_E)k \rangle_H$. Given an operator $X \in \mathcal{L}(E)$ and an operator $S \in \sigma(A)'$, the map $\xi \otimes h \mapsto X\xi \otimes Sh$ defines a bounded operator $X \otimes S$ on $E \otimes_\sigma H$. When $S = I_E$ and $X = \varphi_E(a)$ (for $a \in A$) we get a representation of A on this Hilbert space. We frequently write $a \otimes I_H$ for $\varphi(a) \otimes I_H$.

Definition 2.3. *Let E be a C^*-correspondence over a C^*-algebra A. Then a completely contractive covariant representation of E (or, simply, a c.c. representation of E) on a Hilbert space H is a pair (σ, T), where*

1. *σ is a $*$-representation of A in $B(H)$.*
2. *T is a linear, completely contractive map from E to $B(H)$.*
3. *T is a bimodule map in the sense that $T(a\xi b) = \sigma(a)T(\xi)\sigma(b)$, $\xi \in E$, and $a, b \in A$.*

Such a representation is said to be isometric if, for every $\xi, \eta \in E$, $T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle)$.

It should be noted that there is a natural way to view E as an operator space (by viewing it as a subspace of its linking algebra) and this defines the operator space structure of E to which Definition 2.3 refers when it is asserted that T is completely contractive.

As we showed in [11, lemmas 3.4–3.6], if a completely contractive covariant representation, (σ, T), of E in $B(H)$ is given, then it determines a contraction $\tilde{T} : E \otimes_\sigma H \to H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in E \otimes_\sigma H$. The operator $\tilde{T}$ satisfies

$$\tilde{T}(\varphi(\cdot) \otimes I) = \sigma(\cdot)\tilde{T}. \tag{2.3}$$

In fact we have the following lemma from [13, lemma 2.16].

Lemma 2.4. *The map $(\sigma, T) \to \tilde{T}$ is a bijection between all completely contractive* *covariant representations* (σ, T) *of* E *on the Hilbert space* H *and contractive operators* $\tilde{T} : E \otimes_\sigma H \to H$ *that satisfy equation (2.3). Given* σ *and a contraction* $\tilde{T}$ *satisfying the covariance condition (2.3), we get a completely contractive covariant representation* (σ, T) *of* E *on* H *by setting* $T(\xi)h := \tilde{T}(\xi \otimes h)$.

Moreover, the representation (σ, T) *is an isometric representation if and only if* $\tilde{T}$ *is an isometry.*

Remark 2.5. *In addition to* $\tilde{T}$ *we also require the "generalised higher powers" of* $\tilde{T}$*. These are maps* $\tilde{T}_n : E^{\otimes n} \otimes H \to H$ *defined by the equation* $\tilde{T}_n(\xi_1 \otimes \ldots \otimes \xi_n \otimes h) = T(\xi_1) \cdots T(\xi_n)h$, $\xi_1 \otimes \ldots \otimes \xi_n \otimes h \in E^{\otimes n} \otimes H$*. One checks easily that* $\tilde{T}_n = \tilde{T} \circ (I_E \otimes \tilde{T}) \circ \cdots \circ (I_{E^{\otimes n-1}} \otimes \tilde{T})$, $n > 1$.

3. Regular dilations

In the following we follow the notation of Fowler ([4]). Let P be the semigroup $\mathbb{Z}_+^k$. Suppose $p : X \to P$ is a family of C^*-correspondences over A. Write $X(\mathbf{n})$ for the correspondence $p^{-1}(\mathbf{n})$ for $\mathbf{n} = (n_1, \ldots, n_k) \in P$ and $\varphi_{\mathbf{n}} : A \to \mathcal{L}(X(\mathbf{n}))$ for the left action of A on $X(\mathbf{n})$. We say that X is a *product system over* $\mathbb{Z}_+^k$ if X is a semigroup, p is a semigroup homomorphism and, for each $\mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^k \backslash \{\mathbf{0}\}$, the map $(x, y) \in X(\mathbf{n}) \times X(\mathbf{m}) \to xy \in X(\mathbf{n}+\mathbf{m})$ extends to an isomorphism $\theta_{\mathbf{n},\mathbf{m}}$ of correspondences from $X(\mathbf{n}) \otimes X(\mathbf{m})$ onto $X(\mathbf{n}+\mathbf{m})$. We also require that $X(\mathbf{0}) = A$ and that the multiplications $X(\mathbf{0}) \times X(\mathbf{n}) \to X(\mathbf{n})$ and $X(\mathbf{n}) \times X(\mathbf{0}) \to X(\mathbf{n})$ are given by the left and right actions of A on $X(\mathbf{n})$.

The associativity of the multiplication means that, for every $\mathbf{n}, \mathbf{m}, \mathbf{p} \in \mathbb{Z}_+^k$,

$$\theta_{\mathbf{n}+\mathbf{m},\mathbf{p}}(\theta_{\mathbf{n},\mathbf{m}} \otimes I_{\mathbf{p}}) = \theta_{\mathbf{n},\mathbf{m}+\mathbf{p}}(I_{\mathbf{n}} \otimes \theta_{\mathbf{m},\mathbf{p}}) \tag{3.1}$$

where, for $\mathbf{m} \in \mathbb{Z}_+^k$, $I_{\mathbf{m}}$ stands for the identity of $X(\mathbf{m})$. We shall write $\mathbf{e}_i$ for the element in $\mathbb{Z}_+^k$ whose ith entry is 1 and all other entries are 0 and, for a subset $u \subset \{1, \ldots, k\}$, we write $\mathbf{e}(u) = \sum\{\mathbf{e}_i : i \in u\}$.

Given a product system X over $\mathbb{Z}_+^k$, we set $E_i = X(\mathbf{e}_i)$ for $1 \le i \le k$. It will be convenient to write E_i^n for the n-fold tensor product $E_i^{\otimes n}$ and to identify $X(\mathbf{n})$ (for $\mathbf{n} \in \mathbb{Z}_+^k$) with $E_1^{n_1} \otimes E_2^{n_2} \otimes \cdots \otimes E_k^{n_k}$ (where these tensor products are the balanced tensor products over A). That means, in particular, that the isomorphisms $\theta_{\mathbf{e}_i,\mathbf{e}_j}$, for $i \le j$, are identity maps. Setting $t_{i,j} = \theta_{\mathbf{e}_i,\mathbf{e}_j} : E_i \otimes E_j \to E_j \otimes E_i$ for $i \ge j$ (and $t_{i,j} = t_{j,i}^{-1}$ for $i < j$), one can check that the family $\{t_{i,j} : 1 \le i, j \le k\}$ satisfies

$$(t_{j,i} \otimes I_{\mathbf{e}_l})(I_{\mathbf{e}_j} \otimes t_{l,i})(t_{l,j} \otimes I_{\mathbf{e}_i}) = (I_{\mathbf{e}_i} \otimes t_{l,j})(t_{l,i} \otimes I_{\mathbf{e}_j})(I_{\mathbf{e}_l} \otimes t_{j,i}) \tag{3.2}$$

for every $1 \le i, j, l \le k$. One can also check (but we omit the tedious computation) that, given k correspondences $E_1, \ldots, E_k$ over the C^*-algebra A and a family $\{t_{i,j} : 1 \le i, j \le k\}$ such that $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ is an isomorphism, $t_{i,j} = t_{j,i}^{-1}$ and $t_{i,i}$ is the identity map, it determines, in a unique way, a product system X (with $X(\mathbf{n}) = E_1^{n_1} \otimes \cdots \otimes E_k^{n_k}$) whose isomorphisms $\{\theta_{\mathbf{n},\mathbf{m}}\}$ satisfy $\theta_{\mathbf{e}_i,\mathbf{e}_j} = id$ if $i \le j$ and $\theta_{\mathbf{e}_i,\mathbf{e}_j} = t_{i,j}$ if $i > j$.

Definition 3.1. *A c.c. representation of* X *on a Hilbert space* H *is given by a* *nondegenerate representation σ of A on H and k completely contractive maps $T^{(i)}$: $E_i \to B(H)$ such that, for each $1 \le i \le k$, $(\sigma, T^{(i)})$ is a c.c. representation of E_i and, for i, j, they satisfy the commutation relation*

$$\tilde{T}^{(i)}(I_{E_i} \otimes \tilde{T}^{(j)}) = \tilde{T}^{(j)}(I_{E_j} \otimes \tilde{T}^{(i)}) \circ (t_{i,j} \otimes I_H) \tag{3.3}$$

Recall that we write $\tilde{T}_n^{(i)}$ (where $1 \le i \le k$ and $n \ge 0$) for $\tilde{T}^{(i)}(I_i \otimes \tilde{T}^{(i)}) \cdots (I_i \otimes I_i \otimes \cdots \otimes \tilde{T}^{(i)}) : E_i^n \otimes H \to H$ (where I_i stands for I_{E_i}). Similarly, for $\mathbf{n} \in \mathbb{Z}_+^k$, we write

$$\tilde{T}_{\mathbf{n}} = \tilde{T}_{n_1}^{(1)}(I_{n_1\mathbf{e}_1} \otimes \tilde{T}_{n_2}^{(2)}) \cdots (I_{\mathbf{n}-n_k\mathbf{e}_k} \otimes \tilde{T}_{n_k}^{(k)}) : X(\mathbf{n}) \otimes H \to H. \tag{3.4}$$

The map $T_{\mathbf{n}} : X(\mathbf{n}) \to B(H)$ is then defined by $T_{\mathbf{n}}(\xi)h = \tilde{T}_{\mathbf{n}}(\xi \otimes h)$ (for $h \in H$). It follows from (3.3) that, for $\mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^k$, $\xi \in X(\mathbf{n})$ and $\eta \in X(\mathbf{m})$,

$$T_{\mathbf{n}+\mathbf{m}}(\theta_{\mathbf{n},\mathbf{m}}(\xi \otimes \eta)) = T_{\mathbf{n}}(\xi)T_{\mathbf{m}}(\eta). \tag{3.5}$$

So that Definition 3.1 agrees with the definition stated in Section 1.

For $\mathbf{n} = (n_1, n_2, \dots n_k) \in \mathbb{Z}^k$ we write $\mathbf{n}_+$ for the vector whose ith entry is $\max\{n_i, 0\}$ and $\mathbf{n}_-$ for $\mathbf{n}_+ - \mathbf{n}$. We also write

$$T(\mathbf{n}) = \tilde{T}_{\mathbf{n}_-}^* \tilde{T}_{\mathbf{n}_+} : X(\mathbf{n}_+) \otimes H \to X(\mathbf{n}_-) \otimes H. \tag{3.6}$$

Definition 3.2. *Let $(\sigma, T^{(1)}, \dots, T^{(k)})$ be a c.c. representation of X on H. A regular isometric dilation of $(\sigma, T^{(1)}, \dots, T^{(k)})$ is a representation $(\rho, V^{(1)}, \dots, V^{(k)})$ of X on a Hilbert space K, containing H, such that*
(i) *Each $\tilde{V}^{(i)}$ is an isometry (from $E_i \otimes K$ into K).*
(ii) *H is invariant for every $V^{(i)}(\xi)^*$, $\xi \in E_i$.*
(iii) *H is reducing for ρ and $\rho(a)|H = \sigma(a)$ for $a \in A$.*
(iv) *For every $\boldsymbol{n} \in \mathbb{Z}^k$, $(I_{X(\boldsymbol{n}_-)} \otimes P_H)V(\boldsymbol{n})|X(\boldsymbol{n}_+) \otimes H = T(\boldsymbol{n})$.*
Such a dilation is said to be minimal if the smallest closed subspace of K that contains H and is invariant under all $V^{(i)}(\xi)$, for $\xi \in E_i$, is K.

Note that the word "regular" refers to the fact that we require (iv) to hold for every $\mathbf{n} \in \mathbb{Z}^k$ and not only for $\mathbf{n} \in \mathbb{Z}_+^k$.

In the following, in order to avoid cumbersome notation, we shall often suppress the isomorphisms between $X(\mathbf{n}) \otimes X(\mathbf{m})$ and $X(\mathbf{n} + \mathbf{m})$. For example, the map $I_{\mathbf{p}-\mathbf{e}(u)} \otimes \tilde{T}_{\mathbf{e}(u)}^* \tilde{T}_{\mathbf{e}(u)}$, appearing in the statement of Lemma 3.3 below, is a map from $X(\mathbf{p} - \mathbf{e}(u)) \otimes X(\mathbf{e}(u)) \otimes H$ to itself but we view it there as a map from $X(\mathbf{p}) \otimes H$ to itself, invoking these isomorphisms. Another example is Equation (3.5) which will be frequently used in the form

$$\tilde{T}_{\mathbf{n}+\mathbf{m}} = \tilde{T}_{\mathbf{n}}(I_{\mathbf{n}} \otimes \tilde{T}_{\mathbf{m}}).$$

The following, technical, lemma will be needed in the proof of the next theorem.

Lemma 3.3. *Let $(\sigma, \{T^{(i)}\})$ be a c.c. representation of X on H. Write $R =$*

$(R(\boldsymbol{p},\boldsymbol{q}))_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^k}$ *for the (infinite, operator-valued) matrix defined by*

$$R(\boldsymbol{p},\boldsymbol{q}) = I_{\boldsymbol{q}-(\boldsymbol{q}-\boldsymbol{p})_+} \otimes T(\boldsymbol{q}-\boldsymbol{p}) : X(\boldsymbol{q}) \otimes H \to X(\boldsymbol{p}) \otimes H.$$

Write $S = (S(\boldsymbol{p},\boldsymbol{q}))_{\boldsymbol{p},\boldsymbol{q}\in\mathbb{Z}_+^k}$ *for the matrix defined by* $S(\boldsymbol{p},\boldsymbol{q}) = R(\boldsymbol{p},\boldsymbol{q})$ *if* $\boldsymbol{q} \geq \boldsymbol{p}$ *and* $S(\boldsymbol{p},\boldsymbol{q}) = 0$ *otherwise. Also, let* D *be the diagonal matrix with*

$$D(\boldsymbol{p},\boldsymbol{p}) = \sum_{u\subseteq\{1,\ldots,k\},\boldsymbol{e}(u)\leq\boldsymbol{p}} (-1)^{|u|}(I_{\boldsymbol{p}-\boldsymbol{e}(u)} \otimes \tilde{T}^*_{\boldsymbol{e}(u)}\tilde{T}_{\boldsymbol{e}(u)}) : X(\boldsymbol{p}) \otimes H \to X(\boldsymbol{p}) \otimes H$$

for $\boldsymbol{p} \in \mathbb{Z}_+^k$.

Then

$$R = S^*DS. \tag{3.7}$$

Also, let L *be the (operator-valued) matrix given by* $L(\boldsymbol{n},\boldsymbol{m}) = (-1)^{|v|}I_{\boldsymbol{n}} \otimes T(\boldsymbol{e}(v))$ *if* $\boldsymbol{m}-\boldsymbol{n} = \boldsymbol{e}(v)$ *and* 0 *otherwise. Then* $SL = I$ *and*

$$D = L^*RL. \tag{3.8}$$

Remark 3.4. *Before we turn to the proof, note that, although we multiply here infinite matrices, the sums involved in the computations of the entries of the product are all finite sums. The precise meaning of Equation (3.7) is* $\langle Rh, g\rangle = \langle DSh, Sg\rangle$ *for* $h \in X(\boldsymbol{p}) \otimes H$ *and* $g \in X(\boldsymbol{q}) \otimes H$. *Thus, it holds for all* h, g *in the vector space* $\mathcal{H}_0$, *which is the (algebraic) sum* $\sum_{\boldsymbol{p}\in\mathbb{Z}_+^k} X(p) \otimes H$. *A similar remark applies to Equation (3.8). It thus follows from the lemma that,* R *is positive on this space (in the sense that* $\langle Rh, h\rangle \geq 0$ *for every* $h \in \mathcal{H}_0$*) if and only if* D *is positive (in a similar sense).*

PROOF OF LEMMA 3.3. Given $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}_+^k$, it is easy to check that

$$\sum_{u\subseteq\{1,\ldots,k\},\mathbf{e}(u)\leq\mathbf{n}} (-1)^{|u|} = 0. \tag{3.9}$$

If $\mathbf{n} = \mathbf{0}$, this sum is, of course, 1.

Now compute, for $\mathbf{p},\mathbf{q} \in \mathbb{Z}_+^k$,

$$(S^*DS)(\mathbf{p},\mathbf{q}) = \sum_{\mathbf{l}\leq\mathbf{p}\wedge\mathbf{q}} S(\mathbf{l},\mathbf{p})^*D(\mathbf{l},\mathbf{l})S(\mathbf{l},\mathbf{q}) =$$

$$\sum_{\mathbf{e}(u)\leq\mathbf{l}\leq\mathbf{p}\wedge\mathbf{q}} (-1)^{|u|}(I_{\mathbf{l}} \otimes T(\mathbf{p}-\mathbf{l})^*)(I_{\mathbf{l}-\mathbf{e}(u)} \otimes \tilde{T}^*_{\mathbf{e}(u)}\tilde{T}_{\mathbf{e}(u)})(I_{\mathbf{l}} \otimes T(\mathbf{q}-\mathbf{l})) =$$

$$\sum_{\mathbf{e}(u)\leq\mathbf{l}\leq\mathbf{p}\wedge\mathbf{q}} (-1)^{|u|}I_{\mathbf{l}-\mathbf{e}(u)} \otimes (\tilde{T}^*_{\mathbf{e}(u)+\mathbf{p}-\mathbf{l}}\tilde{T}_{\mathbf{e}(u)+\mathbf{q}-\mathbf{l}}) =$$

$$\sum_{\mathbf{0}\le\mathbf{m}\le\mathbf{p}\wedge\mathbf{q}}\Big(\sum_{\mathbf{e}(u)+\mathbf{m}\le\mathbf{p}\wedge\mathbf{q}}(-1)^{|u|}\Big)(I_{\mathbf{m}}\otimes\tilde{T}^*_{\mathbf{p}+\mathbf{m}}\tilde{T}_{\mathbf{p}+\mathbf{m}}).$$

Applying (3.9), the last sum is equal to

$$I_{\mathbf{p}\wedge\mathbf{q}}\otimes(\tilde{T}^*_{\mathbf{p}-(\mathbf{p}\wedge\mathbf{q})}\tilde{T}_{\mathbf{q}-(\mathbf{p}\wedge\mathbf{q})}) = I_{\mathbf{q}-(\mathbf{q}-\mathbf{p})_+}\otimes T(\mathbf{q}-\mathbf{p}).$$

This proves that $R=S^*DS$.

Now, let L be as in the statement of the lemma and compute

$$(SL)(\mathbf{p},\mathbf{q}) = \sum_{\mathbf{p}\le\mathbf{l}\le\mathbf{q}} S(\mathbf{p},\mathbf{l})L(\mathbf{l},\mathbf{q}) =$$

$$\sum_{\mathbf{p}\le\mathbf{l},\mathbf{q}=\mathbf{l}+\mathbf{e}(v)}(-1)^{|v|}(I_{\mathbf{p}}\otimes T(\mathbf{l}-\mathbf{p}))(I_{\mathbf{l}}\otimes T(\mathbf{e}(v))) =$$

$$\sum_{\mathbf{p}\le\mathbf{l},\mathbf{q}=\mathbf{l}+\mathbf{e}(v)}(-1)^{|v|}I_{\mathbf{p}}\otimes(T(\mathbf{l}-\mathbf{p})(I_{\mathbf{l}-\mathbf{p}}\otimes T(\mathbf{e}(v)))) = \sum(-1)^{|v|}(I_{\mathbf{p}}\otimes T(\mathbf{q}-\mathbf{p})),$$

where the last sum runs over all $v\subseteq\{1\le i\le k : p_i<q_i\}$. The argument at the beginning of the proof shows that this is non zero only if $\mathbf{p}=\mathbf{q}$ and, in that case, it is equal to $I_{\mathbf{q}}$. This shows that $SL=I$ and, consequently, $D=L^*RL$.

■

Theorem 3.5. *A c.c. representation $(\sigma,\{T^{(i)}\})$ of X on H has a regular isometric dilation if and only if, for every $v\subseteq\{1,\dots,k\}$,*

$$\sum_{u\subseteq v}(-1)^{|u|}(I_{\boldsymbol{e}(v)-\boldsymbol{e}(u)}\otimes\tilde{T}^*_{\boldsymbol{e}(u)}\tilde{T}_{\boldsymbol{e}(u)})\ge 0, \tag{3.10}$$

where $|u|$ is the number of elements in u.

The regular isometric dilation, when it exists, can be chosen minimal.

PROOF. Suppose condition (3.10) holds. Write $\mathcal{H}_0$ for the vector space of all finitely supported functions g on $\mathbb{Z}^k_+$ with $g(\mathbf{m})\in X(\mathbf{m})\otimes H$ for all $\mathbf{m}\in\mathbb{Z}^k_+$. On $\mathcal{H}_0$ we consider the following sesquilinear form

$$\langle g,f\rangle = \sum_{\mathbf{n},\mathbf{m}\ge\mathbf{0}}\langle (I_{\mathbf{n}-(\mathbf{n}-\mathbf{m})_+}\otimes T(\mathbf{n}-\mathbf{m}))g(\mathbf{n}),f(\mathbf{m})\rangle. \tag{3.11}$$

Lemma 3.3 (together with condition (3.10)) implies that this form is positive semidefinite. Let $\mathcal{N}$ be the space of all $g\in\mathcal{H}_0$ with $\langle g,g\rangle=0$ and write K for the Hilbert space obtained by completing the quotient $\mathcal{H}_0/\mathcal{N}$ with respect to the inner product defined by (3.11).

We first embedd H into K. For that, define $W:H\to K$ by $Wh=h\delta_{\mathbf{0}}+\mathcal{N}$ where $h\delta_{\mathbf{0}}(\mathbf{0})=h\in H=X(\mathbf{0})\otimes H$ and $h\delta_{\mathbf{0}}(\mathbf{n})=0$ if $\mathbf{n}\ne\mathbf{0}$. Then, for $h,f\in H$, $\langle Wh,Wf\rangle=\langle h\delta_{\mathbf{0}},f\delta_{\mathbf{0}}\rangle=\langle h,f\rangle$. Thus W is an isometry of H into K.

Now, for $a \in A$ and $g \in \mathcal{H}_0$, we set $\rho(a)(g+\mathcal{N}) = f+\mathcal{N}$ where $f(\mathbf{m}) = (\varphi_{X(\mathbf{m})}(a) \otimes I_H)g(\mathbf{m})$. Note that, if $a \in A$ and $\mathbf{n},\mathbf{m} \in \mathbb{Z}_+^k$ satisfy $\mathbf{n} \wedge \mathbf{m} \neq \mathbf{0}$, then $(I_{\mathbf{n}\wedge\mathbf{m}} \otimes T(\mathbf{n}-\mathbf{m}))(\varphi_{X(\mathbf{n})}(a) \otimes I_H) = (\varphi_{X(\mathbf{m})}(a) \otimes I_H)(I_{\mathbf{n}\wedge\mathbf{m}} \otimes T(\mathbf{n}-\mathbf{m}))$ since $\varphi(a)$ acts on the left most factor in $X(\mathbf{n}\wedge\mathbf{m})$. If $\mathbf{n}\wedge\mathbf{m} = \mathbf{0}$ we still have the same equality since, in this case, $T(\mathbf{n}-\mathbf{m})(\varphi_{X(\mathbf{n})}(a) \otimes I_H) = \tilde{T}_{\mathbf{m}}^*\tilde{T}_{\mathbf{n}}(\varphi_{X(\mathbf{n})}(a) \otimes I_H) = \tilde{T}_{\mathbf{m}}^*\sigma(a)\tilde{T}_{\mathbf{n}} = (\varphi_{X(\mathbf{m})}(a) \otimes I_H)\tilde{T}_{\mathbf{m}}^*\tilde{T}_{\mathbf{n}}$. Thus, letting $C(a)$ be the diagonal matrix with $\varphi_{X(\mathbf{n})} \otimes I_H$ in the $\mathbf{n},\mathbf{n}$ entry, we find that $C(a)$ commutes with R (where R is as in Lemma 3.3). Clearly $\|C(a)\| \leq \|a\|$ and, therefore, $C(a)^*RC(a) \leq \|a\|^2R$. It follows that the map $\rho(a)$, defined above, is a well defined bounded operator on K. It is easy to check that ρ is indeed a C^*-representation of A on K.

For $1 \leq i \leq k$ and $\xi \in E_i$, define $V^{(i)}(\xi)(g+\mathcal{N}) = g_i+\mathcal{N}$ where $g_i(\mathbf{n}) = \xi \otimes g(\mathbf{n}-\mathbf{e}_i)$ if $\mathbf{n} \geq \mathbf{e}_i$ and is 0 otherwise. The covariance property of $V^{(i)}$ is easy to check and so is the equality

$$\langle g_i, f_i\rangle = \langle g, \rho(\langle \xi,\eta\rangle)f\rangle$$

for $g,f \in \mathcal{H}_0$. Thus $V^{(i)}(\xi)^*V^{(i)}(\eta) = \rho(\langle\xi,\eta\rangle)$ so that, for each $1 \leq i \leq k$, $(\rho, V^{(i)})$ is an isometric representation of E_i.

Now, for $g \in \mathcal{H}_0$, $\xi \in E_i$ and $h \in H$, we compute

$$\langle g, V^{(i)}(\xi)^*Wh\rangle = \langle V^{(i)}(\xi)g, Wh\rangle = \sum_{\mathbf{n}\geq\mathbf{e}_i} \langle \tilde{T}_{\mathbf{n}}(\xi \otimes g(\mathbf{n}-\mathbf{e}_i)), h\rangle =$$

$$\sum_{\mathbf{n}\geq\mathbf{e}_i} \langle \tilde{T}_{\mathbf{n}-\mathbf{e}_i}(g(\mathbf{n}-\mathbf{e}_i)), T^{(i)}(\xi)^*h\rangle = \langle g, WT^{(i)}(\xi)^*h\rangle.$$

Thus $V^{(i)}(\xi)^*W = WT^{(i)}(\xi)^*$. This proves property (ii) of Definition 3.2. Property (iii) is easy to check and we need only to verify (iv).

Note first that, for $\mathbf{p} \in \mathbb{Z}_+^k$, $\xi \in X(\mathbf{p})$ and $g \in \mathcal{H}_0$, it follows from the definition of $V^{(i)}$ above that $\tilde{V}_{\mathbf{p}}(\xi \otimes g)(\mathbf{n}) = \xi \otimes g(\mathbf{n}-\mathbf{p})$ if $\mathbf{n} \geq \mathbf{p}$ (and it is equal to 0 otherwise). Thus, for $h \in H$, $\tilde{V}_{\mathbf{p}}(\xi \otimes Wh)(\mathbf{n}) = \xi \otimes h$ if $\mathbf{n} = \mathbf{p}$ and 0 otherwise. Therefore, for $\mathbf{n} \in \mathbb{Z}^k$, $\xi \in X(\mathbf{n}_+)$, $\eta \in X(\mathbf{n}_-)$ and $h_1, h_2 \in H$,

$$\langle V(\mathbf{n})(\xi \otimes Wh_1), \eta \otimes Wh_2\rangle = \langle \tilde{V}_{\mathbf{n}_+}(\xi \otimes Wh_1), \tilde{V}_{\mathbf{n}_-}(\eta \otimes Wh_2)\rangle =$$

$$\langle T(\mathbf{n})(\xi \otimes h_1), \eta \otimes h_2\rangle.$$

This proves that this is indeed a regular isometric dilation.

Now assume that $(\sigma, \{T^{(i)}\})$ has an isometric regular dilation $(\rho, \{V^{(i)}\})$ (on K). Let R_V, S_V and D_V be the matrices described in Lemma 3.3 with V replacing T. Since $(\rho, \{V^{(i)}\})$ is an isometric representation, it follows that, for $u \subseteq \{1,\ldots,k\}$, $\tilde{V}_{\mathbf{e}(u)}^*\tilde{V}_{\mathbf{e}(u)}$ is the identity map on $X(\mathbf{e}(u)) \otimes H$. The argument in the first paragraph of the proof of Lemma 3.3 now shows that D_V is the identity matrix and, thus, $R_V = S_V^*D_VS_V \geq 0$. But, since the dilation is regular, the matrix R (as in Lemma 3.3) is a compression of R_V. It follows that $R \geq 0$ and, using Lemma 3.3 again, $D \geq 0$. From this, (3.10) follows.

If a regular, isometric, dilation exists, we can restrict it to the minimal closed subspace containing H and invariant under all $V^{(i)}(\xi)$, $\xi \in E_i$, to get a minimal one. ■

The following lemma is easy to verify but will be useful.

Lemma 3.6. *If $(\sigma, \{V^{(i)}\})$ is an isometric representation (that is, each $\tilde{V}^{(i)}$ is an isometry), then, for $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}_+^k$, $\tilde{V}_{\boldsymbol{m}}^* \tilde{V}_{\boldsymbol{n}} = I_{\boldsymbol{n}\wedge\boldsymbol{m}} \otimes V(\boldsymbol{n}-\boldsymbol{m})$.*

PROOF. Compute $\tilde{V}_{\mathbf{m}}^* \tilde{V}_{\mathbf{n}} = (I_{\mathbf{n}\wedge\mathbf{m}} \otimes \tilde{V}_{\mathbf{m}-\mathbf{m}\wedge\mathbf{n}}^*)\tilde{V}_{\mathbf{m}\wedge\mathbf{n}}^* \tilde{V}_{\mathbf{m}\wedge\mathbf{n}}(I_{\mathbf{n}\wedge\mathbf{m}} \otimes \tilde{V}_{\mathbf{n}-\mathbf{m}\wedge\mathbf{n}}) = I_{\mathbf{m}\wedge\mathbf{n}} \otimes V(\mathbf{n}-\mathbf{m})$. ■

Proposition 3.7. *A minimal, regular, isometric dilation of $(\sigma, \{T^{(i)}\})$ is unique up to unitary equivalence.*

PROOF. Suppose $(\rho, \{V^{(i)}\})$ and $(\tau, \{U^{(i)}\})$ are minimal regular isometric dilations of $(\sigma, \{T^{(i)}\})$ on K and G respectively. For every $\mathbf{n} \in \mathbb{Z}_+^k$ write $K(\mathbf{n}) = \tilde{V}_{\mathbf{n}}(X(\mathbf{n}) \otimes H)$ and $G(\mathbf{n}) = \tilde{U}_{\mathbf{n}}(X(\mathbf{n}) \otimes H)$ (and, for $\mathbf{n} = \mathbf{0}$, $K(\mathbf{0}) = H = G(\mathbf{0})$). Now, let $R(\mathbf{n}) : K(\mathbf{n}) \to G(\mathbf{n})$ be defined by $R(\mathbf{n})\tilde{V}_{\mathbf{n}}(\xi \otimes h) = \tilde{U}_{\mathbf{n}}(\xi \otimes h)$ (for $\xi \in X(\mathbf{n})$ and $h \in H$) and $R(\mathbf{0}) = I_H$. For $\mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^k$, $\xi \in X(\mathbf{n})$, $\eta \in X(\mathbf{m})$ and $h, g \in H$, we have

$$\langle \tilde{V}_{\mathbf{n}}(\xi \otimes h), \tilde{V}_{\mathbf{m}}(\eta \otimes g)\rangle = \langle \tilde{V}_{\mathbf{m}}^* \tilde{V}_{\mathbf{n}}(\xi \otimes h), \eta \otimes g\rangle =$$

$$\langle (I_{\mathbf{m}\wedge\mathbf{n}} \otimes V(\mathbf{n}-\mathbf{m}))(\xi \otimes h), \eta \otimes g\rangle = \langle (I_{\mathbf{m}\wedge\mathbf{n}} \otimes T(\mathbf{n}-\mathbf{m}))(\xi \otimes h), \eta \otimes g\rangle,$$

where the second equality follows from Lemma 3.6 and last one follows from Definition 3.2 (iv). A similar computation holds for U, in place of V, and we get $\langle R(\mathbf{n})k_{\mathbf{n}}, R(\mathbf{m})k_{\mathbf{m}}\rangle = \langle k_{\mathbf{n}}, k_{\mathbf{m}}\rangle$ for every $k_{\mathbf{n}} \in K(\mathbf{n})$ and $k_{\mathbf{m}} \in K(\mathbf{m})$. This shows that each $R(\mathbf{n})$ is well defined and isometric and, also, that there is a unitary operator $R : K \to G$ such that $R|K(\mathbf{n}) = W(\mathbf{n})$ for $\mathbf{n} \in \mathbb{Z}_+^k$. Fix $1 \le i \le k$, $\mathbf{n} \in \mathbb{Z}_+^k$, $\eta \in X(\mathbf{n})$, $\xi \in E_i$ and $h \in H$. Then

$$RV^{(i)}(\xi)\tilde{V}_{\mathbf{n}}(\eta \otimes h) = R\tilde{V}_{\mathbf{e}_i}(I_{\mathbf{e}_i} \otimes \tilde{V}_{\mathbf{n}})(\xi \otimes \eta \otimes h) = R\tilde{V}_{\mathbf{n}+\mathbf{e}_i}(\xi \otimes \eta \otimes h) =$$

$$\tilde{U}_{\mathbf{n}+\mathbf{e}_i}(\xi \otimes \eta \otimes h) = \tilde{U}_{\mathbf{e}_i}(I_{\mathbf{e}_i} \otimes \tilde{U}_{\mathbf{n}})(\xi \otimes \eta \otimes h) = U^{(i)}(\xi)\tilde{U}_{\mathbf{n}}(\eta \otimes h).$$

It follows from the minimality assumption that, for all $1 \le i \le k$ and $\xi \in E_i$, $RV^{(i)}(\xi) = U^{(i)}(\xi)R$. Similarly, one checks that, for $a \in A$, $R\rho(a) = \tau(a)R$. ■

Definition 3.8. *We say that a representation $(\sigma, \{T^{(i)}\})$ is a doubly commuting representation if, for every $i \neq j$ (in $\{1, \ldots, k\}$), we have*

$$\tilde{T}^{(j)*}\tilde{T}^{(i)} = (I_{\boldsymbol{e}_j} \otimes \tilde{T}^{(i)})(I_{\boldsymbol{e}_i} \otimes \tilde{T}^{(j)*}). \tag{3.12}$$

More precisely, $\tilde{T}^{(j)*}\tilde{T}^{(i)} = (I_{\boldsymbol{e}_j} \otimes \tilde{T}^{(i)})(t_{i,j} \otimes I_H)(I_{\boldsymbol{e}_i} \otimes \tilde{T}^{(j)*})$ where $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ is the isomorphism as in Equation (3.2).

The proof of the next lemma is straightforward and is omitted.

Lemma 3.9. *Let $(\sigma, T^{(i)})$ be a doubly commuting representation. Then*

(i) *For $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}_+^k$ with $\boldsymbol{n} \wedge \boldsymbol{m} = \boldsymbol{0}$,*

$$(I_{\boldsymbol{m}} \otimes \tilde{T}_{\boldsymbol{n}})(I_{\boldsymbol{n}} \otimes \tilde{T}^*_{\boldsymbol{m}}) = \tilde{T}^*_{\boldsymbol{m}} \tilde{T}_{\boldsymbol{n}}.$$

In particular, for $\boldsymbol{p} \in \mathbb{Z}^k$,

$$(I_{\boldsymbol{p}_-} \otimes \tilde{T}_{\boldsymbol{p}_+})(I_{\boldsymbol{p}_+} \otimes \tilde{T}^*_{\boldsymbol{p}_-}) = T(\boldsymbol{p}).$$

(ii) *If $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{n} \in \mathbb{Z}_+^k$ with $\boldsymbol{p} \le \boldsymbol{n}$ and $\boldsymbol{q} \wedge \boldsymbol{p} = \boldsymbol{0}$, then*

$$(I_{\boldsymbol{n}-\boldsymbol{p}+\boldsymbol{q}} \otimes \tilde{T}^*_{\boldsymbol{p}} \tilde{T}_{\boldsymbol{p}})(I_{\boldsymbol{n}} \otimes \tilde{T}^*_{\boldsymbol{q}} \tilde{T}_{\boldsymbol{q}}) = I_{\boldsymbol{n}-\boldsymbol{p}} \otimes \tilde{T}^*_{\boldsymbol{p}+\boldsymbol{q}} \tilde{T}_{\boldsymbol{p}+\boldsymbol{q}}.$$

(iii) *For $u \subseteq v \subseteq \{1, \dots, k\}$ and $l \notin v$,*

$$(I_{\boldsymbol{e}(v)-\boldsymbol{e}(u)+\boldsymbol{e}_l} \otimes \tilde{T}^*_{\boldsymbol{e}(u)} \tilde{T}_{\boldsymbol{e}(u)})(I_{\boldsymbol{e}(v)+\boldsymbol{e}_l} \otimes I_H - (I_{\boldsymbol{e}(v)} \otimes \tilde{T}^{(l)*} \tilde{T}^{(l)})) =$$

$$(I_{\boldsymbol{e}(v)-\boldsymbol{e}(u)+\boldsymbol{e}_l} \otimes \tilde{T}^*_{\boldsymbol{e}(u)} \tilde{T}_{\boldsymbol{e}(u)}) - (I_{\boldsymbol{e}(v)-\boldsymbol{e}(u)} \otimes \tilde{T}^*_{\boldsymbol{e}(u)+\boldsymbol{e}_l} \tilde{T}_{\boldsymbol{e}(u)+\boldsymbol{e}_l}).$$

(iv) *Let $j \neq l$ in $\{1, \dots, k\}$ and $\{j, l\} \subseteq w \subseteq \{1, \dots, k\}$. Then*

$$(I_{\boldsymbol{e}(w)-\boldsymbol{e}_j} \otimes \tilde{T}^{(j)*} \tilde{T}^{(j)})(I_{\boldsymbol{e}(w)-\boldsymbol{e}_l} \otimes \tilde{T}^{(l)*} \tilde{T}^{(l)}) = I_{\boldsymbol{e}(w)-\boldsymbol{e}_l-\boldsymbol{e}_j} \otimes \tilde{T}^*_{\boldsymbol{e}_j+\boldsymbol{e}_l} \tilde{T}_{\boldsymbol{e}_j+\boldsymbol{e}_l}$$

$$= (I_{\boldsymbol{e}(w)-\boldsymbol{e}_l} \otimes \tilde{T}^{(l)*} \tilde{T}^{(l)})(I_{\boldsymbol{e}(w)-\boldsymbol{e}_j} \otimes \tilde{T}^{(j)*} \tilde{T}^{(j)}).$$

Theorem 3.10. *If the representation $(\sigma, \{T^{(i)}\})$ is doubly commuting then it has a regular isometric dilation and the regular isometric dilation that is minimal is doubly commuting.*

PROOF. To show that it has a regular isometric dilation, we should verify condition (3.10) of Theorem 3.5. In fact, we claim that, for every $v \subseteq \{1, \dots, k\}$, we have

$$\sum_{u \subseteq v} (-1)^{|u|} (I_{\mathbf{e}(v)-\mathbf{e}(u)} \otimes \tilde{T}^*_{\mathbf{e}(u)} \tilde{T}_{\mathbf{e}(u)}) = \prod_{i \in v} (I_{\mathbf{e}(v)} \otimes I_H - (I_{\mathbf{e}(v)-\mathbf{e}_i} \otimes \tilde{T}^{(i)*} \tilde{T}^{(i)})). \quad (3.13)$$

Since, by Lemma 3.9 (iv), the operators in the product commute, this will show that the condition of Theorem 3.5 holds.

We shall prove the claim by induction on the number of elements in v. If $|v| = 2$, we can write $v = \{j, l\}$ and then the claim follows easily from Lemma 3.9 (iv). Now assume we know it for v and $w = v \cup \{l\}$ where $l \notin v$. Tensoring (3.13) (for v) by $I_{\mathbf{e}_l}$, we get

$$\sum_{u \subseteq v} (-1)^{|u|} (I_{\mathbf{e}(w)-\mathbf{e}(u)} \otimes \tilde{T}^*_{\mathbf{e}(u)} \tilde{T}_{\mathbf{e}(u)}) = \prod_{i \in v} (I_{\mathbf{e}(w)} \otimes I_H - (I_{\mathbf{e}(w)-\mathbf{e}_i} \otimes \tilde{T}^{(i)*} \tilde{T}^{(i)})).$$

Thus

$$\prod_{i\in w}(I_{\mathbf{e}(w)}\otimes I_H-(I_{\mathbf{e}(w)-\mathbf{e}_i}\otimes\tilde{T}^{(i)*}\tilde{T}^{(i)}))=$$

$$(\sum_{u\subseteq v}(-1)^{|u|}(I_{\mathbf{e}(w)-\mathbf{e}(u)}\otimes\tilde{T}^*_{\mathbf{e}(u)}\tilde{T}_{\mathbf{e}(u)}))(I_{\mathbf{e}(w)}\otimes I_H-(I_{\mathbf{e}(w)-\mathbf{e}_l}\otimes\tilde{T}^{(l)*}\tilde{T}^{(l)})).$$

Using Lemma 3.9 (iii), this is equal to

$$\sum_{u\subseteq v}(-1)^{|u|}((I_{\mathbf{e}(v)-\mathbf{e}(u)+\mathbf{e}_l}\otimes\tilde{T}^*_{\mathbf{e}(u)}\tilde{T}_{\mathbf{e}(u)})-(I_{\mathbf{e}(v)-\mathbf{e}(u)}\otimes\tilde{T}^*_{\mathbf{e}(u)+\mathbf{e}_l}\tilde{T}_{\mathbf{e}(u)+\mathbf{e}_l}))=$$

$$\sum_{u\subseteq w}(-1)^{|u|}(I_{\mathbf{e}(w)-\mathbf{e}(u)}\otimes\tilde{T}^*_{\mathbf{e}(u)}\tilde{T}_{\mathbf{e}(u)}).$$

This completes the proof of the claim and shows that the representation has an isometric regular dilation. In this case, it has an isometric regular dilation $(\rho,\{V^{(i)}\})$ (on K) that is minimal in the sense that

$$\bigvee\{\tilde{V}_{\mathbf{n}}(X(\mathbf{n})\otimes H):\ \mathbf{n}\in\mathbb{Z}^k_+\}=K. \tag{3.14}$$

To prove that the representation $(\rho,\{V^{(i)}\})$ is doubly commuting, we fix $i\neq j$ and we should prove the equality

$$\tilde{V}^{(j)*}\tilde{V}^{(i)}=(I_{\mathbf{e}_j}\otimes\tilde{V}^{(i)})(I_{\mathbf{e}_i}\otimes\tilde{V}^{(j)*}).$$

On both sides of this equality we have operators from $E_i\otimes K$ to $E_j\otimes K$. It follows from the minimality condition that

$$\bigvee\{(I_{\mathbf{e}_i}\otimes\tilde{V}_{\mathbf{n}})(X(\mathbf{n}+\mathbf{e}_i)\otimes H):\ \mathbf{n}\in\mathbb{Z}^k_+\}=E_i\otimes K. \tag{3.15}$$

Thus, it suffices to show that, for every $\mathbf{n},\mathbf{m}\in\mathbb{Z}^k_+$, $\xi\in X(\mathbf{n}+\mathbf{e}_i)$, $\eta\in X(\mathbf{m}+\mathbf{e}_j)$ and $h,g\in H$,

$$\langle\tilde{V}^{(j)*}\tilde{V}^{(i)}(I_{\mathbf{e}_i}\otimes\tilde{V}_{\mathbf{n}})(\xi\otimes h),(I_{\mathbf{e}_j}\otimes\tilde{V}_{\mathbf{m}})(\eta\otimes g)\rangle= \tag{3.16}$$

$$\langle(I_{\mathbf{e}_j}\otimes\tilde{V}^{(i)})(I_{\mathbf{e}_i}\otimes\tilde{V}^{(j)*})(I_{\mathbf{e}_i}\otimes\tilde{V}_{\mathbf{n}})(\xi\otimes h),(I_{\mathbf{e}_j}\otimes\tilde{V}_{\mathbf{m}})(\eta\otimes g)\rangle.$$

The left-hand-side of this equality is equal to $\langle(I_{\mathbf{e}_j}\otimes\tilde{V}^*_{\mathbf{m}})\tilde{V}^{(j)*}\tilde{V}^{(i)}(I_{\mathbf{e}_i}\otimes\tilde{V}_{\mathbf{n}})(\xi\otimes h),\eta\otimes g\rangle=\langle\tilde{V}^*_{\mathbf{m}+\mathbf{e}_j}\tilde{V}_{\mathbf{n}+\mathbf{e}_i}(\xi\otimes h),\eta\otimes g\rangle=\langle V(\mathbf{n}+\mathbf{e}_j-\mathbf{m}-\mathbf{e}_i)(\xi\otimes h),\eta\otimes g\rangle$ where the last equality follows from Lemma 3.6. Thus, what we need to prove is

$$\langle(I_{\mathbf{e}_j}\otimes\tilde{V}^{(i)})(I_{\mathbf{e}_i}\otimes\tilde{V}^{(j)*})(I_{\mathbf{e}_i}\otimes\tilde{V}_{\mathbf{n}})(\xi\otimes h),(I_{\mathbf{e}_j}\otimes\tilde{V}_{\mathbf{m}})(\eta\otimes g)\rangle= \tag{3.17}$$

$$\langle V(\mathbf{n}+\mathbf{e}_j-\mathbf{m}-\mathbf{e}_i)(\xi\otimes h),\eta\otimes g\rangle.$$

If $\mathbf{e}_j\leq\mathbf{n}$ (or $\mathbf{e}_i\leq\mathbf{m}$) then this is easy to verify. Thus we are left with the case when $n_j=0=m_i$.

For that, we claim that, for $\mathbf{n} \in \mathbb{Z}_+^k$ and $j \in \{1, \ldots, k\}$ with $\mathbf{n} \wedge \mathbf{e}_j = 0$, we have

$$\tilde{V}^{(j)*}\tilde{V}_{\mathbf{n}}|X(\mathbf{n}) \otimes H = (I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{n}})(I_{\mathbf{n}} \otimes \tilde{V}^*_{\mathbf{e}_j})|X(\mathbf{n}) \otimes H.$$

Note that the ranges of the operators in this equation lie in $E_j \otimes K$. Using (3.15), (which is a consequence of the minimality) it suffices to show, for every $\mathbf{p} \in \mathbb{Z}_+^k$, $\xi \in X(\mathbf{n})$, $\eta \in X(\mathbf{p} + \mathbf{e}_j)$ and $h, g \in H$,

$$\langle \tilde{V}^{(j)*}\tilde{V}_{\mathbf{n}}(\xi \otimes h), (I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{p}})(\eta \otimes g)\rangle = \tag{3.18}$$

$$\langle (I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{n}})(I_{\mathbf{n}} \otimes \tilde{V}^*_{\mathbf{e}_j})(\xi \otimes h), (I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{p}})(\eta \otimes g)\rangle.$$

This is now straightforward to prove using Lemma 3.6 and Lemma 3.9(i).

Now we turn to prove Equation (3.17). The left hand side of that equation is

$$\langle (I_{\mathbf{e}_i} \otimes \tilde{V}^{(j)*})(I_{\mathbf{e}_i} \otimes \tilde{V}_{\mathbf{n}})(\xi \otimes h), (I_{\mathbf{e}_j} \otimes \tilde{V}^{(i)*})(I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{m}})(\eta \otimes g)\rangle =$$

$$\langle (I_{\mathbf{e}_i} \otimes (\tilde{V}^{(j)*}\tilde{V}_{\mathbf{n}}))(\xi \otimes h), (I_{\mathbf{e}_j} \otimes (\tilde{V}^{(i)*}\tilde{V}_{\mathbf{m}}))(\eta \otimes g)\rangle.$$

Applying the claim, this is equal to

$$\langle (I_{\mathbf{e}_i+\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{n}})(I_{\mathbf{n}+\mathbf{e}_i} \otimes \tilde{V}^*_{\mathbf{e}_j})(\xi \otimes h), (I_{\mathbf{e}_i+\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{m}})(I_{\mathbf{m}+\mathbf{e}_j} \otimes \tilde{V}^*_{\mathbf{e}_i})(\eta \otimes g) =$$

$$\langle (I_{\mathbf{e}_i+\mathbf{e}_j+\mathbf{n}\wedge\mathbf{m}} \otimes V(\mathbf{n}-\mathbf{m}))(I_{\mathbf{n}+\mathbf{e}_i} \otimes \tilde{T}^*_{\mathbf{e}_j})(\xi \otimes h), (I_{\mathbf{m}+\mathbf{e}_j} \otimes \tilde{T}^*_{\mathbf{e}_i})(\eta \otimes g)\rangle.$$

By regularity, this is equal to

$$\langle (I_{\mathbf{m}+\mathbf{e}_j} \otimes \tilde{T}_{\mathbf{e}_i})(I_{\mathbf{e}_i+\mathbf{e}_j+\mathbf{n}\wedge\mathbf{m}} \otimes T(\mathbf{n}-\mathbf{m}))(I_{\mathbf{n}+\mathbf{e}_i} \otimes \tilde{T}^*_{\mathbf{e}_j})(\xi \otimes h), (\eta \otimes g)\rangle$$

and, applying Lemma 3.9(i), we get

$$\langle (I_{\mathbf{m}+\mathbf{e}_j} \otimes \tilde{T}_{\mathbf{e}_i})(I_{\mathbf{e}_i+\mathbf{e}_j+\mathbf{m}} \otimes \tilde{T}_{(\mathbf{n}-\mathbf{m})_+})(I_{\mathbf{e}_i+\mathbf{e}_j+\mathbf{n}} \otimes \tilde{T}^*_{(\mathbf{n}-\mathbf{m})_-})(I_{\mathbf{n}+\mathbf{e}_i} \otimes \tilde{T}^*_{\mathbf{e}_j})(\xi \otimes h),$$

$$(\eta \otimes g)\rangle = \langle (I_{\mathbf{m}+\mathbf{e}_j} \otimes \tilde{T}_{(\mathbf{n}-\mathbf{m})_+ + \mathbf{e}_i})(I_{\mathbf{n}+\mathbf{e}_i} \otimes \tilde{T}^*_{(\mathbf{n}-\mathbf{m})_- + \mathbf{e}_j})(\xi \otimes h), \eta \otimes g\rangle =$$

$$\langle (I_{\mathbf{m}+\mathbf{e}_j} \otimes \tilde{T}_{(\mathbf{n}+\mathbf{e}_i-\mathbf{m}-\mathbf{e}_j)_+})(I_{\mathbf{n}+\mathbf{e}_i} \otimes \tilde{T}^*_{(\mathbf{n}+\mathbf{e}_i-\mathbf{m}-\mathbf{e}_j)_-})(\xi \otimes h), \eta \otimes g\rangle.$$

Using Lemma 3.9(i) and the regularity of the dilation, we find that the last expression is equal to

$$\langle V(\mathbf{n} + \mathbf{e}_i - \mathbf{m} - \mathbf{e}_j)(\xi \otimes h), \eta \otimes g\rangle$$

proving (3.17).

■

Lemma 3.11. *An isometric representation $(\rho, \{V^{(i)}\})$ is doubly commuting if and only if, for every $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}_+^k$,*

$$\tilde{V}_{\boldsymbol{n}}\tilde{V}^*_{\boldsymbol{n}}\tilde{V}_{\boldsymbol{m}}\tilde{V}^*_{\boldsymbol{m}} = \tilde{V}_{\boldsymbol{n}\vee\boldsymbol{m}}\tilde{V}^*_{\boldsymbol{n}\vee\boldsymbol{m}}. \tag{3.19}$$

Proof. Assume that the representation is doubly commuting and compute, using Lemma 3.6, for $\mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^k$,

$$\tilde{V}_{\mathbf{n}} \tilde{V}_{\mathbf{n}}^* \tilde{V}_{\mathbf{m}} \tilde{V}_{\mathbf{m}}^* = \tilde{V}_{\mathbf{n}} (I_{\mathbf{n}-(\mathbf{m}-\mathbf{n})_-} \otimes V(\mathbf{m}-\mathbf{n})) \tilde{V}_{\mathbf{m}}^*.$$

Since the representation is doubly commuting, this is equal to

$$\tilde{V}_{\mathbf{n}} (I_{\mathbf{n}} \otimes \tilde{V}_{(\mathbf{m}-\mathbf{n})_+})(I_{\mathbf{m}} \otimes \tilde{V}_{(\mathbf{m}-\mathbf{n})_-}^*) \tilde{V}_{\mathbf{m}}^* = \tilde{V}_{\mathbf{n}+(\mathbf{m}-\mathbf{n})_+} \tilde{V}_{\mathbf{m}+(\mathbf{m}-\mathbf{n})_-}^* =$$

$$\tilde{V}_{\mathbf{n} \vee \mathbf{m}} \tilde{V}_{\mathbf{n} \vee \mathbf{m}}^*$$

proving one direction. For the other direction, assume that (3.19) holds and fix $i \neq j$ in $\{1, \ldots, k\}$. Then

$$\tilde{V}_{\mathbf{e}_j} (I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{e}_i} \tilde{V}_{\mathbf{e}_i}^*) \tilde{V}_{\mathbf{e}_j}^* = \tilde{V}_{\mathbf{e}_i+\mathbf{e}_j} \tilde{V}_{\mathbf{e}_i+\mathbf{e}_j}^* = \tilde{V}_{\mathbf{e}_i} \tilde{V}_{\mathbf{e}_i}^* \tilde{V}_{\mathbf{e}_j} \tilde{V}_{\mathbf{e}_j}^*.$$

Multiplying on the left by $\tilde{V}_{\mathbf{e}_i}^*$ and on the right by $\tilde{V}_{\mathbf{e}_j}$ and using the fact that the representation is isometric, we get $\tilde{V}_{\mathbf{e}_i}^* \tilde{V}_{\mathbf{e}_j} (I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{e}_i} \tilde{V}_{\mathbf{e}_i}^*) = \tilde{V}_{\mathbf{e}_i}^* \tilde{V}_{\mathbf{e}_j}$. Since $\tilde{V}_{\mathbf{e}_j}(I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{e}_i}) = \tilde{V}_{\mathbf{e}_i+\mathbf{e}_j} = \tilde{V}_{\mathbf{e}_i}(I_{\mathbf{e}_i} \otimes \tilde{V}_{\mathbf{e}_j})$, we have

$$\tilde{V}_{\mathbf{e}_i}^* \tilde{V}_{\mathbf{e}_j} = \tilde{V}_{\mathbf{e}_i}^* (\tilde{V}_{\mathbf{e}_j} (I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{e}_i}))(I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{e}_i}^*) = \tilde{V}_{\mathbf{e}_i}^* (\tilde{V}_{\mathbf{e}_i} (I_{\mathbf{e}_i} \otimes \tilde{V}_{\mathbf{e}_j}))(I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{e}_i}^*) =$$

$$(I_{\mathbf{e}_i} \otimes \tilde{V}_{\mathbf{e}_j})(I_{\mathbf{e}_j} \otimes \tilde{V}_{\mathbf{e}_i}^*)$$

proving that the representation is doubly commuting. ■

Remark 3.12. *An isometric representation satisfying (3.19) is referred to in the literature as a Nica-covariant representation (see [4] or [15]). Thus, the lemma shows that being Nica-covariant is equivalent to being an isometric doubly commuting representation.*

An important representation of X is the Fock representation. It is defined as in [4]. We write

$$\mathcal{F}(X) = \sum_{\mathbf{n} \in \mathbb{Z}_+^k} \oplus X(\mathbf{n}).$$

As mentioned in [4], this is a C^*-correspondence over A with left action given by

$$\varphi_\infty(a)(\oplus x_{\mathbf{n}}) = (\oplus \varphi_{\mathbf{n}}(a) x_{\mathbf{n}}).$$

We can define a representation L of X on $\mathcal{F}(X)$ by setting

$$L(x)(\oplus x_{\mathbf{n}}) = \oplus(x \otimes x_{\mathbf{n}}) \ , \ \ \oplus x_{\mathbf{n}} \in \mathcal{F}(X). \tag{3.20}$$

Note that, strictly speaking, this is not what we defined as a representation above (since $\mathcal{F}(X)$ is not a Hilbert space) but we can "fix" it by representing $\mathcal{L}(\mathcal{F}(X))$ on a Hilbert space.

Let $\mathcal{T}_c(X)$ be the C^*-algebra generated by the operators $\{L(x) : x \in X\}$.

If π is a faithful representation of A on a Hilbert space H, then $\mathcal{F}(X) \otimes_\pi H$ is a Hilbert space and the map $T \mapsto T \otimes I_H$ is a faithful representation of $\mathcal{L}(\mathcal{F}(X))$ on $\mathcal{F}(X) \otimes_\pi H$ called *the induced representation*. Its restriction to $\mathcal{T}_c(X)$ is a faithful representation of $\mathcal{T}_c(X)$ denoted $Ind(\pi)$.

In [4, Theorem 6.3], Fowler proved the following.

Theorem 3.13. *([4]) There is a C^*-algebra, denoted $\mathcal{T}_{cov}(X)$, and an isometric representation $i_X : X \to \mathcal{T}_{cov}(X)$ such that $\mathcal{T}_{cov}(X)$ is generated by $i_X(X)$ and $(\mathcal{T}_{cov}(X), i_X)$ is universal for Nica-covariant isometric representations of X, in the sense that:*

(a) *There is a faithful representation θ of $\mathcal{T}_{cov}(X)$ on a Hilbert space such that $\theta \circ i_X$ is a Nica-covariant isometric representation of X ; and*

(b) *for every Nica-covariant isometric representation (σ, T) of X, there is a C^*-representation $T \times \sigma$ of $\mathcal{T}_{cov}(X)$ such that $T = (T \times \sigma) \circ i_X$.*

Up to canonical isomorphism, $(\mathcal{T}_{cov}(X), i_X)$ is the unique pair with this property.

The following definition can be found in [4, definition 5.7]. Recall that, for a Hilbert C^*-module E, $\mathcal{K}(E)$ is the closed ideal in $\mathcal{L}(E)$ generated by the (adjointable) operators $\xi \otimes \eta^*$, for $\xi, \eta \in E$, defined by $(\xi \otimes \eta^*)\zeta = \xi\langle \eta, \zeta \rangle$, $\zeta \in E$.

Definition 3.14. *We say that X is compactly aligned if, whenever $T \in \mathcal{K}(X(\boldsymbol{n}))$ and $S \in \mathcal{K}(X(\boldsymbol{m}))$, we have*

$$(S \otimes I_{\boldsymbol{n} \vee \boldsymbol{m} - \boldsymbol{m}})(T \otimes I_{\boldsymbol{n} \vee \boldsymbol{m} - \boldsymbol{n}}) \in \mathcal{K}(X(\boldsymbol{n} \vee \boldsymbol{m})).$$

Clearly, if, for every $\mathbf{n} \in \mathbb{Z}_+^k$, $\mathcal{K}(X(\mathbf{n})) = \mathcal{L}(X(\mathbf{n}))$ then X is compactly aligned.

The proof of the following result can be dug out of [4].

Theorem 3.15. *Suppose X is compactly aligned and each $X(\boldsymbol{n})$ ($\boldsymbol{n} \in \mathbb{Z}_+^k$) is essential (that is, $\varphi_{X(\boldsymbol{n})}(A)X(\boldsymbol{n})$ is dense in $X(\boldsymbol{n})$) then the pair $(\mathcal{T}_c(X), L)$ is canonically isomorphic to $(\mathcal{T}_{cov}(X), i_X)$. Thus, $(\mathcal{T}_c(X), L)$ is universal for Nica-covariant (equivalently, for doubly commuting) isometric representations of X.*

PROOF. Here we just indicate how to read the proof from the results of [4]. There, the author constructs a C^*-algebra denoted $B_P \times_{\tau, X} P$ that contains $\mathcal{T}_{cov}(X)$ (Theorem 6.3 there). Let π be a faithful nondegenerate representation of A on a Hilbert space H and write Ψ for $Ind(\pi) \circ L$. This is an isometric, Nica-covariant, representation of X on $\mathcal{F}(X) \otimes_\pi H$ (see [4, lemma 5.3]). It gives rise to a representation, denoted $L^\Psi \times \Psi$, of $B_P \times_{\tau, X} P$ on $\mathcal{F}(X) \otimes_\pi H$ whose restriction to $\mathcal{T}_{cov}(X)$ is the Nica-covariant representation that $Ind(\pi) \circ L$ induces on $\mathcal{T}_{cov}(X)$ (by its universal property). In [4, corollary 7.7] it is shown that $L^\Psi \times \Psi$ is a faithful representation. It follows that $Ind(\pi) \circ L$ gives rise to a faithful representation of $\mathcal{T}_{cov}(X)$ on $\mathcal{F}(X) \otimes_\pi H$. Its image is equal to the image of $Ind(\pi)$ and, thus, composing it with $Ind(\pi)^{-1}$, we get a *-isomorphism from $\mathcal{T}_{cov}(X)$ onto $\mathcal{T}_c(X)$ that carries i_X to L. ■

Definition 3.16. *The Banach algebra generated by* $\{L(x) : x \in X(\boldsymbol{n}), \boldsymbol{n} \in \mathbb{Z}_+^k\}$ *will be called* the concrete tensor algebra *of* X *and will be written* $\mathcal{T}_{+,c}(X)$.

In [21] we defined the tensor algebra $\mathcal{T}_+(X)$, associated with X, as an algebra satisfying a certain universal property (for c.c. representations of X). When $k = 1$, it coincides with the concrete tensor algebra $\mathcal{T}_{+,c}(X)$. In general, the concrete tensor algebra does not have that universal property. Nevertheless, it satisfies the following.

Corollary 3.17. *Let* X *be a compactly aligned product system of essential correspondences. For every c.c. doubly commuting representation* $(\sigma, \{T^{(i)}\})$ *of* X *on a Hilbert space* H*, there is a completely contractive representation* $T \times \sigma$ *of* $\mathcal{T}_{+,c}(X)$ *on* H *such that, for every* $1 \le i \le k$ *and every* $\xi \in X(\boldsymbol{e}_i)$,

$$(T \times \sigma)(L(\xi)) = T^{(i)}(\xi).$$

Proof. Let $(\rho, \{V^{(i)}\})$ be the minimal regular isometric dilation of $(\sigma, \{T^{(i)}\})$ (on, say, K). By Theorem 3.10 this isometric representation is doubly commuting. We see in Lemma 3.11 that it is Nica covariant. It then follows from Theorem 3.15 that there is a C^*-representation π of $\mathcal{T}_c(X)$ on K such that $V = \pi \circ L$. Thus, for every $1 \le i \le k$ and every $\xi \in X(\mathbf{e}_i)$, $\pi(L(\xi)) = V^{(i)}(\xi)$. Writing $T \times \sigma$ for $P_H \pi(\cdot)|H$, we see that $T \times \sigma$ is a completely contractive map of $\mathcal{T}_c(X)$ into $B(H)$. Since $K \ominus H$ is invariant under $V^{(i)}$ for all $1 \le i \le k$ (and, thus, invariant for $\tilde{V}_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}_+^k$) the map $T \times \sigma$ is multiplicative on $\mathcal{T}_{+,c}(X)$ and defines a completely contractive representation. ■

4. Examples

4.1. The case $k = 1$

In this case, we have a single C^*-correspondence E over the C^*-algebra A and $X(n) = E^{\otimes n}$, $n \in \mathbb{Z}_+$. The algebra $\mathcal{T}_{+,c}(X)$ was denoted by $\mathcal{T}_+(E)$ in [11] and its representations were studied there. Of course, in this case, every representation is doubly commuting. It was shown in [11, theorem 3.3] that every c.c. representation has a (unique) minimal isometric dilation. In [11, theorem 3.10] it was shown that every c.c. representation of E gives rise to a (unique) completely contractive representation of $\mathcal{T}_+(E)$. Thus, Theorem 3.10 and Corollary 3.17 generalise these results of [11].

4.2. The case $A = E_i = \mathbb{C}$

Now set $A = \mathbb{C}$ and, for each $1 \le i \le k$, $E_i = \mathbb{C}$ (with the obvious correspondence structure). In order to define the product system X (over $\mathbb{Z}_+^k$) we need to specify, for every $1 \le i, j \le k$, an isomorphism of correspondences $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ (with $t_{j,i} = t_{i,j}^{-1}$ and $t_{i,i} = id$). This amounts to fixing complex numbers $\lambda_{i,j}$ with $|\lambda_{i,j}| = 1$, $\lambda_{i,i} = 1$ and $\lambda_{j,i} = \lambda_{i,j}^{-1}$ and setting $t_{i,j}(a \otimes b) = \lambda_{i,j} b \otimes a$. (Note that (3.2) is satisfied.)

So, suppose we fix these numbers and this defines X. Using (3.3), a c.c. representation of X is now a k-tuple $(T^{(1)}, T^{(2)}, \ldots, T^{(k)})$ of contractions in $B(H)$ (for some Hilbert space H) that satisfy

$$T^{(i)}T^{(j)} = \lambda_{i,j}T^{(j)}T^{(i)} \tag{4.1}$$

for all i, j. It is easy to check that this representation is doubly commuting if and only if

$$T^{(i)*}T^{(j)} = \overline{\lambda_{i,j}}T^{(j)}T^{(i)*} \tag{4.2}$$

for all $i \neq j$.

The case where $\lambda_{i,j} = 1$ for all i, j was studied extensively and Theorem 3.10 and Corollary 3.17 are well known in this case (see, for example, [6; 14, chapter I, section 9; 22]). The algebra $\mathcal{T}_{+,c}(X)$ in this case is isomorphic to $A(\mathbb{D}^k)$ and Corollary 3.17 amounts to the validity of the von Neumann inequality (for doubly commuting k-tuples).

If some of the $\lambda_{i,j}$'s are different from 1, $\mathcal{T}_{+,c}(X)$ is a non commutative subalgebra of $B(l_2(\mathbb{Z}_+^k))$. It is the Banach algebra generated by the isometries $\{S_i : 1 \leq i \leq k\}$ where (writing $\delta_{\mathbf{n}}$ for the function in $l_2(\mathbb{Z}_+^k)$ that is 1 on $\mathbf{n}$ and 0 elsewhere)

$$S_i\delta_{\mathbf{n}} = \lambda(\mathbf{n}, i)\delta_{\mathbf{n}+\mathbf{e}_i} \tag{4.3}$$

where $\lambda(\mathbf{n}, i) = \prod_{j<i} \lambda_{j,i}^{n_j}$. (Note that the isomorphism of $E_i \otimes X(\mathbf{n}) = E_i \otimes E_1^{n_1} \otimes E_2^{n_2} \otimes \cdots \otimes E_k^{n_k}$ and $X(\mathbf{n} + \mathbf{e}_i) = E_1^{n_1} \otimes E_2^{n_2} \otimes \cdots \otimes E_i^{n_i+1} \otimes \cdots \otimes E_k^{n_k}$ sends $1 \otimes 1 \otimes \cdots \otimes 1$ to $\lambda(\mathbf{n}, i)(1 \otimes \cdots \otimes 1)$).

For $\lambda := \{\lambda_{i,j}\}$ as above, we write $\mathcal{T}_{+,c}(\lambda)$ for the algebra $\mathcal{T}_{+,c}(X)$ associated with the product system X defined by λ (generated by the operators S_i defined in (4.3)).

The following Corollary is immediate from Theorem 3.10 and Corollary 3.17. Part (ii) can be viewed as a generalized von Neumann inequality.

Corollary 4.1. *Fix* $\lambda = \{\lambda_{i,j} : |\lambda_{i,j}| = 1,\ \lambda_{j,i} = \lambda_{i,j}^{-1}, \lambda_{i,i} = 1\}$ *and let* $T^{(1)}, T^{(2)}, \ldots, T^{(k)}$ *be contractions in* $B(H)$ *that satisfy (4.1) and (4.2) above. Then*

(i) *there are isometries* $U_1, U_2, \ldots, U_k$ *(in* $B(K)$*, for some Hilbert space* K*) that satisfy (4.1) and (4.2) and form a regular dilation of* $T^{(1)}, T^{(2)}, \ldots$ $T^{(k)}$*; and*

(ii) *there is a completely contractive representation* π *of the algebra* $\mathcal{T}_{+,c}(\lambda)$ *such that* $\pi(S_i) = T^{(i)}$ *for all* $1 \leq i \leq k$. *(Where* S_i *are the operators defined in (4.3)). Thus, for every non commutative polynomial* p *of* k *variables,*

$$\|p(T^{(1)}, \ldots, T^{(k)})\| \leq \|p(S_1, \ldots, S_k)\|.$$

If $dimH = 1$, (4.1) implies (4.2) and we get the following.

Corollary 4.2. *The characters of* $\mathcal{T}_{+,c}(\lambda)$ *(that is, the one dimensional representations of the algebra) are in one-to-one correspondence with the set* $\{t = (t_1, t_2, \ldots, t_k) \in \mathbb{C}^k :\ |t_i| \leq 1 \text{ for all } 1 \leq i \leq k,\ t_it_j = 0 \text{ whenever } \lambda_{i,j} \neq 1\}$.

Now take $k = 2$ and write $P_i = I - S_i S_i^*$. Then we have the following.

Corollary 4.3. *Let $k = 2$ and assume that $\lambda := \lambda_{1,2}$ is not a root of unity. Let J be the ideal of the C^*-algebra $\mathcal{T}_c(X)$ generated by P_1 and P_2. Then $\mathcal{T}_c(X)/J$ is isomorphic to the irrational rotation C^*-algebra A_θ (with $e^{2\pi i\theta} = \lambda$).*

Proof. Write q for the quotient map. Since $S_1 S_2 = \lambda S_2 S_1$, the same relation holds for $q(S_1)$ and $q(S_2)$. But these are unitary operators and, thus, generate A_θ. ■

4.3. The case $E_i = {}_{\alpha_i} A$
Now fix a set of k commuting *-automorphisms α_i, $1 \le i \le k$, of A. We write ${}_{\alpha_i}A$ for the C^*-correspondence over A defined as follows. As a space, it is A. The left and right actions are defined by $\varphi(a)cb = \alpha_i(a)cb$ (for $a, b \in A$ and $c \in_{\alpha_i} A$) and the inner product is $\langle c_1, c_2\rangle = c_1^* c_2$. Now let E_i be ${}_{\alpha_i}A$. Note that, for automorphisms α, β of A, ${}_\alpha A \otimes_\beta A \cong_{\beta\alpha} A$ (via $a \otimes b \mapsto \beta(a)b$). Since we assumed that the automorphisms α_i and α_j commute, we can combine these isomorphisms to get an isomorphism $t_{i,j} :_{\alpha_i} A \otimes_{\alpha_j} A \to_{\alpha_j} A \otimes_{\alpha_i} A$. In fact, $t_{i,j}$ can be written explicitely: $t_{i,j}(a \otimes b) = \alpha_i^{-1}\alpha_j(a) \otimes b$. It is easy to check that condition (3.2) holds and, therefore, this defines a product system X.

Suppose $(\sigma, \{T^{(i)}\})$ is a c.c. representation of X on H with a nondegenerate representation σ of A. fix i and a (positive, contractive) approximate unit $\{u_\lambda\}$ in A and consider, for $b \in A$, $T^{(i)}(u_\lambda)\sigma(b) = T^{(i)}(u_\lambda b)$. Since the operators on the right converge (in norm, to $T^{(i)}(b)$), the net $\{T^{(i)}(u_\lambda)\}$ has a strong operator limit T_i. Then T_i is a contraction and, for $b \in A$, $T^{(i)}(b) = T_i\sigma(b)$. For every $a, b \in A$, $T_i\sigma(\alpha_i(b))\sigma(a) = T^{(i)}(\alpha_i(b)a) = T^{(i)}(\varphi(b)a) = \sigma(b)T^{(i)}(a) = \sigma(b)T_i\sigma(a)$. Thus, for every $b \in A$,

$$T_i\sigma(\alpha_i(b)) = \sigma(b)T_i. \tag{4.4}$$

Now, consider the commutation relation (3.3). Apply the left hand side to $a \otimes b \otimes h \in_{\alpha_i} A \otimes_{\alpha_j} A \otimes H$ to get $\tilde{T}^{(i)}(a \otimes T^{(j)}(b)h) = T^{(i)}(a)T^{(j)}(b)h = T_i\sigma(a)T_j\sigma(b)h = T_iT_j\sigma(\alpha_j(a)b)h$. Applying the right hand side to the same element, we get $\tilde{T}^{(j)}(\alpha_i^{-1}\alpha_j(a) \otimes T^{(i)}(b)h) = T^{(j)}(\alpha_i^{-1}\alpha_j(a))T^{(i)}(b)h = T_j\sigma(\alpha_i^{-1}\alpha_j(a))T_i\sigma(b)h = T_jT_i\sigma(\alpha_j(a)b)h$. Thus the commutation relation is equivalent to $T_iT_j = T_jT_i$ for every i, j. It follows that every representation of X is given by a (non degenerate) representation σ of A on H and by a k-tuple of commuting contractions in $B(H)$ satisfying (4.4).

Now it is easy to check that such a representation is doubly commuting if and only if the k-tuple is doubly commuting; that is, $T_iT_j^* = T_j^*T_i$ for every $i \neq j$.

In order to apply Corollary 3.17, note that, although $\mathcal{L}(X(\mathbf{n})) \neq \mathcal{K}(X(\mathbf{n}))$ whenever A is non unital, the product system X is easily seen to be compactly aligned.

Now, it follows from Corollary 3.17 that, given a representation σ of A on H and a doubly commuting k-tuple of contractions $(T_1, \ldots, T_k)$ satisfying (4.4), there is a completely contractive representation of $\mathcal{T}_{+,c}(X)$ on H sending $L(a)$ to $\sigma(a)$, if $a \in A = X(\mathbf{0})$, and to $T_i\sigma(a)$ if $a \in_{\alpha_i} A = X(\mathbf{e}_i)$.

In order to relate the algebra $\mathcal{T}_{+,c}(X)$ to the analytic crossed product studied in [10], we write $\gamma_i = \alpha_i^{-1}$, $1 \leq i \leq k$, and note that $\gamma_1, \ldots, \gamma_k$ define an action γ of $\mathbb{Z}^k$ on A. The *analytic crossed product* algebra $A \times_\alpha \mathbb{Z}^k_+$ is a subalgebra of the C^* crossed product $A \times_\alpha \mathbb{Z}^k$. The C^* crossed product is defined as the completion of the algebra $\ell^1(\mathbb{Z}^k, A)$ (with product defined by convolution and the involution and C^*-norm are the natural ones). The analytic crossed product is then the Banach subalgebra generated by the functions $\delta_{\mathbf{n},a}$ (for $\mathbf{n} \in \mathbb{Z}^k_+$ and $a \in A$) defined by $\delta_{\mathbf{n},a}(\mathbf{m}) = a$ if $\mathbf{n} = \mathbf{m}$ and 0 otherwise.

For every $a \in A$, define $\sigma(a) = \delta_{\mathbf{0},a}$ and, for $b \in_{\alpha_i} A$, set $T^{(i)}(b) = \delta_{\mathbf{e}_i,\alpha_i^{-1}(b)}$ to get an isometric, doubly commuting, representation of X. It follows from Theorem 3.15 that it yields a C^*-representation π of $\mathcal{T}_c(X)$ into (in fact, onto) $A \times_\gamma \mathbb{Z}^k$. Restricting π to $\mathcal{T}_{+,c}$, we get a completely contractive homomorphism

$$\pi_0 : \mathcal{T}_{+,c}(X) \to A \times_\gamma \mathbb{Z}^k_+.$$

Now let τ be a faithful (nondegenerate) representation of A on a Hilbert space H and write $V = Ind(\tau) \circ L$. By [4, Lemma 5.3], this is an isometric, Nica-covariant (hence, doubly commuting) representation of X on $\mathcal{F}(X) \otimes_\tau H$. Using the results of [10], it induces a completely contractive representation of $A \times_\gamma \mathbb{Z}^k$ on $\mathcal{F}(X) \otimes_\tau H$. Combining it with $Ind(\tau)^{-1}$, we get a completely contractive homomorphism

$$\rho : A \times_\gamma \mathbb{Z}^k \to \mathcal{T}_{+,c}(X).$$

Since ρ and π_0 are the inverse of each other, we conclude

Corollary 4.4. *For the product system X defined by $\alpha_1, \ldots, \alpha_k$ as above, the concrete tensor algebra is completely isometrically isomorphic to the analytic crossed product $A \times_\gamma \mathbb{Z}^k_+$ where γ is the action induced by $\{\gamma_i = \alpha_i^{-1}\}$.*

Remark 4.5. *The reason we need to consider $A \times_\gamma \mathbb{Z}^k_+$ instead of $A \times_\alpha \mathbb{Z}^k_+$ can be seen by comparing our covariance condition (4.4) with the covariance relation (1.2) in [10].*

Finally, note that, in the construction of X associated with $\alpha_1, \alpha_2, \ldots, \alpha_k$ as above, we could also add a "twist" to the multiplication, either by complex numbers (as in Subsection 4.2) or by a family of unitaries in the center of A (satisfying a certain "cocycle" identity that derives from (3.2)).

4.4. The case $A = \mathbb{C}$

Now assume that $A = \mathbb{C}$ and, thus, each E_i (and each $X(\mathbf{m})$) is a Hilbert space. The isomorphisms $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ are given by unitary operators (satisfying the associativity condition (3.2)). For simplicity, we assume here that each E_i is finite dimensional and write d_i for its dimension and $\mathbf{d}$ for $(d_1, \ldots, d_k)$. (Note that the product system is compactly aligned even in the infinite dimensional case). Also, we fix an orthonormal basis $\{e_l^{(i)} : 1 \leq l \leq d_i\}$ for E_i .

Note that the algebra $\mathcal{A}_{\mathbf{d},\theta}$, studied in [7, section 4], is the algebra $\mathcal{T}_{+,c}(X)$ defined in Definition 3.16 if each $t_{i,j}$ is induced from a permutation $\theta_{i,j}$ on $\{1,\ldots,d_i\}\times\{1,\ldots,d_j\}$ in the sense that

$$t_{i,j}(e_l^{(i)}\otimes e_m^{(j)})=e_r^{(j)}\otimes e_s^{(i)} \tag{4.5}$$

whenever $\theta_{i,j}(l,m)=(s,r)$. (And we write θ for the family $\{\theta_{i,j}\}$ of these permutations, noting that it is assumed to satisfy an "associativity" condition that can be derived from (3.2)). In [7, theorem 4.1], the authors studied the one-dimensional representations of the algebra $\mathcal{A}_{\mathbf{d},\theta}$ (that is, its characters). It is shown there that every one dimensional representation of X gives rise to such a character (and vice versa).

For general representations (not necessarily one dimensional) we restrict ourselves to the doubly commuting ones. In order to present the consequences of Theorem 3.10 and Corollary 3.17 to the product system X with $A=\mathbb{C}$, we need the following definitions.

It will be convenient to write $[m]$ $(1\leq m\in\mathbb{Z})$ for the set $\{1,\ldots,m\}$.

Definition 4.6. (i) *A* row contraction *of length n on H is an n-tuple $T=(T_1,\ldots,T_n)$ of operators in $B(H)$ satisfying $\sum_{i=1}^n T_iT_i^*\leq I$. Such a row contraction is a* row isometry *provided each T_i is an isometry.*

(ii) *Let $u=(u_{(i,j)(l,p)})_{(i,j),(l,p)\in[n]\times[m]}$ be a unitary matrix (of size $nm\times nm$), and T and S be row contractions of lengths n and m, respectively, on H. We say that the (ordered) pair (T,S) u-*doubly commutes *if, for all $1\leq i\leq n$ and $1\leq j\leq m$,*

(a) $T_iS_j=\sum_{(p,l)\in[n]\times[m]}u_{(i,j)(p,l)}S_lT_p$, *and*

(b) $S_j^*T_i=\sum_{(p,l)\in[n]\times[m]}u_{(i,l)(p,j)}T_pS_l^*$.

Note that, once an orthonormal basis $\{e_l^{(i)}:1\leq l\leq d_i\}$ is fixed for every E_i, a unitary matrix u of size $d_id_j\times d_id_j$ as in Definition 4.6(ii), defines an isomorphism t from $E_i\otimes E_j$ onto $E_j\otimes E_i$ by

$$t(e_q^{(i)}\otimes e_m^{(j)})=\sum_{(p,l)\in[d_i]\times[d_j]}u_{(q,m)(p,l)}e_l^{(j)}\otimes e_p^{(i)}. \tag{4.6}$$

Theorem 4.7. *Let $\{u^{(i,j)}:i,j\in[k]\}$ be a family of unitary matrices that define (via (4.6)) a family $\{t_{i,j}\}$ of isomorphisms satisfying (3.2) and let $(T^{(1)},\ldots,T^{(k)})$ be a k-tuple of row contractions on H such that, for every $i\neq j$, $(T^{(i)},T^{(j)})$ $u^{(i,j)}$-doubly commutes. Then it has a simultaneous (regular) dilation to a k-tuple $(V^{(1)},\ldots,V^{(k)})$ of row isometries such that, for every $i\neq j$, $(V^{(i)},V^{(j)})$ $u^{(i,j)}$-doubly commutes.*

Proof. The theorem follows immediately from Theorem 3.10 once it is observed that each $T^{(i)}$ defines a c.c. representation of E_i, condition (a) of Definition 4.6(ii) amounts to condition (3.3) (that is, to the fact that the k-tuple defines a representation of X) and condition (b) of Definition 4.6(ii) amounts to the assumption that the representation is doubly commuting. ■

Remark 4.8. *It is easy to check that, if each matrix $u^{(i,j)}$ above is diagonal, condition (3.2) is always satisfied.*

Applying Corollary 3.17 we get.

Corollary 4.9. *Every k-tuple as in Theorem 4.7 defines a completely contractive representation of $\mathcal{T}_{+,c}(X)$.*

Specialising to the situation studied in [7, Section 4], we get the following.

Corollary 4.10. *Suppose $\theta = \{\theta_{i,j}\}$ is a family of permutations as in [7] (defining a product system X via (4.5)) and $(T^{(1)}, \ldots, T^{(k)})$ is a k-tuple of row contractions on H such that, for every $i \neq j$ in $[k]$ and every $(l,m) \in [d_i] \times [d_j]$,*

(a) $T_l^{(i)} T_m^{(j)} = T_s^{(j)} T_r^{(i)}$ *where* $(r,s) = \theta_{i,j}(l,m)$, *and*

(b) $T_m^{(j)*} T_l^{(i)} = \sum_{(r,m)=\theta_{i,j}(l,s)} T_r^{(i)} T_s^{(j)*}$.

Then there is a completely contractive representation π of $\mathcal{A}_{\boldsymbol{d},\theta}$ $(=\mathcal{T}_{+,c}(X))$ on H mapping each $L_{e_l^{(i)}}$ (in the notation of [7], which is $L(e_l^{(i)})$ in the sense of (3.20)) to $T_l^{(i)}$.

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