A CONTRACTING HOMOTOPY FOR BARDZELL'S RESOLUTION

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Abstract

A contracting homotopy for the minimal projective resolution of a path algebra with monomial relations over its enveloping algebra is constructed.

1. Introduction

We consider Bardzell's construction [2] of the minimal projective resolution $\mathbf{P}$ of a monomial algebra A as a left module over its enveloping algebra $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$, and define a right A-linear map c such that $dc + cd = \mathrm{id} - \sigma\varepsilon$, where $\varepsilon : \mathbf{P} \to A$ is the augmentation of the resolution and σ is a right A-linear section of ε. This contracting homotopy has a simple combinatorial definition.

2. Notation

Throughout the paper k will be a field. Let Δ be a finite quiver, with vertex set Δ_0 and arrow set Δ_1. For an arrow a, let $o(a)$ denote its original vertex, and $t(a)$ its terminal vertex. A *path* of length $m \geq 1$ in Δ is a sequence $a_1 a_2 \cdots a_m$ of (not necessarily distinct) arrows in Δ_1, such that $t(a_i) = o(a_{i+1})$ for all $1 \leq i \leq m-1$. The vertices $v \in \Delta_0$ will be regarded as paths of length 0, with $o(v) = t(v) = v$. The set of all paths is denoted by Δ_*. The notation $|u|$ is used for the length of a path u. The maps o and t are now extended to paths by $o(a_1 a_2 \cdots a_m) = o(a_1)$, and $t(a_1 a_2 \cdots a_m) = t(a_m)$. The *path algebra* $k\Delta$ is the vector space spanned by the paths in Δ, where the product of two paths $u \cdot v$ is the concatenated path uv if $t(u) = o(v)$, and 0 otherwise. By a *monomial* we mean the image of a path in the path algebra $k\Delta$. Let u, v be paths in Δ. We say that u is a *factor* of v if there are paths s, t such that $v = sut$; we say that u is a *proper factor* of v if u is a factor of v and $u \neq v$; we say that u is a *left factor* of v if there is a path t such that $v = ut$; lastly we say that u is a *right factor* of v if there is a path s such that $v = su$. Given a quiver Δ we define the opposite quiver Δ^{op} by $\Delta_0^{\mathrm{op}} = \Delta_0$ and $\Delta_1^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in \Delta_1\}$ with $o(a^{\mathrm{op}}) = t(a)$ and $t(a^{\mathrm{op}}) = o(a)$. For a path $w = a_1 \cdots a_n$ we define $w^{\mathrm{op}} = a_n^{\mathrm{op}} \cdots a_1^{\mathrm{op}}$, and for a set of paths V, we let $V^{\mathrm{op}} = \{v^{\mathrm{op}} \mid v \in V\}$.

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3. Bardzell's resolution

Let A be a k-algebra with a presentation $A = k\Delta/(V)$, where (V) is a two-sided ideal generated by a finite set of paths V in Δ. Without loss of generality we will assume that $|v| \geq 2$ for all $v \in V$ and that for no two elements of V is one a proper factor of the other. In contrast to Bardzell, however, we will not assume that A is finite-dimensional. The monomials u that are not members of the ideal (V) form a k-basis for A and we will denote it by $B(A)$.

The generators of the modules in Bardzell's resolution can be described using the concept of *chains* (see Anick [1]). We will define them using a graph from Ufnarovski [3]. Thus, let $\Gamma = (\mathcal{V}, \mathcal{E})$ be the directed graph with vertices as follows:

$$\mathcal{V} = \Delta_0 \cup \Delta_1 \cup \{u \mid u \text{ is a proper right factor of some } v \in V\},$$

and edges $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where

$$\mathcal{E}_1 = \{e \to a \mid e \in \Delta_0, a \in \Delta_1, e = o(a)\},$$

$$\mathcal{E}_2 = \{u \to v \mid t(u) = o(v), uv \in (V), w \notin (V) \text{ for all proper left factors } w \text{ of } uv\}.$$

The set of *right i-chains* now consists of all paths $w \in \Delta_*$ for which there is a directed path $v_{-1} \to v_0 \to \cdots \to v_i$ in Γ with $v_{-1} \in \Delta_0$ and $w = v_{-1}v_0 \cdots v_i$. From the graph it is easy to see that there is no other sequence $u_{-1}, \ldots, u_i$ of paths in Δ such that $u_{-1} \cdots u_i = w$ and $u_{-1} \cdots u_j$ is a j-chain for all $j \leq i$, since that would imply the existence of two edges $v_j \to u_{j+1}$ and $v_j \to v_{j+1}$ in Γ with one of u_{j+1} and v_{j+1} a proper left factor of the other.

We can also define a left version, this could be done by dualising the above construction in the obvious way or, equivalently, by letting the set of *left i-chains* of the quiver Δ with relations V be W^{op}, where W is the set of right i-chains of the quiver Δ^{op} with relations V^{op}.

Interestingly, we get exactly the same paths in this way, (see [2, lemma 3.1]); Bardzell gives it without proof: we will prove it for completeness.

Lemma 1. *A path is a left n-chain if, and only if, it is a right n-chain.*

Proof. The statement is obviously true for $n = -1$, so we assume that $n \geq 0$ in the following. Assume that paths $u_0, \ldots, u_n$ are given such that $u_{n-i} \cdots u_n$ is a left i-chain for all $i = 0, \ldots, n$. We then claim that we can factorise each u_i as $u_i = u_i' u_i''$, where $|u_i'| \geq 1$ in such a way that $u_0' u_0'' \cdots u_{i-1}' u_{i-1}'' u_i'$ is a right i-chain for all i. We will show this by induction on i; the claim is clearly true for $i = 0$, and since $u_i' u_i'' u_{i+1} = u_i u_{i+1}$ has an element of V as a left factor, we can write $u_{i-1}'' u_i' u_i'' u_{i+1} = vrw$, where $r \in V$ and $|w|$ maximal. If we now can show that $u_{i-1}'' u_i' u_i''$ is a proper factor of vr, the claim follows that by choosing u_{i+1}' such that $u_{i-1}'' u_i u_{i+1}' = vr$, but if this was not the case, we would have a proper right factor of $u_{i-1} u_i$ containing an element of V as a factor, contradicting that $u_{i-1} \cdots u_n$ is a left $(n-i+1)$-chain. Since $|u_n| = 1$, $|u_n''| = 0$ and thus $u_0 \cdots u_n$ is a right n-chain. The converse statement that every right n-chain is a left n-chain follows by symmetry. ■

Thus we will simply talk about *chains* instead of left or right chains, and note that for an n-chain w there is a unique sequence $v_n, \ldots, v_{-1}$ of paths in Δ such that $w = v_n \cdots v_{-1}$ and $v_i \cdots v_{-1}$ is an i-chain for all i.

We will use the notation $V^{(i)}$ for the set of i-chains, in particular $V^{(-1)} = \Delta_0$, $V^{(0)} = \Delta_1$, and $V^{(1)} = V$. We write $kV^{(i)}$ for the vector space spanned by the i-chains, which also has a natural $k\Delta_0$-bimodule structure.

If u is an i-chain with $i \geq 0$, we define the *left (right) head*, $\mathrm{head}_{\mathrm{L}}$ ($\mathrm{head}_{\mathrm{R}}$), and *tail*, $\mathrm{tail}_{\mathrm{L}}$, ($\mathrm{tail}_{\mathrm{R}}$), of u by $u = \mathrm{head}_{\mathrm{L}}(u) \cdot \mathrm{tail}_{\mathrm{L}}(u) = \mathrm{tail}_{\mathrm{R}}(u) \cdot \mathrm{head}_{\mathrm{R}}(u)$, where $\mathrm{tail}_{\mathrm{L}}(u)$ and $\mathrm{tail}_{\mathrm{R}}(u)$ are $(i-1)$-chains.

Bardzell [2, theorem 4.1] has shown that the minimal A^{e}-projective resolution of A as a left A^{e}-module is the following:

Let $\mathbf{P}$ be the complex

$$\cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0,$$

where $P_i = A \otimes_{k\Delta_0} kV^{(i-1)} \otimes_{k\Delta_0} A$ and the differential d is defined for an $(i-1)$-chain v by

$$d_i(1 \otimes v \otimes 1) = \begin{cases} \mathrm{head}_{\mathrm{L}}(v) \otimes \mathrm{tail}_{\mathrm{L}}(v) \otimes 1 - 1 \otimes \mathrm{tail}_{\mathrm{R}}(v) \otimes \mathrm{head}_{\mathrm{R}}(v), & i \text{ odd,} \\ \sum_{\substack{v_i \in \Delta_* \\ v_1 v_2 v_3 = v \\ v_2 \in V^{(i-2)}}} v_1 \otimes v_2 \otimes v_3, & i \text{ even.} \end{cases}$$

The augmentation map $\varepsilon : \mathbf{P} \to A$ is defined on elements in degree zero by $\varepsilon(u \otimes e \otimes v) = uev$ and it has a right A-linear section $\sigma : A \to P_0$ defined by $\sigma(u) = 1 \otimes 1 \otimes u$ such that $\varepsilon\sigma = \mathrm{id}$, the augmentation is zero in degrees ≥ 1. The elements $u \otimes v \otimes w$, where $u, w \in B(A)$, $t(u) = o(v)$, $t(v) = o(w)$ and $v \in V^{(i-1)}$ form a k-basis for P_i.

4. The contracting homotopy

We define the $k\Delta_0$–A-bilinear map c on the basis element $u \otimes v \otimes e$, where v is an $(i-1)$-chain and $|e| = 0$ by

$$c(u \otimes v \otimes e) = \sum_{\substack{w_i \in \Delta_* \\ w_1 w_2 w_3 = uv \\ w_2 \in V^{(i)}}} w_1 \otimes w_2 \otimes w_3.$$

Theorem 1. *The map c satisfies the identity*

$$dc + cd = \mathrm{id} - \sigma\varepsilon,$$

and is thus a contracting homotopy of $\mathbf{P}$.

Proof. This follows from Lemmas 7, 9 and 10. ■

The above theorem also gives an alternate proof of the exactness of **P**.

We begin by proving a fundamental fact about chains.

Lemma 2. *If u and v are n-chains, then u is not a proper factor of v.*

PROOF. Suppose that $u = u_{-1} \cdots u_{n-1}u_n$ is a proper factor of $v = v_{-1} \cdots v_{n-1}v_n$ and that n is the smallest such integer; n must then be ≥ 2. Let t_1 and t_2 be paths such that $t_1u_{-1} \cdots u_nt_2 = v_{-1} \cdots v_n$. The minimality of n implies that $v_{-1} \cdots v_j$ is a left factor of $t_1u_{-1} \cdots u_j$ for all $j = 0, \ldots, n-1$, so $u_{n-1}u_n$ is a factor of $v_{n-1}v_n$. From the above it follows that $|t_2| = 0$, since $|t_2| > 0$ would preclude the existence of the edge $v_{n-1} \to v_n$ in the graph Γ. A symmetric argument, using Lemma 1, shows that $|t_1| = 0$ as well. ■

We will in the following use two technical lemmas [2, lemmas 3.2 and 3.3], which Bardzell uses to show that **P** is a complex, so we state them here.

Lemma 3. *If u_1u_2 and u_2u_3 are $(2i+1)$-chains, with $|u_2| \geq 1$, then there is a $(2i+2)$-chain v that is a factor of $u_1u_2u_3$.*

Lemma 4. *Let u be a $2i$-chain, then the set of factorisations $u = u_1u_2u_3$ with u_2 a $(2i-1)$-chain has exactly two elements, one with $|u_1| = 0$ and one with $|u_3| = 0$.*

Now we are able to prove the needed properties of the map c.

Lemma 5. *Let $u \otimes v \otimes e$ be a basis element of P_{2i+1} with $|e| = 0$ and $i \geq 0$. Then*

$$c(u \otimes v \otimes e) = \begin{cases} u_1 \otimes u_2v \otimes e, & \textit{if there is a factorisation } u = u_1u_2 \\ & \textit{such that } u_2v \in V^{(2i+1)}, \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. Assume that $c(u \otimes v \otimes e) \neq 0$, which means that there is a basis element $u_1 \otimes u_2v_1 \otimes v_2$ in P_{2i+2}, where $u = u_1u_2$, $v = v_1v_2$.

Consider $w = \mathrm{tail}_{\mathrm{L}}^2(u_2v_1) = \mathrm{tail}_{\mathrm{L}}(\mathrm{tail}_{\mathrm{L}}(u_2v_1))$; since u and therefore u_2 does not contain a relation, we get that w is a factor of v_1 and thus of v. It can not occur as a left factor of v, since u_2 does not contain a relation; thus by Lemma 4 we can conclude that it occurs as a right factor of v. From this it follows that $v_1 = v$ and $v_2 = e$, so the statement is proven. ■

Lemma 6. *Let u and v be paths in Δ with v a $2i$-chain. If there are factorisations $u = u_1u_2$ and $v = v_1v_2$ such that u_2v_1 is a $2i$-chain and $|v_2| \geq 1$, then* $\mathrm{head}_{\mathrm{R}}(v)$ *is a factor of v_2.*

PROOF. Assume that $\mathrm{head}_{\mathrm{R}}(v)$ is a not factor of v_2, then v_2 is a proper factor of $\mathrm{head}_{\mathrm{R}}(v)$, which implies that $\mathrm{tail}_{\mathrm{R}}(v)$ is a proper factor of v_1 and thus of u_2v_1. From Lemma 4 it must be either a right factor, which would imply that $v_2 = \mathrm{head}_{\mathrm{R}}(v)$, or it is a left factor, which implies that $v_1 = v$ and $v_2 = e$. Both cases contradict our assumption, so we can conclude that $\mathrm{head}_{\mathrm{R}}(v)$ is a factor of v_2. ∎

Let us now prove the theorem for the case of elements of degree 0.

Lemma 7. *Let $u \otimes e \otimes e$ be a basis element of P_0 with $|e| = 0$. Then*

$$(dc + cd)(u \otimes e \otimes e) = (\mathrm{id} - \sigma\varepsilon)(u \otimes e \otimes e).$$

PROOF. Let $u = a_1 \cdots a_n$ with $a_i \in \Delta_1$ and let $f = o(u)$, then $(dc+cd)(u \otimes e \otimes e) = dc(u \otimes e \otimes e) = d(\sum_{i=1}^{n} a_1 \cdots a_{i-1} \otimes a_i \otimes a_{i+1} \cdots a_n) = u \otimes e \otimes e - f \otimes f \otimes u = u \otimes e \otimes e - \sigma\varepsilon(u \otimes e \otimes e)$. ∎

Next we will prove the theorem for elements of odd degree. We start with a preliminary lemma.

Lemma 8. *Let u, v be paths in Δ with $u \in B(A)$ and $v \in V^{(2i+1)}$. If w_1, w_2, w_3 are paths with $uv = w_1 w_2 w_3$, where $w_1, w_3 \in B(A)$ and $w_2 \in V^{(2i)}$, then $w_2 w_3$ is a factor of v.*

PROOF. Our first claim is that $\mathrm{tail}_{\mathrm{L}}(w_2)w_3$ must be a right factor of v. To see this we write $v = \mathrm{tail}^2_{\mathrm{R}}(v)v'$, then $\mathrm{tail}^2_{\mathrm{R}}(v)$ is a $(2i-1)$-chain as is $\mathrm{tail}_{\mathrm{L}}(w_2)$. From Lemma 6 we thus find that if $\mathrm{tail}_{\mathrm{L}}(w_2)w_3$ is not a right factor of v, then v' is a factor of w_3 and hence $w_3 \in (V)$, which contradicts the assumption that $w_3 \in B(A)$, thus $\mathrm{tail}_{\mathrm{L}}(w_2)w_3$ is indeed a right factor of v.

Next, assume that $\mathrm{head}_{\mathrm{R}}(v)$ is a right factor of w_3; then $\mathrm{tail}_{\mathrm{L}}(w_2)$ is a factor of $\mathrm{tail}_{\mathrm{R}}(v)$. It cannot occur as a left factor since then we would have $w_3 = u$ and again $w_3 \notin B(A)$.

Thus we may conclude that either $\mathrm{head}_{\mathrm{R}}(v)$ is not a right factor of w_3, or $\mathrm{tail}_{\mathrm{L}}(w_2)$ occurs as a right factor of $\mathrm{tail}_{\mathrm{R}}(v)$. In both these cases we can see that $w_2 w_3$ is a right factor of v. ∎

Lemma 9. *Let $u \otimes v \otimes e$ be a basis element of P_{2i+1} with $|e| = 0$ and $i \geq 0$. Then*

$$(dc + cd)(u \otimes v \otimes e) = u \otimes v \otimes e.$$

PROOF. We begin by considering the case where there is a factorisation $u = u'u''$ such that $u''v \in V^{(2i+1)}$. In this case, using Lemmas 5 and 8, we see that

$$dc(u \otimes v \otimes e) = \sum u_1 \otimes u_2 v_1 \otimes v_2, \tag{4.1}$$

where the sum runs over all factorisations $u = u_1 u_2$, $v = v_1 v_2$ such that $u_2 v_1 \in V^{(2i)}$. It is easy to see that all the terms in the sum above actually are nonzero.

On the other hand, when we consider the application of d we see that $u\,\mathrm{head}_{\mathrm{L}}(v) \otimes \mathrm{tail}_{\mathrm{L}}(v) \otimes e = 0$, so

$$cd(u \otimes v \otimes e) = -c(u \otimes \mathrm{tail}_{\mathrm{R}}(v) \otimes \mathrm{head}_{\mathrm{R}}(v)) = -\sum u_1 \otimes u_2 v_1 \otimes v_2, \tag{4.2}$$

where the sum runs over all factorisations $u = u_1 u_2$, $v = v_1 v_2$ such that $u_2 v_1 \in V^{(2i)}$ and $\mathrm{head}_{\mathrm{R}}(v)$ is a right factor of v_2. By Lemma 6 we then see that the terms from (4.1) and from (4.2) all cancel, except $u \otimes v \otimes e$, which proves the case.

We then turn to the case where there is no factorisation $u = u_1 u_2$ such that $u_2 v \in V^{(2i+1)}$. In this case we directly get via Lemma 5 that $dc(u \otimes v \otimes e) = 0$, so what we have to prove is that $cd(u \otimes v \otimes e) = u \otimes v \otimes e$.

Now,

$$\begin{aligned} cd(u \otimes v \otimes e) &= c(u\,\mathrm{head}_{\mathrm{L}}(v) \otimes \mathrm{tail}_{\mathrm{L}}(v) \otimes e) - c(u \otimes \mathrm{tail}_{\mathrm{R}}(v) \otimes \mathrm{head}_{\mathrm{R}}(v)) \\ &= \sum u_1 \otimes u_2 v_1 \otimes v_2, \end{aligned} \tag{4.3}$$

where the sum runs over all factorisations $u = u_1 u_2$, $v = v_1 v_2$ such that $u_2 v_1 \in V^{(2i)}$ and v_2 is a proper right factor of $\mathrm{head}_{\mathrm{R}}(v)$. The last condition and Lemma 6 imply that $v_2 = e$ so the lemma is proven in this case as well. ■

Let us now turn to the case of elements in even degrees ≥ 2.

Lemma 10. *Let $u \otimes v \otimes e$ be a basis element of P_{2i+2}, with $|e| = 0$ and $i \geq 0$. Then*

$$(dc + cd)(u \otimes v \otimes e) = u \otimes v \otimes e.$$

Proof. We begin the proof by observing that, if $uv = w_1 w_2 w_3$, where $w_2 \in V^{(2i+1)}$, and $w_3 \in (V)$, then by Lemma 2 $v = \mathrm{tail}^2_{\mathrm{R}}(v) v'$ must be a right factor of $\mathrm{tail}^2_{\mathrm{L}}(w_2) w_3$. This means that there is an element of V that is a factor of u, which contradicts the assumption that $u \in B(A)$. This means that $w_1 \otimes w_2 \otimes w_3 \neq 0$ whenever $w_2 \in V^{(2i+1)}$.

First we assume that we can write $uv = w_1 w_2 w_3$, where $w_2 \in V^{(2i+1)}$ and $|w_3| \geq 1$. We will begin by proving the lemma in this case, and return to the case where there is no such factorisation later.

Now consider the following set of factorisations of uv:

$$F = \{(w_1, w_2, w_3) \mid w_i \in \Delta_*,\ w_1 w_2 w_3 = uv, w_2 \in V^{(2i+1)}\}.$$

Write $F = \{(w_{1,1}, w_{1,2}, w_{1,3}), \ldots, (w_{k,1}, w_{k,2}, w_{k,3})\}$ where $|w_{1,3}| < |w_{2,3}| < \cdots < |w_{k,3}|$. By Lemma 3 there is for each $i = 1, \ldots, k-1$ a factorisation $w_{i,1} = w'_{i,1} w''_{i,1}$ such that $w''_{i,1} w_{i,2}$ is a $(2i{+}2)$-chain with $\mathrm{tail}_{\mathrm{L}}(w''_{i,1} w_{i,2}) = w_{i,2}$ and $\mathrm{tail}_{\mathrm{R}}(w''_{i,1} w_{i,2}) = w_{i+1,2}$. Now we get

$$c(u \otimes v \otimes e) = \sum_{i=1}^{k-1} w'_{i,1} \otimes w''_{i,1} w_{i,2} \otimes w_{i,3}, \tag{4.4}$$

since, by Lemma 3, there can be no factorisations $uv = w'_1 w'_2 w'_3$ with $w'_2 \in V^{(2i+1)}$ other than the ones present in the sum. Thus

$$dc(u \otimes v \otimes e) = u \otimes v \otimes e - w_{k,1} \otimes w_{k,2} \otimes w_{k,3}. \tag{4.5}$$

We now turn to $cd(u \otimes v \otimes e)$. If $w_{i,1} \otimes w_{i,2} \otimes w_{i,3}$ should appear as a term in $cd(u \otimes v \otimes e)$, it would have to be as $c(w_{i,1} \operatorname{head}_{\mathrm{L}}(w_{i,2}) \otimes \operatorname{tail}_{\mathrm{L}}(w_{i,2}) \otimes w_{i,3})$ by Lemma 5. For $i = 1, \ldots, k-1$ we have by Lemma 3 that $w_{i,2}$ is the right factor of a $(2i+2)$-chain, which in turn is a right factor of $w_{i,1}w_{i,2}$. This implies that $w_{i,1} \operatorname{head}_{\mathrm{L}}(w_{i,2})$ contains an element of V as a factor and thus is zero in A. Thus the only term that can occur in $cd(u \otimes v \otimes e)$ is $w_{k,1} \otimes w_{k,2} \otimes w_{k,3}$, and it will indeed occur by our initial observation.

So far we have proven the lemma in the first case, and we turn to the second case; when there is no factorisation $uv = w_1 w_2 w_3$ with $w_2 \in V^{(2i+1)}$ and $|w_3| \geq 1$.

In this case it is easy to see that $c(u \otimes v \otimes e) = 0$, and that $cd(u \otimes v \otimes e) = u \otimes v \otimes e$ which then proves the second case of the lemma. ■

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