APPROXIMATION BY WEIGHTED COMPOSITION OPERATORS ON $C(X)$

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Abstract

Given a pair of compact Hausdorff spaces X and Y, this article centers around the approximation of arbitrary continuous linear mappings from $C(X)$ into $C(Y)$ by weighted composition operators. Optimal results are obtained for compact operators and also for positive weakly compact operators.

1. Introduction

A continuous linear mapping $\varphi : A \to B$ between two complex Banach algebras A and B is said to be *disjointness preserving* or *separating* provided that $\varphi(f)\varphi(g) = 0$ for all $f, g \in A$ with $fg = 0$. The following definition is due to Dolinar [3] and provides a quantitative extension of this notion. Given an arbitrary $\varepsilon \geq 0$, the mapping $\varphi : A \to B$ is said to be ε-*disjointness preserving* if

$$\|\varphi(f)\varphi(g)\| \leq \varepsilon\|f\|\|g\| \qquad \text{for all } f, g \in A \text{ with } fg = 0.$$

Evidently, φ is ε-disjointness preserving for any $\varepsilon \geq \|\varphi\|^2$, but much more can be said, as will be seen below. Let $\varepsilon(\varphi)$ denote the infimum of the set of all $\varepsilon \geq 0$ for which φ is ε-disjointness preserving. Clearly, $\varepsilon(\varphi)$ is the smallest ε for which φ is ε-disjointness preserving, and the identity $\varepsilon(\varphi) = 0$ holds precisely when φ is disjointness preserving.

A natural problem is the approximation of ε-disjointness preserving operators by disjointness preserving operators. More precisely, for arbitrary $\delta > 0$, one wants to find an $\varepsilon > 0$ such that, for each bounded linear operator $\varphi : A \to B$ with

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$\varepsilon(\varphi) \leq \varepsilon$, there exists a disjointness preserving operator $\psi : A \to B$ for which $\|\varphi - \psi\| \leq \delta$. The investigation of this problem was initiated by Dolinar [3] and is in the spirit of the work of Johnson [8; 9] on the approximation of approximately multiplicative linear mappings by algebra homomorphisms.

In this article, we consider ε-disjointness preserving mappings between the Banach algebras $C(X)$ and $C(Y)$ of all continuous complex-valued functions on the compact Hausdorff spaces X and Y, respectively. For this case, it is well known that the disjointness preserving operators are precisely the weighted composition operators. Thus our main goal is the approximation of certain continuous linear mappings in terms of weighted composition operators. Such operators have a long history in functional analysis, for instance in connection with the Banach–Stone theorem, the representation of isometries between certain function spaces, and the description of the extreme points in certain convex sets of bounded linear operators from $C(X)$ into $C(Y)$; see [1].

In Section 2, we first consider the case of ε-disjointness preserving linear functionals on $C(X)$. This will then allow us to handle the general case of operators from $C(X)$ into $C(Y)$ by a canonical pointwise approach in Section 3. Some of our techniques are inspired by the work of Dolinar [3] and Johnson [8], but our choice of approximation is different from the one considered in [3] and leads to considerably better error estimates. In fact, our approach preserves positivity and provides the best approximation in the case of functionals, compact operators, and positive weakly compact operators.

2. Approximation of functionals

Let X be a compact Hausdorff space, let Σ denote the σ-algebra of all Borel subsets of X, and let $M(X)$ stand for the space of all regular complex Borel measures on Σ. Also, let $\|\cdot\|_\infty$ denote the supremum norm on $C(X)$.

Throughout this section, we consider a continuous linear functional $\varphi : C(X) \to \mathbb{C}$ and the corresponding representing measure $\lambda \in M(X)$ provided by the Riesz representation theorem. Thus

$$\varphi(f) = \int_X f \,\mathrm{d}\lambda \qquad \text{for all } f \in C(X)$$

and $\|\varphi\| = |\lambda|(X)$, where $|\lambda|$ denotes the total variation of λ. The following result provides a useful formula for $\varepsilon(\varphi)$ in terms of the representing measure λ.

Proposition 2.1. $\varepsilon(\varphi) = \gamma(\varphi)\left(\|\varphi\| - \gamma(\varphi)\right)$, *where*

$$\gamma(\varphi) := \sup\left\{|\lambda|(A) : A \in \Sigma \text{ with } |\lambda|(A) \leq \|\varphi\|/2\right\}.$$

Moreover, $\varepsilon(\varphi) \leq \|\varphi\|^2/4$, *and equality holds precisely when* $\gamma(\varphi) = \|\varphi\|/2$.

Proof. To see that φ is ε-disjointness preserving for $\varepsilon = \gamma(\varphi)\left(\|\varphi\| - \gamma(\varphi)\right)$, let $f, g \in C(X)$ be given such that $fg = 0$. Since the open sets $A := \{x \in X : f(x) \neq 0\}$ and $B := \{x \in X : g(x) \neq 0\}$ are disjoint, we conclude that $|\lambda|(A) \leq |\lambda|(X)/2$ or

$|\lambda|(B) \le |\lambda|(X)/2$. Without loss of generality, we may assume that the first case occurs; this ensures that $|\lambda|(A) \le \gamma(\varphi)$. We then estimate that

$$\begin{aligned}|\varphi(f)\varphi(g)| &\le \left(\int_A |f|\,\mathrm{d}|\lambda|\right)\left(\int_B |g|\,\mathrm{d}|\lambda|\right)\\ &\le |\lambda|(A)\,|\lambda|(B)\,\|f\|_\infty\,\|g\|_\infty\\ &\le |\lambda|(A)\,|\lambda|(X\setminus A)\,\|f\|_\infty\,\|g\|_\infty\\ &= |\lambda|(A)\,(|\lambda|(X) - |\lambda|(A))\,\|f\|_\infty\,\|g\|_\infty\,.\end{aligned}$$

Now observe that, for arbitrary $c \ge 0$, the quadratic polynomial p given by $p(t) := t\,(c-t)$ is strictly increasing on $(-\infty, c/2]$, that p attains its global maximum on $(-\infty,\infty)$ at the point $c/2$, and that $p(c/2) = c^2/4$. With the choice $c := |\lambda|(X) = \|\varphi\|$, we infer that

$$\begin{aligned}|\lambda|(A)\,(|\lambda|(X) - |\lambda|(A)) &\le \gamma(\varphi)\,(|\lambda|(X) - \gamma(\varphi))\\ &= \gamma(\varphi)\,(\|\varphi\| - \gamma(\varphi))\\ &\le \|\varphi\|^2/4.\end{aligned}$$

In particular, it follows that φ is ε-disjointness preserving for $\varepsilon = \gamma(\varphi)\,(\|\varphi\| - \gamma(\varphi))$.

We next establish that every $\varepsilon \ge 0$ for which φ is ε-disjointness preserving necessarily satisfies $\varepsilon \ge \gamma(\varphi)\,(\|\varphi\| - \gamma(\varphi))$. By the preceding discussion of the polynomial p, it suffices to show that

$$\varepsilon \ge |\lambda|(A)\,(|\lambda|(X) - |\lambda|(A))$$

for each $A \in \Sigma$ with $|\lambda|(A) \le |\lambda|(X)/2$. Since the desired estimate is trivial when $|\lambda|(A) = 0$, we may assume, in addition, that $|\lambda|(A) > 0$. Then, given an arbitrary δ with $0 < \delta < |\lambda|(A)$, we choose, by the regularity of the measure $|\lambda|$, a compact set $K \subseteq A$ such that $|\lambda|(K) \ge |\lambda|(A) - \delta$ and a compact set $L \subseteq X \setminus A$ such that $|\lambda|(L) \ge |\lambda|(X\setminus A) - \delta$. Since K and L are disjoint and the compact Hausdorff space X is normal, there exist open sets $U, V \subseteq X$ for which $K \subseteq U$, $L \subseteq V$, and $U \cap V = \emptyset$. Finally, Urysohn's lemma provides a function $f \in C(X)$ with $0 \le f \le 1$ on X, $f \equiv 1$ on K, and $\operatorname{supp} f \subseteq U$ and, similarly, a function $g \in C(X)$ with $0 \le g \le 1$ on X, $g \equiv 1$ on L, and $\operatorname{supp} g \subseteq V$. Note that $\|f\|_\infty = \|g\|_\infty = 1$, because our conditions on A and δ ensure that both K and L are non-empty. Moreover, $fg = 0$, since, by construction, the supports of f and g are disjoint. Finally, by the polar decomposition of complex measures, a standard consequence of the Radon–Nikodym theorem, there exists a Borel function w on X for which $\mathrm{d}\lambda = w\,\mathrm{d}|\lambda|$ and $|w| \equiv 1$ on X; see theorem 6.12 of [11]. Since φ is ε-disjointness preserving, we conclude that

$$\begin{aligned}\varepsilon \geq |\varphi(f\,\overline{w})\,\varphi(g\,\overline{w})| &= \left(\int_X f\,\overline{w}\,\mathrm{d}\lambda\right)\left(\int_X g\,\overline{w}\,\mathrm{d}\lambda\right)\\ &= \left(\int_X f\,\mathrm{d}|\lambda|\right)\left(\int_X g\,\mathrm{d}|\lambda|\right) \geq \left(\int_K f\,\mathrm{d}|\lambda|\right)\left(\int_L g\,\mathrm{d}|\lambda|\right)\\ &= |\lambda|\,(K)\,|\lambda|\,(L) \geq (|\lambda|\,(A)-\delta)\,(|\lambda|\,(X\setminus A)-\delta)\,.\end{aligned}$$

Passing now to the limit as $\delta \to 0$, we obtain that $\varepsilon \geq |\lambda|\,(A)\,|\lambda|\,(X\setminus A)$, the desired estimate. This completes the proof of the formula $\varepsilon(\varphi) = \gamma(\varphi)\,(\|\varphi\| - \gamma(\varphi))$. The final statements are now clear from the properties of the polynomial p discussed above. ■

In the estimate $0 \leq \varepsilon(\varphi) \leq \|\varphi\|^2/4$, the two extreme cases when $\varepsilon(\varphi) = 0$ and when $\varepsilon(\varphi) = \|\varphi\|^2/4$ are of particular interest. As a first application of the preceding result, we are able to present a short new proof of a well-known characterization of the disjointness preserving linear functionals on $C(X)$; see, for instance, [7]. We also obtain several simple classes of examples for which $\varepsilon(\varphi) = \|\varphi\|^2/4$. As usual, the Dirac measure at a point $x \in X$ will be denoted by δ_x.

Corollary 2.2. *The following assertions hold.*

(a) *The functional φ is disjointness preserving precisely when $\lambda = \alpha\,\delta_x$ for some $\alpha \in \mathbb{C}$ and $x \in X$.*

(b) *If the measure λ is continuous, in the sense that $\lambda(\{x\}) = 0$ for all $x \in X$, then $\varepsilon(\varphi) = \|\varphi\|^2/4$.*

(c) *If the measure λ is discrete and has finite support, so that $\lambda = \alpha_1\,\delta_{x_1} + \cdots + \alpha_n\,\delta_{x_n}$ with finitely many distinct points $x_1, \ldots, x_n \in X$ and arbitrary scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, then the identity $\varepsilon(\varphi) = \|\varphi\|^2/4$ holds precisely when there exists a subset J of $\{1, \ldots, n\}$ for which*

$$\sum_{j\in J} |\alpha_j| = \frac{1}{2}\sum_{j=1}^{n} |\alpha_j|\,.$$

In particular, if λ is the sum of an even number of Dirac measures on X, then $\varepsilon(\varphi) = \|\varphi\|^2/4$.

(d) *If X contains at least two points, then, for every $\hat{\varepsilon} \in [0, 1/4]$, there exists some $\hat{\varepsilon}$-disjointness preserving functional φ on $C(X)$ for which $\|\varphi\| = 1$ and $\varepsilon(\varphi) = \hat{\varepsilon}$.*

Proof. (a) By Proposition 2.1, the case $\varepsilon(\varphi) = 0$ occurs if and only if $\gamma(\varphi) = 0$, which means precisely that $|\lambda|$ attains only the values 0 and $|\lambda|\,(X)$. Clearly, all multiples of point masses satisfy this condition. Conversely, we may suppose that λ has the property that $|\lambda|\,(A) \in \{0, 1\}$ for all $A \in \Sigma$, and we consider the following two cases. If $|\lambda|\,(\{x\}) = 0$ for all $x \in X$, then we may choose, by regularity, for each $x \in X$, an open neighbourhood U_x of x such that $|\lambda|\,(U_x) < 1$. By the condition on λ, this implies that actually $|\lambda|(U_x) = 0$ for all $x \in X$. Since the compact Hausdorff space X may be covered by a finite union of the sets U_x, we conclude that $|\lambda| = 0$ and therefore that $\lambda = 0$. Thus, in this case, λ is the zero multiple of any Dirac measure on X. In the remaining case, there exists some $x \in X$ for which $|\lambda|(\{x\}) > 0$ and hence $|\lambda|(\{x\}) = 1$, again by the condition on λ. Thus $|\lambda| = \delta_x$ and consequently $\lambda = \alpha\,\delta_x$ for some $\alpha \in \mathbb{C}$.

(b) By a standard property of continuous measures, there exists some $A \in \Sigma$ such that $|\lambda|(A) = |\lambda|(X)/2 = \|\varphi\|/2$; see [5, section 11.44]. Thus $\gamma(\varphi) = \|\varphi\|/2$, by the definition of γ, and therefore $\varepsilon(\varphi) = \|\varphi\|^2/4$, by the last assertion of Proposition 2.1.

(c) Because $|\lambda| = |\alpha_1|\,\delta_{x_1} + \cdots + |\alpha_n|\,\delta_{x_n}$, the statement is again immediate from the final claim of the preceding result.

(d) We fix distinct points $u, v \in X$ and define $\lambda_t := t\,\delta_u + (1-t)\,\delta_v$ for arbitrary $0 \le t \le 1/2$. Clearly, the corresponding linear functional φ_t satisfies $\gamma(\varphi_t) = t$. Thus $\{\varepsilon(\varphi_t) : t \in [0, 1/2]\} = p([0, 1/2]) = [0, 1/4]$, where, as above, $p(t) = t(1-t)$. ■

Proposition 2.3. *There exists a point $x \in X$ such that*

$$|\lambda(\{x\})| \ge \sqrt{\|\varphi\|^2 - 4\,\varepsilon(\varphi)}. \tag{2.1}$$

Moreover, if $\varepsilon(\varphi) < (2/9)\,\|\varphi\|^2$, then there exists a unique point $x \in X$ for which

$$|\lambda(\{x\})| = \left(\|\varphi\| + \sqrt{\|\varphi\|^2 - 4\,\varepsilon(\varphi)}\right)/2 > 2\,|\lambda|(X)/3. \tag{2.2}$$

PROOF. Without loss of generality, we may assume that λ is not the zero measure, so that $\|\varphi\| = |\lambda|(X) > 0$. Then $\mu := |\lambda| / |\lambda|(X)$ is a probability measure on X, and the corresponding linear functional ψ on $C(X)$ satisfies $\gamma(\varphi) = \|\varphi\|\,\gamma(\psi)$. By Proposition 2.1, it follows that

$$\varepsilon(\varphi) = \gamma(\varphi)\,(\|\varphi\| - \gamma(\varphi)) = \|\varphi\|^2\,\varepsilon(\psi).$$

Hence it suffices to prove the proposition in the case that λ is a probability measure.

To establish the first assertion in this case, let $\delta := \sqrt{1 - 4\,\varepsilon(\varphi)}$, and assume that $\lambda(\{x\}) < \delta$ for all $x \in X$. Then, by the regularity of λ, for each $x \in X$ there exists an open neighbourhood U_x of x such that $\lambda(U_x) < \delta$. By the compactness of X, we obtain finitely many points $x_1, \ldots, x_n \in X$ such that $X = U_{x_1} \cup \cdots \cup U_{x_n}$. Let $V_0 := \emptyset$ and $V_j := U_{x_1} \cup \cdots \cup U_{x_j}$ for $j = 1, \ldots, n$. Clearly, $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = X$ and hence $\lambda(V_n) = 1$. Let j be the largest integer in $\{0, 1, \ldots, n-1\}$ for which $\lambda(V_j) \le (1-\delta)/2$. Then we conclude that

$$(1-\delta)/2 < \lambda(V_{j+1}) = \lambda(V_j \cup U_{x_{j+1}}) \le \lambda(V_j) + \lambda(U_{x_{j+1}}) < (1-\delta)/2 + \delta = (1+\delta)/2.$$

Choosing either V_{j+1} or $X \setminus V_{j+1}$, we obtain a set $A \in \Sigma$ for which $(1-\delta)/2 < \lambda(A) \le 1/2$. Thus $\gamma(\varphi) > (1-\delta)/2$. Since the polynomial p given by $p(t) := t(1-t)$

strictly increases on $[0,1/2]$, we conclude from Proposition 2.1 and the definition of δ that

$$\varepsilon(\varphi)=\gamma(\varphi)\,(1-\gamma(\varphi))>\frac{1-\delta}{2}\left(1-\frac{1-\delta}{2}\right)=\frac{1-\delta^2}{4}=\varepsilon(\varphi),$$

the desired contradiction. Hence, for some $x\in X$, condition (2.1) holds for the probability measure λ.

To prove the second claim, suppose now that λ is a probability measure for which $\varepsilon(\varphi)<2/9$, and consider a point $x\in X$ for which $\lambda(\{x\})\geq\sqrt{1-4\,\varepsilon(\varphi)}$. If $\lambda(\{x\})\leq 1/2$, then clearly $\varepsilon(\varphi)>0$. Also, $\lambda(\{x\})\leq\gamma(\varphi)$ and hence $\sqrt{1-4\,\varepsilon(\varphi)}\leq\gamma(\varphi)$. Consequently, $p\left(\sqrt{1-4\,\varepsilon(\varphi)}\right)\leq p\,(\gamma(\varphi))$ and therefore

$$\sqrt{1-4\,\varepsilon(\varphi)}\left(1-\sqrt{1-4\,\varepsilon(\varphi)}\right)\leq\gamma(\varphi)\,(1-\gamma(\varphi))=\varepsilon(\varphi),$$

again by Proposition 2.1. But an elementary computation shows that this estimate for $\varepsilon(\varphi)$ is equivalent to $\varepsilon(\varphi)\geq 2/9$ and hence contradicts our current condition on $\varepsilon(\varphi)$. Thus $\lambda(\{x\})>1/2$. But then a set $A\in\Sigma$ satisfies $\lambda\,(A)\leq 1/2$ if and only if $A\subseteq X\setminus\{x\}$. This entails that

$$1-\lambda(\{x\})=\lambda\,(X\setminus\{x\})=\gamma(\varphi),$$

by the definition of $\gamma(\varphi)$, and therefore

$$\lambda(\{x\})=1-\gamma(\varphi)=\left(1+\sqrt{1-4\,\varepsilon(\varphi)}\right)/2>2/3,$$

because $\varepsilon(\varphi)=\gamma(\varphi)\,(1-\gamma(\varphi))$ and $\varepsilon(\varphi)<2/9<1/4$. Thus x satisfies condition (2.2) for the probability measure λ. Since $\lambda(\{x\})>1/2$, it is clear that x is uniquely determined by this condition. ■

Note that the existence of some $x\in X$ for which (2.1) holds provides an alternative proof of the fact that $\varepsilon(\varphi)=\|\varphi\|^2/4$ whenever λ is a continuous measure. We also obtain yet another proof of the fact that each disjointness preserving continuous linear functional on $C(X)$ is represented by a multiple of a Dirac measure. Evidently, estimate (2.2) is sharper than estimate (2.1).

Simple examples show that the condition $\varepsilon(\varphi)<(2/9)\,\|\varphi\|^2$ is essential for the conclusion of the last part of Proposition 2.3. Indeed, for the probability measure $\lambda:=1/3\,(\delta_u+\delta_v+\delta_w)$ with three distinct points $u,v,w\in X$, the corresponding functional φ satisfies $\gamma(\varphi)=1/3$ and hence $\varepsilon(\varphi)=2/9$, but there is no point $x\in X$ for which $\lambda(\{x\})>2/3$.

Lemma 2.4. *There exists some $x\in X$ for which*

$$|\lambda(\{x\})|\geq|\lambda(\{u\})|\qquad\text{for all } u\in X. \tag{2.3}$$

Moreover, if ψ denotes the continuous linear functional on $C(X)$ that is represented *by the measure $\lambda(\{x\})\,\delta_x$ for a point x with property* (2.3)*, then ψ is a best disjointness preserving linear approximation of φ and satisfies*

$$\|\varphi-\psi\| = |\lambda|\,(X) - |\lambda(\{x\})| = |\lambda|\,(X\setminus\{x\}).$$

Conversely, any best disjointness preserving linear approximation of φ is represented by a measure of the form $\lambda(\{x\})\,\delta_x$ for some $x\in X$ with property (2.3).

PROOF. To establish the first claim, without loss of generality we may assume that λ is not a continuous measure. Then $s := \sup\{|\lambda(\{u\})| : u\in X\}$ is strictly positive. If $s\neq|\lambda(\{x\})|$ for all $x\in X$, then there exists a sequence of points $x_n\in X$ for which the sequence of numbers $|\lambda(\{x_n\})|$ strictly increases to s. But then

$$\sum_{n=1}^{\infty}|\lambda(\{x_n\})| = \sum_{n=1}^{\infty}|\lambda|\,(\{x_n\}) \le |\lambda|\,(X) < \infty$$

and hence $|\lambda(\{x_n\})|\to 0$ as $n\to\infty$, a contradiction to $s>0$. Consequently, there exists some $x\in X$ that satisfies condition (2.3).

Now, given an arbitrary point $x\in X$, we consider, for each $\alpha\in\mathbb{C}$, the linear functional ψ_α on $C(X)$ that is represented by $\alpha\,\delta_x$. Clearly, $\lambda-\alpha\,\delta_x = \mu+\nu$ with the choice $\mu := \lambda-\lambda(\{x\})\delta_x$ and $\nu := (\lambda(\{x\})-\alpha)\,\delta_x$. Since the measures μ and ν are mutually singular, it follows from the definition of the total variation that $|\lambda-\alpha\,\delta_x| = |\mu+\nu| = |\mu|+|\nu|$ and therefore

$$\|\varphi-\psi_\alpha\| = |\lambda-\alpha\,\delta_x|\,(X) = |\mu|\,(X)+|\nu|\,(X) = |\lambda|\,(X\setminus\{x\}) + |\lambda(\{x\})-\alpha|\,.$$

Hence, for fixed $x\in X$, the norm of $\varphi-\psi_\alpha$ is minimal exactly when $\alpha=\lambda(\{x\})$, and the smallest value of these norms is $|\lambda|\,(X)-|\lambda(\{x\})|$. Evidently, here the optimal case occurs precisely when x has property (2.3). Since, by part (a) of Corollary 2.2, all disjointness preserving continuous linear functionals on $C(X)$ are represented by multiples of Dirac measures, the remaining assertion is now immediate. ■

The following result improves [3, theorem 1]. Our proof is based on a different choice of the approximating disjointness preserving functional. The new approach actually provides the best approximation and also leads to a precise formula for the error.

Theorem 2.5. *Suppose that $\varepsilon(\varphi) < (2/9)\,\|\varphi\|^2$, and let $x\in X$ be the unique point for which condition* (2.2) *is satisfied. Then the continuous linear functional ψ on $C(X)$ represented by $\lambda(\{x\})\,\delta_x$ is the unique best disjointness preserving linear approximation of φ. Moreover,*

$$\|\varphi-\psi\| = \left(\|\varphi\| - \sqrt{\|\varphi\|^2 - 4\,\varepsilon(\varphi)}\right)/2 \le \sqrt{\varepsilon(\varphi)/2}.$$

PROOF. Except for the very last estimate, the assertions are immediate from

Proposition 2.3 and Lemma 2.4. But a straightforward computation shows that the remaining inequality is equivalent to $\varepsilon(\varphi) \leq (2/9)\,\|\varphi\|^2$. ■

Note that, in general, the error estimate $\|\varphi - \psi\| \leq \sqrt{\varepsilon(\varphi)/2}$ is weaker than the exact formula for $\|\varphi - \psi\|$ provided in Theorem 2.5. In fact, using this formula, it is easily verified that $\|\varphi - \psi\| < \sqrt{\varepsilon(\varphi)/2}$ whenever $0 < \varepsilon(\varphi) < (2/9)\,\|\varphi\|^2$.

As another advantage over [3], we mention that our approach respects positivity. Indeed, if the functional φ is positive, in the sense that $\varphi(f) \geq 0$ for all $f \in C(X)$ with $f \geq 0$ on X, then λ is a positive measure, and it is easily seen that the disjointness preserving approximation ψ provided by Theorem 2.5 satisfies $0 \leq \psi \leq \varphi$ on the positive cone of $C(X)$.

On the other hand, in Section 3 we will show that, in the setting of Theorem 2.5, it is sometimes useful to consider the approximation of φ by the linear functional η on $C(X)$ represented by $\lambda(X)\,\delta_x$. In fact, this is precisely the approach pursued in [3]. The proof of Lemma 2.4 confirms that

$$\|\varphi - \eta\| = |\lambda|\,(X \setminus \{x\}) + |\lambda(X \setminus \{x\})| \leq 2\,|\lambda|\,(X \setminus \{x\}) \leq 2\,\|\varphi - \psi\|\,,$$

with equality holding whenever φ is positive. Thus Theorem 2.5 leads to the error estimate $\|\varphi - \eta\| \leq 2\sqrt{\varepsilon(\varphi)/2}$. This approximation will be employed in Theorem 3.8.

3. Approximation of operators

In this section, let X and Y be compact Hausdorff spaces, and consider an arbitrary continuous linear operator $\varphi : C(X) \to C(Y)$. For each $y \in Y$, let φ_y denote the continuous linear functional on $C(X)$ given by

$$\varphi_y(f) := \varphi(f)(y) \qquad \text{for all } f \in C(X).$$

Thus $\varphi_y = \rho_y \circ \varphi$, where $\rho_y \in C(Y)^*$ denotes the functional given by evaluation at y. Also, let $\lambda_y \in M(X)$ denote the regular complex Borel measure representing the functional φ_y. Evidently, $\varepsilon(\varphi_y) \leq \varepsilon(\varphi)$. In fact, it is easily seen that $\varepsilon(\varphi) = \sup\{\varepsilon(\varphi_y) : y \in Y\}$. Similarly, it is clear that $\|\varphi\| = \sup\{\|\varphi_y\| : y \in Y\}$.

If the point $y \in Y$ satisfies $\varepsilon(\varphi) < (2/9)\,\|\varphi_y\|^2$, then, by Proposition 2.3, there exists a unique point $x \in X$ for which $|\lambda_y(\{x\})| > 2\,|\lambda_y|\,(X)/3$. Let $\alpha(y)$ denote this particular point x, and let ψ_y denote the linear functional on $C(X)$ represented by the measure $\lambda_y(\{\alpha(y)\})\,\delta_{\alpha(y)}$. By Theorem 2.5, ψ_y is the unique best disjointness preserving linear approximation of φ_y. The mapping α will be referred to as the *support mapping* of φ.

Proposition 3.1. *The set* $Y_\varphi := \left\{y \in Y : \varepsilon(\varphi) < (2/9)\,\|\varphi_y\|^2\right\}$ *is an open subset of* Y*, and the support mapping* $\alpha : Y_\varphi \to X$ *is continuous.*

Proof. To see that Y_φ is open, let $y \in Y_\varphi$ be given. Then $3\sqrt{\varepsilon(\varphi)/2} < |\varphi(f)(y)|$ for some $f \in C(X)$ with $\|f\|_\infty \leq 1$. By the continuity of $\varphi(f)$, it follows that

$3\sqrt{\varepsilon(\varphi)/2} < |\varphi(f)(u)| \le \|\varphi_u\|$ for all u in some open neighbourhood U of y. We conclude that $U \subseteq Y_\varphi$ and hence that Y_φ is open.

Now assume that α fails to be continuous. Then there exist a point $y \in Y_\varphi$ and an open neighbourhood V of $\alpha(y)$ in X such that, for each open neighbourhood U of y, there is a point $u \in U \cap Y_\varphi$ for which $\alpha(u) \notin V$. We choose, by normality, a function $f \in C(X)$ with the property that $0 \le f \le 1$ on X, $f(\alpha(y)) = 1$, and $f \equiv 0$ on $X \setminus V$.

We claim that the quantity $\delta := \sqrt{\varepsilon(\varphi)/2} \ge 0$ satisfies both $|\varphi(f)(y)| > \delta$ and $|\varphi(f)(u)| \le \delta$ for each $u \in Y_\varphi$ for which $\alpha(u) \notin V$. This will establish an obvious contradiction to the continuity of $\varphi(f)$ at the point y and hence will complete the proof.

To prove the first estimate, we employ Proposition 2.3 and Theorem 2.5 to obtain that

$$\begin{aligned}
|\varphi(f)(y)| &\ge |\lambda_y(\{\alpha(y)\})| - |\varphi(f)(y) - \lambda_y(\{\alpha(y)\})| \\
&= |\lambda_y(\{\alpha(y)\})| - |\varphi_y(f) - \psi_y(f)| \\
&\ge |\lambda_y(\{\alpha(y)\})| - \|\varphi_y - \psi_y\| \\
&= \frac{\|\varphi_y\| + \sqrt{\|\varphi_y\|^2 - 4\,\varepsilon(\varphi_y)}}{2} - \frac{\|\varphi_y\| - \sqrt{\|\varphi_y\|^2 - 4\,\varepsilon(\varphi_y)}}{2} \\
&= \sqrt{\|\varphi_y\|^2 - 4\,\varepsilon(\varphi_y)} \\
&\ge \sqrt{\|\varphi_y\|^2 - 4\,\varepsilon(\varphi)}.
\end{aligned}$$

Because $y \in Y_\varphi$ and therefore $\|\varphi_y\|^2 > (9/2)\,\varepsilon(\varphi)$, we infer that $|\varphi(f)(y)| > \sqrt{\varepsilon(\varphi)/2} = \delta$, as claimed.

Finally, consider an arbitrary point $u \in Y_\varphi$ for which $\alpha(u) \notin V$. Then $f(\alpha(u)) = 0$ and therefore $\psi_u(f) = \lambda_u(\{\alpha(u)\})\, f(\alpha(u)) = 0$. Again by Theorem 2.5, we conclude that

$$|\varphi(f)(u)| = |(\varphi_u - \psi_u)(f)| \le \|\varphi_u - \psi_u\| \le \sqrt{\varepsilon(\varphi_u)/2} \le \delta,$$

which establishes the second desired estimate. ■

When φ is a positive operator, in the sense that $\varphi(f) \ge 0$ on Y for all $f \in C(X)$ with $f \ge 0$ on X, there is a simple description of the set Y_φ. Evidently, the operator φ is positive precisely when, for each $y \in Y$, the functional φ_y is positive on $C(X)$, or, equivalently, when the representing measure λ_y is a positive measure on X. In this case,

$$\|\varphi_y\| = \lambda_y(X) = \int_X 1\,\mathrm{d}\lambda_y = \varphi(1)(y) \qquad \text{for all } y \in Y$$

and $\|\varphi\| = \|\varphi(1)\|_\infty = \max \varphi(1)$. In particular, it follows that the assignment $y \mapsto \|\varphi_y\|$ is continuous on Y, and that Y_φ is non-empty precisely when

$$\max \varphi(1) > 3\sqrt{\varepsilon(\varphi)/2},$$

while $Y_\varphi = Y$ precisely when $\min \varphi(1) > 3\sqrt{\varepsilon(\varphi)/2}$.

As an immediate consequence of the preceding results, we obtain the standard representation of disjointness preserving operators in our setting; see [6; 7].

Proposition 3.2. *A bounded linear operator $\varphi : C(X) \to C(Y)$ is disjointness preserving if and only if there exist an open subset Y_0 of Y and continuous functions $\alpha : Y_0 \to X$ and $h : Y \to \mathbb{C}$ such that $Y_0 = \{y \in Y : h(y) \neq 0\}$ and*

$$\varphi(f)(y) = \begin{cases} h(y)f(\alpha(y)), & \text{if } y \in Y_0, \\ 0, & \text{if } y \in Y \setminus Y_0, \end{cases}$$

for each $f \in C(X)$. Moreover, this representation of a disjointness preserving operator is unique.

Proof. Evidently, every weighted composition operator of the indicated form maps $C(X)$ into $C(Y)$ and is a disjointness preserving bounded linear operator. Conversely, suppose that φ is disjointness preserving. Then $\varepsilon(\varphi) = 0$ and hence $Y_\varphi = \{y \in Y : \varphi_y \neq 0\}$. By Proposition 3.1, the set $Y_0 := Y_\varphi$ is open, and the support mapping $\alpha : Y_0 \to X$ of φ is continuous. Let $h(y) := \lambda_y(\{\alpha(y)\})$ for all $y \in Y_0$, and $h(y) := 0$ for all $y \in Y \setminus Y_0$. Clearly, $Y_0 = \{y \in Y : h(y) \neq 0\}$, and, by Theorem 2.5, φ is represented by h and α in the desired form. In particular, it follows that $h = \varphi(1) \in C(Y)$. The proof of the uniqueness assertion is straightforward and therefore omitted. ■

Returning to the case of an arbitrary bounded linear operator φ from $C(X)$ into $C(Y)$, the preceding results lead to a canonical candidate for the approximation of φ by a disjointness preserving operator, namely the weighted composition operator induced by the support mapping α and the complex-valued function h on Y_φ given by

$$h(y) := \lambda_y(\{\alpha(y)\}) \qquad \text{for all } y \in Y_\varphi.$$

As shown in the proof of Proposition 3.2, h is continuous provided that φ is disjointness preserving, but, in general, the continuity of h turns out to be a non-trivial issue. As will be seen below, certain compactness conditions on φ ensure that h is continuous. For the basic theory of compact and weakly compact operators from $C(X)$ to $C(Y)$, we refer to [2, chapter VI] and [4, chapter VI]. In particular, if X is extremally disconnected (= Stonean) and Y is metrizable, then it follows from [2, corollary VI.2.12] that every bounded linear mapping from $C(X)$ to $C(Y)$ is weakly compact. The following preliminary result deals with the continuity properties of certain functions that are closely related to h.

Let $\tau : Y \to M(X)$ be given by $\tau(y) := \lambda_y$ for all $y \in Y$, and, for each Borel set $A \subseteq X$, let $\tau_A : Y \to \mathbb{C}$ denote the function given by $\tau_A(y) := \tau(y)(A) = \lambda_y(A)$ for all $y \in Y$.

Proposition 3.3. *For every bounded linear operator $\varphi : C(X) \to C(Y)$, the following assertions hold.*

(a) *τ is continuous with respect to the weak-* topology $\sigma(M(X), C(X))$ on $M(X)$.*
(b) *τ is continuous with respect to the weak topology $\sigma(M(X), M(X)^*)$ on $M(X)$ precisely when φ is weakly compact.*
(c) *τ is continuous with respect to the variation norm on $M(X)$ precisely when φ is compact.*
(d) *If φ is weakly compact, then τ_A is continuous for each Borel set A in X. The converse holds when X is metrizable.*

PROOF. Assertions (a), (b) and (c) follow from [4, theorem VI.7.1], while the first part of (d) is a consequence of (b). In fact, it suffices to note that, for each Borel set $A \subseteq X$, the definition $\sigma_A(\mu) := \mu(A)$ for all $\mu \in M(X)$ yields a functional $\sigma_A \in M(X)^*$ with the property that $\sigma_A \circ \tau = \tau_A$.

Alternatively, the first part of assertion (d) follows from a classical result due to Bartle, Dunford, and Schwartz. Indeed, if φ is weakly compact, then there exists, by [4, theorem VI.7.3], a vector measure ν defined on Σ and having values in $C(Y)$ such that

$$\varphi(f) = \int_X f \, \mathrm{d}\nu \qquad \text{for all } f \in C(X)$$

and $\rho \circ \nu \in M(X)$ for all $\rho \in C(Y)^*$. For each $y \in Y$, it follows that

$$\varphi(f)(y) = \left(\int_X f \, \mathrm{d}\nu \right)(y) = \int_X f \, \mathrm{d}(\rho_y \circ \nu) \qquad \text{for all } f \in C(X),$$

hence $\lambda_y = \rho_y \circ \nu$, and therefore $\lambda_y(A) = \nu(A)(y)$ for each Borel set A in X. Thus $\tau_A = \nu(A) \in C(Y)$ for all such A, as desired.

To establish the final claim, suppose that X is metrizable and that $\tau_A \in C(Y)$ for all $A \in \Sigma$. Clearly, the definition $\nu(A) := \tau_A$ yields an additive set function $\nu : \Sigma \to C(Y)$. Moreover, for each $\rho \in C(Y)^*$ with representing measure $\eta \in M(Y)$, we obtain that

$$(\rho \circ \nu)(A) = \int_Y \tau_A \, \mathrm{d}\eta = \int_Y \lambda_y(A) \, \mathrm{d}\eta(y) \qquad \text{for all } A \in \Sigma.$$

Because $\lambda_y \in M(X)$ and $|\lambda_y(A)| \leq |\lambda_y|(A) \leq \|\varphi_y\| \leq \|\varphi\| < \infty$ for all $y \in Y$ and $A \in \Sigma$, the Lebesgue dominated convergence theorem ensures that $\rho \circ \nu$ is countably additive. By a classical result due to Pettis, [4, theorem IV.10.1], we conclude that ν is a strongly countably additive vector measure, a fact that may also be derived from Dini's theorem. Since X is metrizable, it follows from [4, exercise III.9.22] that, for each $\rho \in C(Y)^*$, the complex measure $\rho \circ \nu$ is regular. Consequently, by [4, theorem VI.7.3], the definition

$$\psi(f) := \int_X f \, \mathrm{d}\nu \qquad \text{for all } f \in C(X)$$

yields a weakly compact operator $\psi : C(X) \to C(Y)$. But, for each $y \in Y$, we have

$(\rho_y \circ \nu)(A) = \tau_A(y) = \lambda_y(A)$ for all $A \in \Sigma$, and therefore

$$\psi(f)(y) = \int_X f \,\mathrm{d}\,(\rho_y \circ \nu) = \int_X f \,\mathrm{d}\lambda_y = \varphi(f)(y) \qquad \text{for all } f \in C(X).$$

Thus $\psi = \varphi$, so that φ is weakly compact as claimed. ■

In the disjointness preserving case, we obtain the following useful characterization which extends the first part of [10, theorem A]. Here we say that the support mapping α is *locally constant on* Y_φ if, for each $y \in Y_\varphi$, there exists an open neighbourhood U of φ in Y_φ such that α is constant on U. Evidently, by elementary topology, this condition forces α to be constant on each connected component of Y_φ, but the converse fails to be true in general, since the connected components need not be open. In fact, as observed by Kamowitz [10], there exist simple examples of non-compact weighted composition operators φ for which the corresponding support mapping is constant on each connected component of Y_φ.

Proposition 3.4. *Suppose that φ is disjointness preserving with support mapping α. Then the following assertions are equivalent:*

(i) *φ is compact;*
(ii) *φ is weakly compact;*
(iii) *τ_A is continuous for each Borel set A in X;*
(iv) *$\tau_{\{x\}}$ is continuous on Y_φ for each x in the range of α;*
(v) *α is locally constant on Y_φ.*

Proof. By Proposition 3.3, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. Now suppose that (iv) holds, and consider the canonical representation of the disjointness preserving operator φ from Proposition 3.2. Thus $Y_\varphi = Y_0$, and $\lambda_y = h(y)\delta_{\alpha(y)} \neq 0$ for all $y \in Y_\varphi$, while $\lambda_y = 0$ for all $y \in Y \setminus Y_\varphi$. To establish that α is locally constant on Y_φ, we fix an arbitrary point $y \in Y_\varphi$ and let $x := \alpha(y)$. Since $\tau_{\{x\}}$ is continuous at y and satisfies $\tau_{\{x\}}(y) = \lambda_y(\{\alpha(y)\}) = h(y) \neq 0$, there exists an open neighbourhood U of y in the open set Y_φ such that $\tau_{\{x\}}(u) \neq 0$ for all $u \in U$. Because

$$\tau_{\{x\}}(u) = \lambda_u(\{\alpha(y)\}) = h(u)\delta_{\alpha(u)}(\{\alpha(y)\}),$$

we conclude that $\alpha(u) = \alpha(y)$ for all $u \in U$. Thus (iv) entails (v).

Finally suppose that (v) holds. By Proposition 3.3, it remains to prove the continuity of τ at an arbitrary point $y \in Y$ with respect to the variation norm on $M(X)$. If $y \in Y_\varphi$, then we choose, for each $\varepsilon > 0$, an open neighbourhood U of y in Y_φ such that $\alpha(u) = \alpha(y)$ and $|h(u) - h(y)| < \varepsilon$ for all $u \in U$. It follows that $\tau(u) - \tau(y) = \lambda_u - \lambda_y = (h(u) - h(y))\,\delta_{\alpha(y)}$, and consequently,

$$\|\tau(u) - \tau(y)\| = |h(u) - h(y)| < \varepsilon \qquad \text{for all } u \in U$$

as desired. On the other hand, if $y \notin Y_\varphi$, then we have both $h(y) = 0$ and $\tau(y) = 0$. Again by the continuity of h, there exists, for each $\varepsilon > 0$, an open neighbourhood

U of y in Y such that $|h(u)| < \varepsilon$ for all $u \in U$. Thus $\|\tau(u) - \tau(y)\| = \left\|h(u)\delta_{\alpha(u)}\right\| = |h(u)| < \varepsilon$ for all $u \in U \cap Y_\varphi$, while $\tau(u) = \tau(y) = 0$ for all $u \in U \setminus Y_\varphi$. This completes the proof of the continuity of τ. Thus (v) implies (i). ■

We now address the question of the continuity of the mapping h associated with a positive bounded linear mapping $\varphi : C(X) \to C(Y)$.

Proposition 3.5. *Suppose that the operator φ is positive. Then the function h given by*

$$h(y) := \lambda_y\left(\{\alpha(y)\}\right) \qquad \text{for all } y \in Y_\varphi$$

is upper semi-continuous on Y_φ. Moreover, the following assertions are equivalent:

(i) *$\tau_{\{x\}}$ is continuous on Y_φ for each x in the range of α;*
(ii) *α is locally constant and h is continuous on Y_φ.*

PROOF. To establish the upper semi-continuity of the non-negative function h on Y_φ, let $y \in Y_\varphi$ and $s \in \mathbb{R}$ be given so that $h(y) < s$. To show that this estimate holds on some open neighbourhood of y, we first choose a real number t for which $h(y) < t < s$ and then, by the regularity of the measure λ_y, some open neighbourhood U of $\alpha(y)$ in X such that

$$h(y) = \lambda_y\left(\{\alpha(y)\}\right) \leq \lambda_y(U) < t.$$

By the normality of the compact Hausdorff space X and Urysohn's lemma, there exists a continuous function $f : X \to [0,1]$ that satisfies $\operatorname{supp} f \subseteq U$ as well as $f \equiv 1$ on some open neighbourhood V of $\alpha(y)$ in X. Finally, taking Proposition 3.1 into account, we may employ the continuity of the functions α and $\varphi(f)$ at the point y to obtain an open neighbourhood W of y in Y_φ such that $\alpha(w) \in V$ and $\varphi(f)(w) - \varphi(f)(y) < s - t$ for all $w \in W$. For arbitrary $w \in W$, we conclude that

$$\begin{aligned} h(w) = \lambda_w(\{\alpha(w)\}) &\leq \int_X f \, \mathrm{d}\lambda_w = \varphi(f)(w) \\ &< \varphi(f)(y) + s - t = \int_U f \, \mathrm{d}\lambda_y + s - t \leq \lambda_y(U) + s - t \end{aligned}$$

and therefore $h(w) < s$. Thus h is upper semi-continuous on Y_φ.

Next suppose that (i) holds, consider an arbitrary point $y \in Y_\varphi$, let $x := \alpha(y)$, and observe that

$$\tau_{\{x\}}(y) = \lambda_y(\{x\}) = \lambda_y(\{\alpha(y)\}) > \frac{2}{3}\lambda_y(X) = \frac{2}{3}\varphi(1)(y),$$

by the definition of the support mapping α of φ. Since both functions $\tau_{\{x\}}$ and $\varphi(1)$ are continuous at y, there exists an open neighbourhood U of y in Y_φ such that

$$\lambda_u(\{x\}) = \tau_{\{x\}}(u) > \frac{2}{3}\varphi(1)(u) = \frac{2}{3}\lambda_u(X) \qquad \text{for all } u \in U.$$

Again by the definition of α, we conclude that $\alpha(u) = x$ for all $u \in U$, so that α is constant on U. Moreover, $h(u) = \lambda_u(\{\alpha(u)\}) = \lambda_u(\{x\}) = \tau_{\{x\}}(u)$ for all $u \in U$, which ensures the continuity of h at y. Thus (i) $\Rightarrow$ (ii).

Finally suppose that (ii) is satisfied, and consider an element $x \in X$ of the form $x = \alpha(y)$ for some $y \in Y_\varphi$. If U denotes an open neighbourhood of y in Y_φ such that $\alpha(u) = x$ for all $u \in U$, then it follows that $\tau_{\{x\}} \equiv h$ on U and hence that $\tau_{\{x\}}$ is continuous at y. Thus (ii) $\Rightarrow$ (i) as desired. ■

In particular, it follows from Propositions 3.3 and 3.5 that h is continuous on Y_φ provided that φ is both positive and weakly compact, but more can be said. Note that the continuity condition on φ in the following result holds not only when φ is positive, but also, by part (c) of Proposition 3.3, when φ is compact. Thus Theorem 3.6 applies, in particular, to arbitrary compact and to positive weakly compact operators.

Theorem 3.6. *Suppose that $\varphi : C(X) \to C(Y)$ is a weakly compact linear operator for which the assignment $y \mapsto \|\varphi_y\|$ is continuous on Y_φ. Then the following assertions hold.*

(a) *The support mapping α of φ is locally constant on Y_φ, and the weight function h given by*

$$h(y) := \lambda_y\left(\{\alpha(y)\}\right) \qquad \text{for all } y \in Y_\varphi$$

is continuous on Y_φ.

(b) *If $Y_\varphi = Y$, then the weighted composition operator $\psi : C(X) \to C(Y)$ given by*

$$\psi(f)(y) := h(y)f(\alpha(y)) \qquad \text{for all } y \in Y$$

is a compact operator that satisfies $\|\varphi - \psi\| \leq \sqrt{\varepsilon(\varphi)/2}$ and provides the minimal distance of φ to the class of disjointness preserving bounded linear operators from $C(X)$ to $C(Y)$; moreover, if φ is positive, then $0 \leq \psi \leq \varphi$.

(c) *If $Y_\varphi \neq Y$, then, for each $\delta > 0$, there exists a compact weighted composition operator $\psi : C(X) \to C(Y)$ with the property that $\|\varphi - \psi\| \leq (3 + \delta)\sqrt{\varepsilon(\varphi)/2}$; moreover, if φ is positive, then ψ may be chosen so that $0 \leq \psi \leq \varphi$.*

Proof. (a) To establish this assertion, we employ part (d) of Proposition 3.3 and recycle an argument from Proposition 3.5. Given $y \in Y_\varphi$, let $x := \alpha(y)$, and note that $\left|\tau_{\{x\}}(y)\right| = |\lambda_y(\{\alpha(y)\})| > 2\,|\lambda_y|\,(X)/3 = 2\,\|\varphi_y\|\,/3$. Hence, by continuity, there exists an open neighbourhood U of y in Y_φ such that $|\lambda_u(\{x\})| = \left|\tau_{\{x\}}(u)\right| > 2\,\|\varphi_u\|\,/3$ for all $u \in U$. Thus $\alpha(u) = x$ for all $u \in U$, so that α is constant on U. Also, $h \equiv \tau_{\{x\}}$ on U, so that h is continuous at y, as desired.

(b) By part (a), in conjunction with Propositions 3.2 and 3.4, ψ is both disjointness preserving and compact. Moreover, the estimate $\|\varphi - \psi\| \leq \sqrt{\varepsilon(\varphi)/2}$ is immediate from the construction of ψ and Theorem 2.5. Also, if $\xi : C(X) \to C(Y)$ is any disjointness preserving continuous linear mapping, then, for each $y \in Y$, the functional $\xi_y := \rho_y \circ \xi \in C(X)^*$ is disjointness preserving. Again by Theorem 2.5, it follows that $\|\varphi_y - \psi_y\| \le \|\varphi_y - \xi_y\|$ for all $y \in Y$ and therefore $\|\varphi - \psi\| \le \|\varphi - \xi\|$, as desired. Finally, if φ is positive, then it is easily verified that $0 \le \psi \le \varphi$ in the canonical order for operators from $C(X)$ to $C(Y)$.

(c) Without loss of generality, we may assume that $\varepsilon(\varphi) > 0$. Then the set

$$K := \left\{ y \in Y : \varepsilon(\varphi) \le 2 \|\varphi_y\|^2 / (3+\delta)^2 \right\}$$

is a compact subset of the open set Y_φ. By Urysohn's lemma, there exists a continuous function $g : Y \to [0,1]$ that satisfies $g \equiv 1$ on K and $\operatorname{supp} g \subseteq Y_\varphi$. By Propositions 3.2 and 3.4, the definition

$$\psi(f)(y) := \begin{cases} g(y)h(y)f(\alpha(y)), & \text{if } y \in Y_\varphi, \\ 0, & \text{if } y \in Y \setminus Y_\varphi, \end{cases}$$

for arbitrary $f \in C(X)$ yields a compact disjointness preserving linear operator from $C(X)$ into $C(Y)$. Because $0 \le g \le 1$ on Y, we obtain that $0 \le \psi \le \varphi$ whenever $\varphi \ge 0$. To estimate the distance $\|\varphi - \psi\|$, let $f \in C(X)$ with $\|f\|_\infty \le 1$ be given, and consider for an arbitrary point $y \in Y$ the following three cases.

First, if $y \in K$, then $g(y) = 1$ and hence

$$|\varphi(f)(y) - \psi(f)(y)| = |\varphi_y(f) - \psi_y(f)| \le \sqrt{\varepsilon(\varphi_y)/2} \le \sqrt{\varepsilon(\varphi)/2},$$

by Theorem 2.5. Next, if $y \notin Y_\varphi$, then $g(y) = 0$ and $(2/9)\|\varphi_y\|^2 \le \varepsilon(\varphi)$, so that

$$|\varphi(f)(y) - \psi(f)(y)| = |\varphi_y(f)| \le \|\varphi_y\| \le 3\sqrt{\varepsilon(\varphi)/2}.$$

Finally, if $y \in Y_\varphi \setminus K$, then, by Proposition 2.3 and Theorem 2.5,

$$\begin{aligned} |\varphi(f)(y) - \psi(f)(y)| &\le |\varphi_y(f) - h(y)f(\alpha(y))| + |1 - g(y)|\,|h(y)|\,|f(\alpha(y))| \\ &\le \frac{\|\varphi_y\| - \sqrt{\|\varphi_y\|^2 - 4\,\varepsilon(\varphi_y)}}{2} + \frac{\|\varphi_y\| + \sqrt{\|\varphi_y\|^2 - 4\,\varepsilon(\varphi_y)}}{2} \\ &= \|\varphi_y\| < (3+\delta)\sqrt{\varepsilon(\varphi)/2}, \end{aligned}$$

where the last inequality follows from $y \notin K$. Thus $\|\varphi - \psi\| \le (3+\delta)\sqrt{\varepsilon(\varphi)/2}$. ■

Without any compactness condition on φ, it is still possible to construct a disjointness preserving approximation, but one has to allow for a larger error. Theorem 3.8 below improves a corresponding result from [3]. The following technical lemma will be needed.

Lemma 3.7. *Suppose that* $\varepsilon(\varphi) > 0$, *let* c *be any real number for which* $0 < c < 2/25$, *and let* K *denote the closure of the set* $\left\{ y \in Y : \varepsilon(\varphi) < c\,\|\varphi_y\|^2 \right\}$. *Then* $K \subseteq Y_\varphi$.

PROOF. Write $c = 2/(5+\delta)^2$ for suitable $\delta > 0$, and assume that there exists a point $y \in K \setminus Y_\varphi$. By the definition of Y_φ, it follows that

$$|\varphi(1)(y)| \le \|\varphi_y\| \le 3\sqrt{\varepsilon(\varphi)/2} < (3+\delta)\sqrt{\varepsilon(\varphi)/2}.$$

Hence, by continuity, there exists an open neighbourhood U of y such that $|\varphi(1)(u)| < (3+\delta)\sqrt{\varepsilon(\varphi)/2}$ for all $u \in U$. Because $y \in K$, we obtain some $u \in Y$ that satisfies both

$$(5+\delta)\sqrt{\varepsilon(\varphi)/2} < \|\varphi_u\| \quad \text{and} \quad |\varphi(1)(u)| < (3+\delta)\sqrt{\varepsilon(\varphi)/2}.$$

Clearly, $u \in Y_\varphi$. If $\zeta_u \in C(X)^*$ denotes the functional given by evaluation at $\alpha(u)$, then, by the last observation of the preceding section, we have

$$\|\varphi_u - \varphi_u(1)\zeta_u\| = \|\varphi_u - \lambda_u(X)\zeta_u\| \le 2\sqrt{\varepsilon(\varphi_u)/2} \le 2\sqrt{\varepsilon(\varphi)/2}$$

and consequently

$$\|\varphi_u\| \le \|\varphi_u - \varphi_u(1)\zeta_u\| + |\varphi_u(1)| < (5+\delta)\sqrt{\varepsilon(\varphi)/2},$$

in contradiction to $(5+\delta)\sqrt{\varepsilon(\varphi)/2} < \|\varphi_u\|$. The assertion follows. ■

Theorem 3.8. *For every bounded linear mapping $\varphi : C(X) \to C(Y)$ and every $\delta > 0$, there exists a disjointness preserving bounded linear mapping $\psi : C(X) \to C(Y)$ with the property that $\|\varphi - \psi\| \le (7+\delta)\sqrt{\varepsilon(\varphi)/2}$.*

Proof. We may assume that $\varepsilon(\varphi) > 0$. Let K denote the compact set from Lemma 3.7 that corresponds to $c := 2/(5+\delta)^2$, let $h := \varphi(1)$, and choose some $g \in C(Y)$ such that $0 \le g \le 1$ on Y, $g \equiv 1$ on K, and $\operatorname{supp} g \subseteq Y_\varphi$. Also, for arbitrary $f \in C(X)$, let

$$\psi(f)(y) := \begin{cases} g(y)h(y)f(\alpha(y)), & \text{if } y \in Y_\varphi, \\ 0, & \text{if } y \in Y \setminus Y_\varphi. \end{cases}$$

Then $\psi : C(X) \to C(Y)$ is a disjointness preserving bounded linear mapping. Given $f \in C(X)$ with $\|f\|_\infty \le 1$ and $y \in Y$, we consider, as in the proof of Theorem 3.6, the following three cases. If $y \in K$, then $g(y) = 1$ and therefore

$$|\varphi(f)(y) - \psi(f)(y)| = |\varphi_y(f) - \lambda_y(X)f(\alpha(y))| \le 2\sqrt{\varepsilon(\varphi_y)/2} \le 2\sqrt{\varepsilon(\varphi)/2},$$

again by the concluding remark of the preceding section. Next, if $y \notin Y_\varphi$, then $g(y) = 0$ and $(2/9)\|\varphi_y\|^2 \le \varepsilon(\varphi)$, so that

$$|\varphi(f)(y) - \psi(f)(y)| = |\varphi_y(f)| \le \|\varphi_y\| \le 3\sqrt{\varepsilon(\varphi)/2}.$$

Finally, if $y \in Y_\varphi \setminus K$, then $|\lambda_y(X)| \le \|\varphi_y\| \le (5+\delta)\sqrt{\varepsilon(\varphi)/2}$ and hence

$$\begin{aligned} |\varphi(f)(y) - \psi(f)(y)| &\le |\varphi_y(f) - \lambda_y(X)f(\alpha(y))| + |1-g(y)|\,|\lambda_y(X)|\,|f(\alpha(y))| \\ &\le 2\sqrt{\varepsilon(\varphi_y)/2} + \|\varphi_y\| \le (7+\delta)\sqrt{\varepsilon(\varphi)/2}. \end{aligned}$$

Thus $\|\varphi - \psi\| \leq (7+\delta)\sqrt{\varepsilon(\varphi)/2}$ as desired. ■

The estimate $\|\varphi - \psi\| \leq (7+\delta)\sqrt{\varepsilon(\varphi)/2}$ provided in Theorem 3.8 probably fails to be optimal in general. Indeed, if the canonical weight function h given by $h(y) := \lambda_y(\{\alpha(y)\})$ for all $y \in Y_\varphi$ happens to be continuous on Y_φ, then a simple modification of the preceding arguments leads to a disjointness preserving approximation ψ for which $\|\varphi - \psi\| \leq (5+\delta)\sqrt{\varepsilon(\varphi)/2}$. Thus the most interesting open question in this context is that of the automatic continuity of the weight function h associated with an arbitrary bounded linear mapping $\varphi : C(X) \to C(Y)$. Of course, Propositions 3.2 and 3.4 show that h may well be continuous even if φ fails to be weakly compact, but, in general, the continuity of h remains an open problem even when φ is positive.

References

[1] R.M. Blumenthal, J. Lindenstrauss and R.R. Phelps, Extreme operators into $C(K)$, *Pacific Journal of Mathematics* **15** (1965), 747–56.

[2] J. Diestel and J.J. Uhl, *Vector measures,* Mathematical Surveys 15, American Mathematical Society, Providence, RI, 1977.

[3] G. Dolinar, Stability of disjointness preserving mappings, *Proceedings of the American Mathematical Society* **130** (2001), 129–38.

[4] N. Dunford and J.T. Schwartz, *Linear operators,* Part I, Wiley-Interscience, New York, 1958.

[5] E. Hewitt and K.A. Ross, *Abstract harmonic analysis,* Volume I, Springer-Verlag, Berlin, 1963.

[6] J.E. Jamison and M. Rajagopalan, Weighted composition operator on $C(X, E)$, *Journal of Operator Theory* **19** (1988), 307–17.

[7] K. Jarosz, Automatic continuity of separating linear isomorphisms, *Canadian Mathematical Bulletin* **33** (1990), 139–144.

[8] B.E. Johnson, Approximately multiplicative functionals, *Journal of the London Mathematical Society (2)* **34** (1986), 489–510.

[9] B.E. Johnson, Approximately multiplicative maps between Banach algebras, *Journal of the London Mathematical Society (2)* **37** (1988), 294–316.

[10] H. Kamowitz, Compact weighted endomorphisms of $C(X)$, *Proceedings of the American Mathematical Society* **83** (1981), 517–21.

[11] W. Rudin, *Real and complex analysis,* McGraw-Hill, New York, 1966.