SUBMAXIMAL AND SPECTRAL SPACES

By

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Abstract

A topological space X is said to be *submaximal* if every dense subset of X is open. In this paper, descriptions of submaximal spectral spaces and Stone submaximal spaces are given. Throughout this paper a number of illustrative examples are given.

1. Introduction

Non-Hausdorff spaces play a more significant role than Hausdorff spaces in connection with order. The T_0-axiom (credited to Kolmogoroff) and the T_1-axiom (credited to Riesz) are among the best known of the non-Hausdorff separation axioms. A topological space $(X, \mathcal{T})$ is said to be a T_D*-space* [3] if every one-point set of X is locally closed (that is, every one-point set is the intersection of an open set and a closed set). Classically we have the following implications:

$$T_1 \Rightarrow T_D \Rightarrow T_0.$$

We recall that a topological space X is said to be *submaximal* if every dense subset of X is open [6]. Submaximal spaces have been studied by several authors (see, for

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instance, [1; 2; 7; 13; 15; 17]). Note also, that Bezhanishvili *et al.* [5] have investigated modal logics of submaximal spaces (and related topics); an axiomatization of these modal logics is also given in [5].

It is well known that every submaximal space is a T_0-space. In fact, by [2, theorem 1.2], every submaximal space is a T_D-space.

Recall that an *Alexandroff topology* on a set X is a topology $\mathcal{T}$ for which any intersection of open sets is again open. By the *Alexandroff topology on X associated with an ordering* $\leq$, we mean the topology on X that has as a basis $\{(\downarrow x) \mid x \in X\}$, where $(\downarrow x) = \{y \in X \mid y \leq x\}$. If $x \in X$, then we denote by $(\uparrow x)$ the subset $\{y \in X \mid x \leq y\}$ of X.

Section 2 is devoted to some remarks about submaximal spaces. We exhibit examples distinguishing various notions for submaximal spaces, T_D-spaces and Alexandroff spaces.

In Section 3, spectral submaximal spaces are characterized and an example is given that, in some sense, shows this characterization is sharp.

In [4], A-spectral spaces are considered. A-class spaces pertinent to this paper are considered in Section 4.

2. Remarks on submaximal spaces

Firstly, let us recall a result about submaximal spaces [5, theorem 3.1].

Theorem 2.1. *Let X be a topological space. Then the following statements are equivalent:*

(*i*) *X is submaximal;*

(*ii*) *$\overline{S} \setminus S$ is closed, for each $S \subseteq X$;*

(*iii*) *$\overline{S} \setminus S$ is closed and discrete, for each $S \subseteq X$.*

In particular, the above result implies that every submaximal space is a T_D-space.

Let $(X, \mathcal{T})$ be a T_0-space and $\leq$ be the *specialization order* (that is, $x \leq y$ if and only if $y \in \overline{\{x\}}$). A chain $x_0 < x_1 < ... < x_n$ of elements of X is said to be of length n; the supremum of the lengths is called the *Krull dimension* of $(X, \mathcal{T})$, which we write as $dim_K(X, \mathcal{T})$.

Theorem 2.1 immediately gives the following result.

Proposition 2.2. *Let $(X, \mathcal{T})$ be a T_0-space. Then the following properties hold:*

(1) *if $(X, \mathcal{T})$ is a submaximal space, then $dim_K(X, \mathcal{T}) \leq 1$;*

(2) *if, in addition, $\mathcal{T}$ is an Alexandroff topology, then the following statements are equivalent:*

  (*i*) *$(X, \mathcal{T})$ is a submaximal space;*

  (*ii*) *$dim_K(X, \mathcal{T}) \leq 1$.*

Proof. (1) Let $x \in X$; then, according to Theorem 2.1, $\overline{\{x\}} \setminus \{x\}$ is a closed and discrete subspace of X. Hence, if $y \in \overline{\{x\}} \setminus \{x\}$, then y is a maximal point (in the specialization order) of X. Thus $dim_K(X, \mathcal{T}) \leq 1$.

(2) Note that Bezhanishvili *et al.* [5] have given a proof of this part (see [5, proposition 4.1]). We provide another proof here. Let S be a subset of X. Suppose that S is not closed and $x \in \overline{S} \setminus S$. Since $\mathcal{T}$ is Alexandroff, $\overline{S} = \bigcup((\uparrow y) : \; y \in S)$ and, consequently, there exists $y \in S$ such that $y < x$. Hence $(\uparrow x) = \{x\}$, since $dim_K(X, \mathcal{T}) \leq 1$. Thus $\overline{S} \setminus S = \bigcup((\uparrow x) : \; x \in \overline{S} \setminus S)$ and it follows that $\overline{S} \setminus S$ is closed. Therefore $(X, \mathcal{T})$ is a submaximal space by Theorem 2.1. ∎

Example 2.3. The assumption that $\mathcal{T}$ is an Alexandroff topology in Proposition 2.2 is essential, and there exists a T_0-space with Krull dimension 1 that is not a T_D-space.

Let $\mathbb{Z}$ be the set of the integers, equipped with the topology $\mathcal{T} = \{\emptyset\} \cup \{U \subseteq \mathbb{Z} \mid 0 \in U$ and $\mathbb{Z} \setminus U$ is finite$\}$. Clearly $dim_K(\mathbb{Z}, \mathcal{T}) = 1$, while $(\mathbb{Z}, \mathcal{T})$ is not a T_D-space, since $\{0\}$ is not locally closed.

In fact, more can be said.

Example 2.4. There exists a T_D-space with Krull dimension 1 that is not a submaximal space.

Let $X = \{m_i \mid 0 \leq i \leq \omega\} \cup \{n_i \mid 1 \leq i \leq \omega\}$ and $\leq$ be the order on X defined by $n_\omega \leq m_\omega$ and, for $1 \leq i < \omega$, $n_i \leq m_{i-1}$ and m_i. Let $\mathcal{T}$ be the topology on X whose closed sets are $\emptyset$, finite upper sets and upper sets containing m_ω. Clearly, $dim_K(X, \mathcal{T}) \leq 1$. Next let $x \in X$. There are three cases to be considered. If $x = m_i$ with $0 \leq i \leq \omega$, then $\overline{\{m_i\}} \setminus \{m_i\} = \emptyset$ since $\{m_i\}$ is closed. If $x = n_i$ with $1 \leq i < \omega$, then $\overline{\{n_i\}} \setminus \{n_i\} = \{m_{i-1}, m_i\}$, which is closed. Finally, if $x = n_\omega$, then $\overline{\{n_\omega\}} \setminus \{n_\omega\} = \{m_\omega\}$, which is also closed. Either way, X is a T_D-space. However, for $S = \{n_i \mid \; 1 \leq i \leq \omega\} \cup \{m_\omega\}$, $\overline{S} \setminus S = \{m_i \mid \; 0 \leq i < \omega\}$ is not closed. It follows that X is not a submaximal space.

Example 2.5. There exists a submaximal space that is not Alexandroff. Let X be an infinite set and $m \in X$. We define the topology $\mathcal{T}_m$ on X whose closed sets are finite sets and sets containing m. Since X is infinite, $(X, \mathcal{T}_m)$ is not an Alexandroff space.

Let S be a subset of X. We then have the following two cases:
Case 1: $m \in S$ or S is finite; in this case S is a closed subset of X. Thus $\overline{S} \setminus S = \emptyset$.
Case 2: $m \notin S$ and S is infinite, then $\overline{S} = S \cup \{m\}$. Thus $\overline{S} \setminus S = \{m\}$ is closed.
Therefore, $(X, \mathcal{T}_m)$ is a non-Alexandroff submaximal space.

Remark 2.6. One may provide a simpler example of a submaximal space that is not Alexandroff. It suffices to consider a convergent sequence (take for instance, $X = \{\frac{1}{n} \mid n \in \mathbb{N}^*\} \cup \{0\})$ equipped with the usual topology.

Proposition 2.7. *Let $(X, \mathcal{T})$ be a compact submaximal space. Then $\overline{S} \setminus S$ is finite for every subset S of X.*

PROOF. Since $\overline{S} \setminus S$ is a closed set with a discrete relative topology (see Theorem 2.1), it is compact, and hence finite. ■

According to Kelley [10], a topological space X is said to be a *door space* if every subset of X is either closed or open. It is immediate that every door space is a submaximal space.

For the next example, we recall the digital topology on the set of integers $\mathbb{Z}$. The study of the geometric and topological properties of digital images is the goal of digital topology, and many topological spaces that fail to be T_1 are of significant importance in digital topology. In the process of digitizing a movie, some situations are often represented by subspaces and quotients of locally finite topological spaces, so the study of these topological spaces is important. The digital line is the major building block of the digital n-space.

Example 2.8. There exists a submaximal space of Krull dimension 1 that is not a door space.

The digital line, also known as the *Khalimsky line* is the set of the integers $\mathbb{Z}$, equipped with the topology $\mathcal{K}$, generated by the family $\{\{2n-1, 2n, 2n+1\} \mid n \in \mathbb{Z}\}$, see [11; 12]. Hence, a set $U \subseteq \mathbb{Z}$ is open in $\mathcal{K}$, if and only if, whenever $x \in U$ is an even integer, then $x-1, x+1 \in U$. Clearly, the Khalimsky line is an Alexandroff space with Krull dimension 1. By Proposition 2.2, $(\mathbb{Z}, \mathcal{K})$ is a submaximal space. However, the set $S = \{2n-1, 2n+2\}$ is neither open nor closed.

Again, more can be said.

Example 2.9. There exists a submaximal space of Krull dimension 0 that is not a door space.
Consider the disjoint union of two convergent sequences. For instance, consider $X = \{\frac{1}{n} \mid n \in \mathbb{N}^*\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}^*\} \cup \{0, 2\}$ equipped with the usual topology. Thus X is a submaximal space with Krull dimension 0 that is not a door space (since $\{0\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}^*\}$ is neither open nor closed).

Lemma 2.10. *Let $(X, \mathcal{T})$ be a topological space that has a finite number of accumulation points. If each accumulation point of X is closed, then $(X, \mathcal{T})$ is a submaximal space.*

PROOF. Let S be a subset of X and Y be the set of accumulation points of X. Then, clearly, $\overline{S} \setminus S \subseteq Y$, which is a finite union of closed one-point sets. ■

For a T_0-space $(X, \mathcal{T})$ and $x \in X$, the *height* $ht(x)$ of x is 0 if $y = x$ whenever $x \in \overline{\{y\}}$. Let $X_0 = \{x \in X | ht(x) = 0\}$. Then every element of $X \setminus X_0$ is an accumulation point. For, if $x \in X \setminus X_0$, then there exists $y \in X_0$ such that $x \in$

$\overline{\{y\}} \setminus \{y\}$. In particular, each open set containing x contains y, which implies that x is an accumulation point.

More can be said if $(X, \mathcal{T})$ is a submaximal space.

Lemma 2.11. *Let $(X, \mathcal{T})$ be a submaximal space and $\leq$ be the specialization order of X. Then each accumulation point of X belongs to the set of maximal elements $Max(X)$ (that is, the set of closed points of X) of the partially-ordered set (poset) $(X, \leq)$.*

PROOF. Indeed, let $x \notin Max(X)$ and $S = X \setminus \{x\}$. Suppose that $\overline{S} = X$. Then $X \setminus \{x\}$ is open. Hence $\{x\}$ is a closed subset of $(X, \mathcal{T})$. This is impossible, since $x \notin Max(X)$. Thus, $\overline{S} = S = X \setminus \{x\}$. Therefore, $\{x\}$ is an open subset of $(X, \mathcal{T})$ and it follows that x is not an accumulation point. ■

3. Spectral spaces

According to Hochster [9], a space $(X, \mathcal{T})$ is said to be *spectral* providing

(i) X is a T_0-space,
(ii) X is compact,
(iii) the compact open subsets are closed under finite intersections and form an open basis, and
(iv) every non-empty closed subspace that is *irreducible* (that is to say, it is not the union of two proper closed subsets) is the closure of one of its points (in other words, it has a *generic point*).

Recall that a *Stone space* $(X, \mathcal{T})$ is a compact totally disconnected space (that is, it is compact and, for distinct $x, y \in X$, there exists a clopen set U such that $x \in U$ and $y \notin U$). It is well known that a spectral space X is Stone iff $dim_K(X, \mathcal{T}) = 0$ (that is, X is a spectral T_1-space) [9].

Proposition 3.1. *Let $(X, \mathcal{T})$ be a Stone space. Then the following statements are equivalent:*

(i) *$(X, \mathcal{T})$ is a submaximal space;*
(ii) *$(X, \mathcal{T})$ has a finite number of accumulation points.*

PROOF. Since a Stone space is Hausdorff, each one-point set is closed. By Lemma 2.10, if X has a finite number of accumulation points, then it is a submaximal space.

Conversely, suppose there are infinitely many accumulation points. For any two distinct accumulation points x and y choose a disjoint clopen set U such that $x \in U$ and $y \notin U$. Then infinitely many accumulation points do not belong to U or infinitely many accumulation points do not belong to $X \setminus U$. In the former case, let $l_0 = x$ and $L_0 = U$ and, in the latter case, let $l_0 = y$ and $L_0 = X \setminus U$.

Since infinitely many accumulation points do not belong to L_0, we may proceed inductively to define a disjoint family $(L_i : \ i < \omega)$ of clopen sets and a family of distinct accumulation points $(l_i \in L_i : \ i < \omega)$

Let

$$Y = (X \setminus \bigcup(L_i : \ i < \omega)) \cup (L_i \setminus \{l_i\} : \ i < \omega).$$

If $(X, \mathcal{T})$ is locally closed, then $Y = U \cap V$ for some open set U and closed set V of X.

Consider $\overline{\{l_i : \ i < \omega\}}$. If $\overline{\{l_i : \ i < \omega\}} \subseteq \bigcup(L_i : \ i < \omega)$, then $(L_i : \ i < \omega)$ forms an open cover of $\overline{\{l_i : \ i < \omega\}}$, from which it follows that there is a finite subcover. Since L_i and L_j are disjoint whenever i and j are distinct, this is impossible. In particular, there exists $l \in (P \setminus \bigcup(L_i : \ i < \omega)) \cap \overline{\{l_i : \ i < \omega\}}$. Since $l \in U$, $l_i \in U$ for some $i < \omega$. Since $L_i \setminus \{l_i\} \subseteq V$ and $\overline{L_i \setminus \{l_i\}} = L_i$, $l_i \in V$ too. That is, $l_i \in U \cap V$, which is a contradiction. ■

The anonymous referee of this paper has suggested the following elegant result that generalizes Proposition 3.1.

Theorem 3.2. *Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then the following statements are equivalent:*

(i) *$(X, \mathcal{T})$ is a submaximal space;*

(ii) *$(X, \mathcal{T})$ has a finite number of accumulation points.*

PROOF. It is sufficient to show (i) $\Longrightarrow$ (ii). Suppose that X has infinitely many accumulation points. Since X is Hausdorff, we may consider a set D, which is an infinite (relatively) discrete subset of the set of accumulation points. Then, clearly, $X \setminus D$ is dense in X. But, $X \setminus D$ is not open (that is D is not closed); indeed, X being compact, D has an accumulation point that is necessarily in $X \setminus D$ (because D is (relatively) discrete). This contradicts the fact that X is submaximal. ■

We need to recall the patch topology [9].

By the *patch topology* on X for a spectral space X, we mean the topology that has as a sub-basis for its closed sets the closed sets and compact open sets of the original space (or better, which has the compact open sets and their complements as an open sub-basis). Recall that the patch topology associated with a spectral space is a Stone space [9] and that it is finer than the original topology. A closed set for the patch topology will be called a *patch*.

Theorem 3.3. *Let $(X, \mathcal{T})$ be a spectral space. Then the following statements are equivalent:*

(i) *$(X, \mathcal{T})$ is a submaximal space;*

(ii) *X has finitely many accumulation points and every accumulation point of X belongs to $Max(X)$.*

PROOF. Suppose X has finitely many accumulation points and every accumulation point of X belongs to $Max(X)$. Let S be a subset of X and $x \in \overline{S} \setminus S$. Clearly x is an accumulation point of X. Then $\overline{S} \setminus S$ is finite, since X has only finitely many accumulation points. Therefore $\overline{S} \setminus S = \bigcup(\{x\} \mid x \in \overline{S} \setminus S) = \bigcup(\overline{\{x\}} \mid x \in \overline{S} \setminus S)$, as every accumulation point belongs to $Max(X)$. In particular, $\overline{S} \setminus S$ is closed and $(X, \mathcal{T})$ is a submaximal space.

Conversely, suppose that $(X, \mathcal{T})$ is a submaximal space. It is to be shown that X has finitely many accumulation points, since every accumulation point belongs to $Max(X)$ by Lemma 2.11.

If $\dim_K(X, \mathcal{T}) = 0$, then $(X, \mathcal{T})$ is a Stone space and, by Proposition 3.1, it follows that there are only finitely many accumulation points.

If $\dim_K(X, \mathcal{T}) \neq 0$, then, by Proposition 2.2, $\dim_K(X, \mathcal{T}) = 1$.

Set $X_0 = \{x \in X \mid ht(x) = 0\}$ and $X_1 = X \setminus X_0$ (that is, $X_1 = \{x \in X \mid ht(x) = 1\}$).

Let X_{pat} be the set X equipped with the patch topology. Then X_{pat} is a Stone space. Since the patch topology is finer than the initial topology on X, X_{pat} is a submaximal space. Thus X_{pat} has a finite number of accumulation points, by Proposition 3.1. We denote by Y_0 and Y the set of accumulation points of X_{pat} and $(X, \mathcal{T})$, respectively. Clearly, $Y_0 \subseteq Y$. Suppose there exists $x \in Y \setminus Y_0$ such that $ht(x) = 0$. Then $\{x\}$ is a clopen subset of X_{pat} and not an open subset of $(X, \mathcal{T})$. So $\overline{X \setminus \{x\}}^{pat} = X \setminus \{x\}$, that is $X \setminus \{x\}$ is a patch, and consequently $\overline{X \setminus \{x\}}^{\mathcal{T}} = \uparrow (X \setminus \{x\})$ by [9, corollary 2.1].

But $ht(x) = 0$ implies that $\uparrow (X \setminus \{x\}) = X \setminus \{x\}$. Thus $\overline{X \setminus \{x\}}^{\mathcal{T}} = X \setminus \{x\}$. Then $\{x\}$ is an open subset of $(X, \mathcal{T})$, contradicting the fact that x is an accumulation point of $(X, \mathcal{T})$. Therefore, $Y \setminus Y_0 \subseteq X \setminus X_0 = X_1$.

Suppose that $Y \setminus Y_0$ is infinite, then it has an accumulation point in X_{pat} (since X_{pat} is a compact Hausdorff space). This accumulation point belongs to Y_0 and so $\overline{Y \setminus Y_0}^{pat} \neq Y \setminus Y_0$. Hence $\overline{Y \setminus Y_0}^{\mathcal{T}} \neq Y \setminus Y_0$. Let $S = X \setminus (Y \setminus Y_0)$. Thus $\overline{S}^{\mathcal{T}} \supseteq \uparrow S$. But $Y \setminus Y_0 \subseteq X_1$ implies that $\uparrow S = X$. It follows that $\overline{S}^{\mathcal{T}} \setminus S = X \setminus S = Y \setminus Y_0$ is a closed subset of $(X, \mathcal{T})$ (since $(X, \mathcal{T})$ is a submaximal space), contradicting the fact that $\overline{Y \setminus Y_0}^{\mathcal{T}} \neq Y \setminus Y_0$.

We conclude that $Y \setminus Y_0$ is finite, and thus Y is also finite. ■

From Proposition 2.2 and Theorem 3.3, one might easily suspect that a spectral space $(X, \mathcal{T})$ is a submaximal space iff X has finitely many accumulation points and $\dim_K(X, \mathcal{T}) \leq 1$. That this is not the case will be shown by Example 3.5. However, before proceeding to that example, we will give a class of topological spaces for which it is indeed the case that the property submaximal is equivalent to the fact that the space has a finite number of accumulation points and its Krull dimension is ≤ 1.

We first recall a specific topology on a poset as given by Lewis and Ohm in [14]. According to [14], for a poset $(X, \leq)$ and $m \in X$, the $\mathcal{C}_m$*-topology* on X has as a basis for the closed sets of the topology: (i) finite upper sets not containing m; and (ii) cofinite upper sets containing m.

A topology $\mathcal{T}$ on a poset $(X, \leq)$ is said to be *compatible* with the ordering $\leq$ providing that $\overline{\{x\}} = (\uparrow x)$, for each $x \in X$.

As shown in [14], $(X, \mathcal{C}_m)$ is a spectral space iff the topology $\mathcal{C}_m$ is compatible with the order iff $x \leq m$ whenever $(\uparrow x)$ is infinite. Furthermore, if $(X, \mathcal{C}_m)$ is a spectral space, then there are a number of consequences. For example, as shown in [14], the closed sets of $(X, \mathcal{C}_m)$ are the finite upper sets not containing m and the upper sets containing m. An immediate consequence of this fact is that, for any subset S of X containing m, the $\mathcal{C}_m$-closure of S is $\overline{S} = \uparrow S$, where $\uparrow S = \bigcup((\uparrow x) | \; x \in S)$. Of particular interest to us is the fact that if $x \in X_0 = \{z \in X | \; ht(z) = 0\}$ and $x \neq m$, then x is not an accumulation point. Indeed, $X \setminus \{x\}$ is an upper set containing m, that is $X \setminus \{x\}$ a closed subset of $(X, \mathcal{C}_m)$. Thus $\{x\}$ is open and consequently x is not an accumulation point. In particular, the set of accumulation points of X is contained in $(X \setminus X_0) \cup \{m\}$.

Theorem 3.4. *Let $(X, \leq)$ be a poset and $m \in Max(X)$. If $(X, \mathcal{C}_m)$ is a spectral space, then the following statements are equivalent:*

(*i*) *$(X, \mathcal{C}_m)$ is a submaximal space;*

(*ii*) *$X_1 = \{x \in X | \; ht(x) = 1\}$ is finite and $\dim_K(X, \mathcal{C}_m) \leq 1$.*

Proof. Suppose $(X, \mathcal{C}_m)$ is a submaximal space. Then, by Proposition 2.2, $\dim_K(X, \mathcal{C}_m) \leq 1$. Let $S = X \setminus X_1$. We consider two cases. First, $m \in S$. Then $\overline{S} = \uparrow S$, by the preceding remarks. But $\uparrow S = X$. Hence S is open, since X is a submaximal space. Thus X_1 is a closed subset of $(X, \mathcal{C}_m)$ not containing m, X_1 is finite. Second, let $m \notin S$. Let $S_1 = S \cup \{m\}$, then, S_1 is an open set of $(X, \mathcal{C}_m)$. Hence $X \setminus S_1 = X_1 \setminus \{m\}$ is a closed subset of $(X, \mathcal{C}_m)$ not containing m. Thus $X_1 \setminus \{m\}$ is a finite subset of X, and again X_1 is finite.

Conversely, let S be a subset of X such that $\overline{S} = X$. Once more, there are two cases to consider. First, $m \in S$. Then, by the preceding remarks, $\overline{S} = \uparrow S$ and, so, $\overline{S} \setminus S \subset X_1$. Since $\overline{S} \setminus S = X \setminus S$ is a finite upper subset of X not containing m, $X \setminus S$ is closed. That is, S is an open subset of $(X, \mathcal{C}_m)$. Second, $m \notin S$. According to the preceding remarks, each element of $X_0 \setminus \{m\}$ is open in $(X, \mathcal{C}_m)$, which, since S is dense in X, implies $X \setminus (S \cup \{m\}) \subseteq X_1$. Since $m \in Max(X)$, it follows that $X \setminus S$ is an upper set containing m and, consequently, S is an open subset of $(X, \mathcal{C}_m)$. Either way, we conclude that every dense subset of $(X, \mathcal{C}_m)$ is open, and thus X is a submaximal space. ■

By the remarks preceding Theorem 3.4, if, for $m \in Max(X)$, $(X, \mathcal{C}_m)$ is a spectral space, then it is a submaximal space iff it has finitely many accumulation points and $\dim_K(X) \leq 1$.

Example 3.5. There exists a spectral space with a finite number of accumulation points and Krull dimension ≤ 1, which is not a submaximal space.

For $X = \{m_i | \; 0 \leq i < \omega\} \cup \{m, n\}$, let $(X, \leq)$ denote the poset whose only non-trivial order is $m \leq n$. Consider the space $(X, \mathcal{C}_m)$. Since $(\uparrow x)$ is finite for any $x \in X$, the topology $\mathcal{C}_m$ is compatible with the order by the remarks preceding Theorem 3.4, and it follows that $(X, \mathcal{C}_m)$ is spectral. Clearly it has only a finite number of accumulation points (namely, m and n) and its Krull dimension is 1. Let $S = X \setminus \{m\}$. Then $\overline{S} = X$, but S is not open. In particular, $(X, \mathcal{C}_m)$ is not submaximal.

Observe too that the space $(X, \mathcal{C}_m)$ of Example 3.5 also shows that the hypothesis $m \in Max(X)$ in Theorem 3.4 cannot be dispensed with.

4. A-class spaces

Let $(X, \mathcal{T})$ be a topological space and $(\widetilde{X}, \widetilde{\mathcal{T}})$ its *Alexandroff extension* (that is, $\widetilde{X} = X \cup \{\omega\}$ and $\widetilde{\mathcal{T}}$ the topology on $\widetilde{X}$, whose members are the open sets of X and all subsets U of X such that $X \setminus U$ is a compact closed subset of X). For any particular class of spaces, call a topological space $(X, \mathcal{T})$ an *Alexandroff-class space* (*A-class space*, for short), if $(\widetilde{X}, \widetilde{\mathcal{T}})$ is a member of the specified class. For example, a topological space $(X, \mathcal{T})$ an A-spectral space providing $(\widetilde{X}, \widetilde{\mathcal{T}})$ is a spectral space.

In [4], the authors have discussed when the one-point compactification of a topological space is a spectral space, that is, when is a space an A-spectral space. Since a topological space is a Stone space iff it is a T_1-spectral space and since in addition the one-point compactification of $(X, \mathcal{T})$ is T_1 if and only if $(X, \mathcal{T})$ is T_1, the following Proposition 4.1 is an immediate consequence of the characterization given in [4].

We first recall some terminology from [4] and [8]. Let $(X, \mathcal{T})$ be a topological space and let U be a subset of X. Then U is said to be *intersection compact open* (ICO), if for each compact open subset V of X, $U \cap V$ is compact. Likewise, U is said to be *intersection compact closed* (ICC), if for each compact closed subset V of X, $U \cap V$ is compact. If U is both ICO and ICC, then it is said to be *intersection compact open closed* (ICOC).

Proposition 4.1. *A topological space $(X, \mathcal{T})$ is an A-Stone space iff the following properties hold:*

(*i*) *X is a T_1-space;*
(*ii*) *X has a basis of compact open sets closed under finite intersections;*
(*iii*) *X is sober (that is, every non-empty irreducible closed set has a generic point); and*
(*iv*) *for each compact closed subset U of X, there exists a co-compact and ICOC open subset V of X such that $V \subseteq X \setminus U$.*

A more natural characterization might be the following Theorem 4.2. First we introduce two concepts. A topological space $(X, \mathcal{T})$ is said to be *locally compact clopen* if each point is contained in a compact clopen set of X and T_{0CC} (T_0 *clopen compact*) if given $x \neq y$, there exists a clopen compact set that contains exactly one of them.

Proposition 4.2. *A topological space $(X, \mathcal{T})$ is an A-Stone space iff it is T_{0CC} and locally compact clopen.*

Proof. Let $(X,\mathcal{T})$ be an A-Stone space and $x \in X$. Since $\omega \neq x$, there exists a clopen set U of $\widetilde{X}$ such that $\omega \notin U$ and $x \in U$. Obviously U is compact. Hence U is a compact clopen subset of X and $x \in U$. Let x, y be two distinct points of X. Since $\widetilde{X}$ is Stone, there exists a clopen subset U of $\widetilde{X}$ such that $x \in U$ and $y \notin U$. Either $\omega \notin U$, in which case U is a compact clopen subset of X such that $x \in U$ and $y \notin U$, or else $\omega \in U$, in which case $V = \widetilde{X} \setminus U$ is a compact clopen subset of X such that $x \notin V$ and $y \in V$. Therefore, X is a T_{0CC}-space.

Let $(X,\mathcal{T})$ be locally compact clopen and T_{0CC}. Consider two distinct points x, $y \in \widetilde{X}$. If, say, $x = \omega$, then $y \in X$. Since X is locally compact clopen, there exists a clopen compact set V of X such that $y \in V$. Thus $U = (X \setminus V) \cup \{\omega\}$ is a clopen set of $\widetilde{X}$ such that $x \in U$ and $y \notin U$. Otherwise, $x \neq \omega$ and $y \neq \omega$. Since X is a T_{0CC}-space, there exists a clopen compact subset U of X such that, without loss of generality, $x \in U$ and $y \notin U$. Then U is a clopen set of $\widetilde{X}$ such that $x \in U$ and $y \notin U$. In particular, $(\widetilde{X},\widetilde{\mathcal{T}})$ is a Stone space. ■

Among other spaces considered here we have the following proposition:

Proposition 4.3. *A topological space $(X,\mathcal{T})$ is an A-door space iff every subset of X is either open or compact closed.*

Proof. Let $(X,\mathcal{T})$ be an A-door space and S be a subset of X. Since $\widetilde{X}$ is a door space, S is either an open or closed subset of $\widetilde{X}$. Then S is either an open or compact closed subset of X.

Conversely, let A be a subset of the one-point compactification $\widetilde{X} = X \cup \{\omega\}$ of X. If $\omega \notin A$, then A is a subset of X and, as such, A is either an open or compact closed subset of X, in which case, A is either an open or closed subset of $\widetilde{X}$. Otherwise $\omega \in A$ and $\omega \notin S = \widetilde{X} \setminus A = X \setminus A$. In this case, S is either an open or compact closed subset of X and S is either open or closed subset of $\widetilde{X}$. So $A = \widetilde{X} \setminus S$ is either an open or closed subset of $\widetilde{X}$. ■

Proposition 4.4. *A topological space $(X,\mathcal{T})$ is an A-submaximal space iff every dense subset of X is open and co-compact.*

Proof. Let $(X,\mathcal{T})$ be an A-submaximal space and S a subset of X such that $\overline{S}^X = X$. Then $\overline{S}^X \cup \{\omega\} = \widetilde{X} = \overline{S}^{\widetilde{X}}$. Since X is A-submaximal, S is an open subset of $\widetilde{X}$ and thus an open subset of X. Since $S \cup \{\omega\}$ is a dense subset of $\widetilde{X}$, $S \cup \{\omega\}$ is an open subset of X and so $\widetilde{X} \setminus (S \cup \{\omega\}) = X \setminus S$ is a compact closed subset of X.

Conversely, let S be a dense subset of $\widetilde{X}$. If $\omega \notin S$, then $\overline{S}^X = X$. Therefore, S is an open subset of X and, therefore, of $\widetilde{X}$. If $\omega \in S$, then $S \setminus \{\omega\}$ is a dense subset of X. Therefore $S \setminus \{\omega\}$ is an open and co-compact subset of X and so

$\widetilde{X} \setminus S = X \setminus (S \setminus \{\omega\})$ is a compact closed subset of X. In particular, S is an open subset of $\widetilde{X}$. ■

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