# POLAROID OPERATORS AND GENERALIZED BROWDER–WEYL THEOREMS

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Abstract

A Banach space operator $T \in B(\mathcal{X})$ is polaroid (left polaroid) if isolated points of the spectrum (resp. isolated points $\lambda$ of the approximate point spectrum) of $T$ are poles of the resolvent of $T$ (resp. are such that $(T-\lambda I)$ has finite ascent $\leq d$ and $(T-\lambda I)^{d+1}\mathcal{X}$ is closed). Necessary and sufficient conditions for operators $T \in B(\mathcal{X})$ to satisfy generalized and $a$-generalized Browder and Weyl theorems are given. In the case of polaroid (resp. left polaroid) operators $T$, it is proved that $T$ satisfies generalized Weyl's theorem (resp. generalized $a$–Weyl's theorem) if and only if $T$ satisfies Weyl's theorem (resp. $a$–Weyl's theorem).

## 1. Introduction

A Banach space operator $T \in B(\mathcal{X})$ is *polaroid* if the isolated points of the spectrum of $T$, points $\lambda \in \text{iso}\,\sigma(T)$, are poles of the resolvent of $T$. Let $\Pi(T)$ denote the set of poles of the resolvent of $T$. A necessary and sufficient condition for $\lambda \in \Pi(T)$ is that $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$, where $T-\lambda = T-\lambda I$, and the *ascent of* $T$, $\text{asc}(T)$ (resp. *descent of* $T$, $\text{dsc}(T)$), is the least non-negative integer $n$ such that $T^{-n}(0) = T^{-(n+1)}(0)$ (resp. $T^n\mathcal{X} = T^{n+1}\mathcal{X}$). $T$ has *the single-valued extension property* (SVEP) at a point $\lambda_0 \in \mathbf{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at $\lambda_0$ the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(T-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$. The single valued extension property plays an important role in local spectral theory and Fredholm theory (see [1; 29]). Evidently, every $T$ has SVEP at points in the resolvent $\rho(T) = \mathbf{C} \setminus \sigma(T)$ or the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. It is easily verified that SVEP is inherited by restrictions, and that if $T$ has SVEP and $TX = XY$ for some injection $X$, then $Y$ has SVEP.

An operator $T \in B(\mathcal{X})$ is *Weyl* (resp. *Browder*) if it is Fredholm of index 0 (resp. Fredholm of finite ascent and descent). The *Weyl spectrum* $\sigma_W(T)$ (resp. *Browder spectrum* $\sigma_B(T)$) of $T$ is the set of $\lambda \in \mathbf{C}$ such that $T-\lambda$ is not Weyl (resp. $\lambda \in \mathbf{C}$ such that $T-\lambda$ is not Browder). Let $\text{iso}\,\sigma(T)$ and $\Pi_0(T)$ denote, respectively, the set of isolated points of $\sigma(T)$ and the set of poles of finite rank of the resolvent of $T$. We say that $T$ satisfies *Weyl's theorem* ($Wt$ for short) if $\sigma(T) \setminus \sigma_W(T) = E_0(T)$; $T$ satisfies $a$–Weyl's theorem ($a-Wt$ for short) if $\sigma_a(T) \setminus \sigma_{Wa}(T) = \Pi_0^a(T)$; and

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$T$ satisfies *Browder's theorem* ($Bt$ for short) if $\sigma_B(T) = \sigma_W(T)$. Here $E_0(T) = \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(T-\lambda) = \dim(T-\lambda)^{-1}(0) < \infty\}$, $\sigma_a(T)$ is the approximate point spectrum of $T$, $\sigma_{Wa}(T)$ is the Weyl approximate point spectrum of $T$ [1, p. 151] and $\Pi_0^a(T)$ is the set of left poles (see definition below) of $T$ of finite rank. More generally, $T$ satisfies *generalized Weyl's theorem*, or $gWt$ (resp. *generalized Browder's theorem*, or $gBt$), if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$ (resp. $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$), where $E(T) = \{\lambda \in \mathbf{C} : \lambda \in \text{iso}\,\sigma(T), 0 < \alpha(T-\lambda)\}$ and $\sigma_{BW}(T)$ is the set of complex numbers $\lambda$ for which $T-\lambda$ fails to be '$B$–Weyl'. Berkani [7] has called an operator $T \in B(\mathcal{X})$ '$B$–Fredholm', $T \in \Phi_{BF}(\mathcal{X})$, if there exists a natural number $n$, $n \in \mathbf{N}$, for which the induced operator $T_n : T^n(\mathcal{X}) \longrightarrow T^n(\mathcal{X})$ is Fredholm in the usual sense, and '$B$-Weyl', $T \in \Phi_{BW}(\mathcal{X})$, if in addition $T_n$ has index 0. The implication $gWt \Longrightarrow Wt$ holds, but the reverse implication fails [10].

An operator $T$ is *semi B–Fredholm*, $T \in \Phi_{SBF}(\mathcal{X})$, if $T^n(\mathcal{X})$ is closed for some $n \in \mathbf{N}$ and the induced operator $T_n$ is either *upper semi–Fredholm* or *lower semi–Fredholm* (in the usual sense) [10]. For a $T \in \Phi_{SBF}(\mathcal{X})$, the index of $T$ is defined by $\text{ind}(T) = \text{ind}(T_d)$, where $d \in \mathbf{N}$ is *the degree of stable iteration* of $T$ (see [10, definition 2.2]). Let

$$\Phi_{SBF_+^-}(T) = \{T \in \Phi_{SBF}(\mathcal{X}) : T \text{ is upper B-Fredholm with } \text{ind}(T) \leq 0\},$$

and let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi_{SBF_+^-}(\mathcal{X})\}.$$

We say that a point $\lambda \in \sigma_a(T)$ is a *left pole* (resp. *left pole of finite rank*) of $T$, denoted $\lambda \in \Pi^a(T)$ (resp. $\lambda \in \Pi_0^a(T)$), if $T - \lambda \in LD(\mathcal{X})$ (resp. $T - \lambda \in LD(\mathcal{X})$ and $\alpha(T-\lambda) < \infty$), where $LD(\mathcal{X})$ is the *regularity*

$$LD(\mathcal{X}) = \{T \in B(\mathcal{X}) : d = \text{asc}(T) < \infty \text{ and } T^{d+1}(\mathcal{X}) \text{ is closed}\}.$$

The (left Drazin) spectrum induced by the regularity $LD$ will be denoted by $\sigma_{LD}(.)$. Evidently, $\Pi^a(T) = \{\lambda \in \sigma_a(T) : \text{asc}(T-\lambda) = d < \infty, (T-\lambda)^{d+1}\mathcal{X} \text{ is closed}\}$. Let $E^a(T) = \{\lambda \in \text{iso}\,\sigma_a(T) : 0 < \alpha(T-\lambda)\}$. Following Berkani and Koliha [10] we say that $T$ satisfies *the generalized a–Weyl's theorem*, or $a - gWt$ (resp. generalized *a–Browder's theorem*, or $a - gBt$), if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$ (resp. $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T)$). The following implications hold ([6; 10]): $a - gWt \Longrightarrow a - gBt \Longrightarrow gBt \Longleftrightarrow Bt$ and $a - gWt \Longrightarrow gWt \Longrightarrow Wt$.

This paper considers necessary and sufficient conditions for operators $T \in B(\mathcal{X})$ to satisfy generalized Browder and Weyl theorems, with attention on polaroid and left polaroid operators (i.e., operators $T$ for which points $\lambda \in \sigma_a(T)$ are left poles). It is proved that a necessary and sufficient condition for $T \in B(\mathcal{X})$ to satisfy: (i) $gWt$ is that $T$ has SVEP at points $\lambda \notin \sigma_{BW}(T)$ and $T$ is polaroid at points $\lambda \in E(T)$; (ii) $a - gWt$ is that $T$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$ and $T$ is left polaroid at points $\lambda \in E^a(T)$. Furthermore, if $T$ is polaroid and $T$ has SVEP at points $\lambda \notin \sigma_{BW}(T)$ (resp. $T$ is left polaroid and $T$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$), then $f(T)$ satisfies $gWt$ (resp. $f(T)$ satisfies $a - gWt$) for every $f \in H(\sigma(T))$ (resp.

every $f \in H_c(\sigma(T))$). (*Here $H(\sigma(T))$ is the set of functions $f$ that are analytic on an open neighbourhood of $\sigma(T)$, and $H_c(\sigma(T)) = \{f \in H(\sigma(T)) : f$ is non-constant on each of the components of the set on which it is defined$\}$.*) We also prove that a polaroid operator $T$ satisfies $gWT$ (resp. $a - gWt$) if and only if $T$ satisfies $Wt$ (resp. $a - Wt$). The class of polaroid operators is large. It contains, among other classes, the class of operators for which $H_0(T-\lambda) = (T-\lambda)^{-m_\lambda}(0)$ for some positive integer $m_\lambda$ at every $\lambda \in \mathbf{C}$, considered by Oudghiri [32] and Aiena [2], and the class $\mathcal{CHN}$ considered by the author in [18; 19]. We shall say more about these classes in a later section, noting at this point in time simply that a large number of the more commonly considered classes of operators satisfy $gWt$ and $a - gWt$. In particular, our results generalize corresponding results from [2; 8; 9; 12–16; 18; 19; 21; 25; 26; 28; 31].

## 2. Further notation

In addition to the notation (and terminology) already introduced, we shall use $\sigma_p(T)$ to denote the point spectrum of $T$. Recall that $T$ is said to be *isoloid* if the isolated points of $\sigma(T)$ are eigenvalues of $T$. We shall denote the set of semi–Fredholm points of an operator $T$ by $\Phi_{SF}(T)$, the semi-group of semi–Fredholm operators by $\Phi_{SF}(\mathcal{X})$ and the set of *upper semi–Fredholm* (resp. *lower semi–Fredhom*) operators by $\Phi_{SF_+}(\mathcal{X})$ (resp. $\Phi_{SF_-}(\mathcal{X})$). Let $\Phi_{SF_+^-}(\mathcal{X}) = \{T \in \Phi_{SF_+}(\mathcal{X}) : \text{ind}(T) \le 0\}$ (resp. $\Phi_{SF_-^+}(\mathcal{X}) = \{T \in \Phi_{SF_-}(\mathcal{X}) : \text{ind}(T) \ge 0\}$. We say that $T$ has *uniform descent for* $n \ge d \in \mathbf{N}$, if $R(T) + T^{-n}(0) = R(T) + T^{-d}(0)$ for all $n \ge d$. If, in addition, $R(T) + T^{-d}(0)$ is closed, then $T$ is said to have *topological uniform descent* for $n \ge d$. Evidently, if either of the deficiency indices $\alpha(T)$ and $\beta(T)$ $(= \dim(\mathcal{X}/T\mathcal{X}))$ or the chain lengths $\text{asc}(T)$ and $\text{dsc}(T)$ is finite, then $T$ has uniform descent. An operator $T$ is Drazin invertible if both $\text{asc}(T)$ and $\text{dsc}(T)$ are finite [11]. If $\lambda \in \Pi(T)$, then $T - \lambda$ is Drazin invertible, and hence B–Fredholm.

We say that $T$ has SVEP if it has SVEP at every $\lambda \in \mathbf{C}$. The *quasinilpotent part* $H_0(T - \lambda)$ and the *analytic core* $K(T - \lambda)$ of $(T - \lambda)$ are defined by

$$H_0(T-\lambda) = \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} ||(T-\lambda)^n x||^{\frac{1}{n}} = 0\}$$

and

$$K(T-\lambda) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0$$
$$\text{for which } x = x_0, (T-\lambda)x_{n+1} = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, ...\}.$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T - \lambda)$ such that $(T - \lambda)^{-p}(0) \subseteq H_0(T - \lambda)$ for all $p = 0, 1, 2, ...$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [1].

Let $\sigma_s(T)$ denote the *surjectivity spectrum* of $T$, $RD(\mathcal{X})$ the regularity $\{T \in B(\mathcal{X}) : \text{dsc}(T) = d < \infty, T^d\mathcal{X} \text{ is closed}\}$, $\Pi^s(T) = \{\lambda \in \sigma_s(T) : T - \lambda \in RD(\mathcal{X})\}$ the set of *right poles* of $T$, $\Phi_{SBF_-^+}(\mathcal{X})$ the set of $T \in \Phi_{SBF}(\mathcal{X})$ that is lower semi B–Fredholm with $\text{ind}(T) \ge 0$, and let $\sigma_{SBF_-^+}(T) = \{\lambda : T - \lambda \notin \Phi_{SBF_-^+}(\mathcal{X})\}$.

Then, it follows from a straightforward argument that $\Pi(T) = \Pi^a(T) \cap \Pi^s(T)$ and $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_-^+}(T) = \sigma_{SBF_-^+}(T^*) \cup \sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T^*)$.

## 3. Results

Before going on to our main result we state a number of complementary results, some of them well known. Recall from [1, corollary 2.45] that if $T \in B(\mathcal{X})$ (resp. $T^* \in B(\mathcal{X}^*)$) has SVEP, then $\sigma(T^*) = \sigma_a(T^*) = \sigma_s(T)$ (resp. $\sigma(T) = \sigma_a(T) = \sigma_s(T^*)$). If $\mathrm{dsc}(T-\lambda) < \infty$, then $T$ has SVEP at $\lambda \Longleftrightarrow \mathrm{asc}(T-\lambda) < \infty \Longleftrightarrow \lambda \in \Pi(T) \Longrightarrow \lambda \in \mathrm{iso}\, \sigma(T)$ [1, theorem 3.81]. The following lemma is known, and we include a short proof for the reader's convenience.

**Lemma 3.1.** *If $\lambda \in \Pi^a(T)$, then $T - \lambda$ is of topological uniform descent, $\lambda \in$ iso $\sigma_a(T)$ and $\lambda \notin \sigma_{SBF_+^-}(T)$.*

Proof. We may assume, without loss of generality, that $\lambda = 0$. If $\mathrm{asc}(T) = d < \infty$ and $T^{d+1}\mathcal{X}$ is closed, then for all $n \geq d$, $T\mathcal{X} + (T)^{-d}(0) = T\mathcal{X} + (T)^{-n}(0)$ is closed [30, proposition 4.10.4]; hence $T$ is of topological uniform descent. Assume now that $0 \notin \mathrm{iso}\, \sigma_a(T)$. Then there exists a sequence $\{\lambda_n\} \subset \sigma_p(T)$ such that $\lambda_n \neq 0$ for all $n$ and $\lim_{n\longrightarrow\infty} \lambda_n = 0$. Since $T$ has topological uniform descent for $n \geq d$, there exists an $\epsilon > 0$ such that $(T-\mu)\mathcal{X}$ is closed for all $0 < |\mu| < \epsilon$; hence $\mu \in \sigma_a(T)$ if and only if $\mu \in \sigma_p(T)$. Furthermore, since $(T-\mu)^{-1}(0) \subseteq T^\infty\mathcal{X} (= \cap_{n=1}^\infty T^n\mathcal{X})$ for all $\mu \neq 0$, it follows that $\lambda_n \in \sigma_a(T_1)$, where $T_1 = T|_{T^\infty\mathcal{X}}$. Since $\sigma(T_1)$ is closed, $0 \in \sigma(T_1)$.

The hypothesis $\mathrm{asc}(T) < \infty \Longrightarrow T$ has SVEP at $\lambda \Longrightarrow T_1$ has SVEP at $\lambda$. Already $T_1$ is onto [24, theorem 3.4]; hence the fact that $T_1$ has SVEP at 0 implies that $T_1$ is

injective [1, corollary 2.24]; hence $0 \notin \sigma(T_1)$, a contradiction. Thus $0 \in \mathrm{iso}\, \sigma_a(T)$. To complete the proof, we observe that $\mathrm{asc}(T) = d < \infty \Longrightarrow \mathrm{ind}(T) \leq 0$ [1, theorem 3.4] and $(T)^n\mathcal{X} \cap (T)^{-1}(0) = \{0\}$ [30, lemma 4.10.1]. Hence $\alpha(T|_{T^d\mathcal{X}}) < \infty$ and $\mathrm{ind}(T) \leq 0$. Since $T^n\mathcal{X}$ is closed for all $n \geq d$, $0 \notin \sigma_{SBF_+^-}(T)$. ■

Using Banach space duality, Lemma 3.1 implies the following.

**Corollary 3.2.** *If $\lambda \in \Pi^s(T)$, then $\lambda \in$ iso $\sigma_s(T)$ and $\lambda \notin \sigma_{SBF_-^+}(T)$.*

**Lemma 3.3.** *If $T$ has SVEP at points $\lambda \in \Phi_{BF}(T)$, then $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for all $f \in H(\sigma(T))$.*

Proof. Use [7, theorem 3.4]. ■

It is known that the spectra induced by the regularities $LD(\mathcal{X})$ and $RD(\mathcal{X})$ satisfy the spectral mapping theorem for all analytic functions that are locally non-constant [31]: we prove in the following that a similar statement holds for $\sigma_{SBF_+^-}(T)$, provided that $T$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$. The following lemma is proved in [9, theorem 2.5].

**Lemma 3.4.** *If $T$ is an operator of topological uniform descent, then $T$ has SVEP at $0 \Longleftrightarrow asc(T) < \infty$.*

In addition, Lemma 3.4 implies that if $T$ is an operator of topological uniform descent, then $T^*$ has SVEP at $0 \Longleftrightarrow \mathrm{dsc}(T) < \infty$. The following lemmas gives a sufficient condition for $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.

**Lemma 3.5.** *If $T$ has SVEP at points $\lambda \notin \sigma_{SBF_-^+}(T)$ (resp. $T^*$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$), then $\sigma_{SBF_-^+}(T) = \sigma_{BW}(T)$ (resp. $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$).*

PROOF. If $\lambda \notin \sigma_{SBF_-^+}(T)$ (or $\lambda \notin \sigma_{SBF_+^-}(T)$), then $T - \lambda$ has topological uniform descent. Thus, if $T$ has SVEP at $\lambda$ (resp. $T^*$ has SVEP at $\lambda$), then $\mathrm{asc}(T-\lambda) < \infty$ (resp. $\mathrm{dsc}(T - \lambda) < \infty$). Hence, $\mathrm{ind}(T - \lambda) \leq 0$ (resp. $\mathrm{ind}(T - \lambda) \geq 0$). Since already $\mathrm{ind}(T - \lambda) \geq 0$ (resp. $\mathrm{ind}(T - \lambda) \leq 0$), $\mathrm{ind}(T - \lambda) = 0$ and $\lambda \notin \sigma_{BW}(T)$; hence $\sigma_{BW}(T) \subseteq \sigma_{SBF_-^+}(T)$ and, respectively, $\sigma_{BW}(T) \subseteq \sigma_{SBF_+^-}(T)$. The reverse inclusions being always true, the lemma is proved. ■

**Lemma 3.6.** *If $T$ has SVEP at points $\lambda \in \Phi_{SBF_-}(T)$ (resp. $T^*$ has SVEP at points $\lambda \in \Phi_{SBF_+}(T)$), then $f(\sigma_{SBF_-^+}(T)) = \sigma_{SBF_-^+}(f(T))$ (resp. $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$ for every $f \in H_c(\sigma(T))$).*

PROOF. It is consequent from Lemmas 3.3 and 3.5 that $f(\sigma_{SBF_-^+}(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ (resp. $f(\sigma_{SBF_+^-}(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$) for every $f \in H(\sigma(T))$. Recall from [1, theorem 2.39] that $f(T)$ has SVEP at a point $\lambda \notin \sigma_{SBF_-^+}(f(T))$ (resp. $\lambda \notin \sigma_{SBF_+^-}(f(T))$) if and only if $T$ has SVEP at every $\mu \notin \sigma_{SBF_-^+}(T)$ (resp. $\mu \notin \sigma_{SBF_+^-}(T)$) such that $f(\mu) = \lambda$. Hence, the hypothesis of the lemma implies that $f(T)$ has SVEP at points $\lambda \notin \sigma_{SBF_-^+}(f(T))$ (resp. $f(T^*) = f(T)^*$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(f(T))$). Arguing as in the proof of Lemma 3.5, this implies that $\sigma_{SBF_-^+}(f(T)) = \sigma_{BW}(f(T))$ (resp. $\sigma_{SBF_+^-}(f(T)) = \sigma_{BW}(f(T))$). This completes the proof. ■

Our next lemma is taken from [7, theorem 2.7]

**Lemma 3.7.** *If $T$ is B-Fredholm, then there exist (closed) subspaces $M$ and $N$ of $\mathcal{X}$ such that $\mathcal{X} = M \oplus N$ and $T = T_1 \oplus T_2$, where $T_1 = T|_M$ is Fredholm with $ind(T_1) = ind(T)$ and $T_2 = T|_N$ is nilpotent.*

It is an easy exercise to deduce from the above lemma that if $\lambda \in \text{iso}\,\sigma(T)$ and $T-\lambda \in \Phi_{BW}(T)$, then $\lambda \in \Pi(T)$. The following lemma says $T$ inherits the polaroid property from *algebraically polaroid operators* $p(T)$.

**Lemma 3.8.** *If $T$ is algebraically polaroid, i.e., $p(T)$ is polaroid for some (non-constant) polynomial $p(.)$, then $T$ is polaroid.*

Proof. If $\lambda \in \text{iso}\,\sigma(T)$, then $\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$ and $T-\lambda = (T-\lambda)|_{H_0(T-\lambda)} \oplus (T-\lambda)|_{K(T-\lambda)} = T_1 \oplus T_2$, where $\sigma(T_1) = \{0\}$. Trivially, $\sigma(p(T_1)) = \{p(0)\}$. If we let $p(T_1) - p(0) = cT_1^m g(T_1)$ for some constant $c$, positive integer $m$ and polynomial $g(.)$, then $g(T_1)$ is invertible. Consequently, $T_1$ is nilpotent $\Longrightarrow$ $H_0(T-\lambda) = (T-\lambda)^{-m}(0)$ for some positive integer $m$. Hence $\mathcal{X} = (T-\lambda)^{-m}(0) \oplus K(T-\lambda) \Longrightarrow (T-\lambda)^m\mathcal{X} = 0 \oplus (T-\lambda)^m K(T-\lambda) = K(T-\lambda) \Longrightarrow \mathcal{X} = (T-\lambda)^{-m}(0) \oplus (T-\lambda)^m\mathcal{X} \Longrightarrow \lambda \in \Pi(T)$. This completes the proof.
■

The following propositions give some necessary and sufficient conditions for an operator $T \in B(\mathcal{X})$ to satisfy $gBt$ and $a-gBt$. Let $\triangle(T) = \{\lambda : T-\lambda \in \Phi_{BW}(\mathcal{X}), 0 < \alpha(T-\lambda)\}$ and $\triangle^a(T) = \{\lambda : T-\lambda \in \Phi_{SBF_+^-}(\mathcal{X}), 0 < \alpha(T-\lambda)\}$.

**Proposition 3.9.** (a) *The following statements are equivalent:*
(i) *$T$ satisfies $gBt$;*
(ii) *$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$;*
(iii) *$T$ has SVEP at points $\lambda \notin \sigma_{BW}(T)$;*
(iv) *$\triangle(T) = \Pi(T)$; and*
(v) *$H_0(T-\lambda) = (T-\lambda)^{-d}(0)$, for some positive integer $d$, at points $\lambda \in \triangle(T)$.*

(*b*) *$T$ satisfies $gBt$ if and only if $T$ satisfies $Bt$.*

Proof. (a). (*i*) $\Longleftrightarrow$ (*ii*) is evident.

(*ii*) $\Longleftrightarrow$ (*iii*). If (ii) is satisfied, then $\lambda \notin \sigma_{BW}(T) \Longrightarrow \lambda \in \rho(T)$ (= the resolvent set of $T$) or $\lambda \in \text{iso}\,\sigma(T)$. In either case, $T$ has SVEP at $\lambda$. Conversely, $T$ has SVEP at $\lambda \notin \sigma_{BW}(T) \Longrightarrow T-\lambda$ has SVEP at 0, $T-\lambda \in \Phi_{BF}(\mathcal{X})$ and $\text{ind}(T-\lambda) = 0$. By Lemma 3.4, there exist closed subspaces $M$ and $N$ of $\mathcal{X}$ such that $\mathcal{X} = M \oplus N$ and $T-\lambda = T_1 \oplus T_2$, where $T_1 = (T-\lambda)|_M$ is Fredholm, $T_1$ has SVEP at 0, $\text{ind}(T_1) = 0$ and $T_2 = (T-\lambda)|_N$ is nilpotent. Hence there exists a positive integer $m$ such that $M = T_1^{-m}(0) \oplus T_1^m M$: this follows from an application of [1, theorem 3.16], which implies that $\text{asc}(T_1) = m < \infty$, followed by an application of [1, theorem 3.4(iv)], which implies that $\text{asc}(T_1) = \text{dsc}(T_1) = m < \infty$, followed by an application of [1, theorem 3.81]. Thus $\mathcal{X} = \{(T-\lambda)^{-m}(0) \oplus N\} \oplus (T-\lambda)^m\mathcal{X}$. If we let $d = \max\{m, m_1\}$, where $m_1$ is as in $N = T_2^{-m_1}(0)$, then $\mathcal{X} = (T-\lambda)^{-d}(0) \oplus (T-\lambda)^d\mathcal{X} \Longrightarrow \lambda \in \Pi(T) \Longrightarrow \sigma(T) \setminus \sigma_{BW}(T) \subseteq \Pi(T)$. Since $\lambda \in \Pi(T) \Longrightarrow (\lambda \in \text{iso}\,\sigma(T)$ and) $T-\lambda \in \Phi_{BW}(\mathcal{X})$, $\Pi(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$.

$(iii) \Longrightarrow (iv)$. Trivially, $\Pi(T) \subseteq \triangle(T)$. Let $\lambda \in \triangle(T)$. If $(iii)$ is satisfied, then ($T - \lambda \in \Phi_{BW}(T)$ and) $T$ has SVEP at $\lambda$, and this by the argument above (see $(iii) \Longrightarrow (ii)$) implies that $\lambda \in \Pi(T)$.

$(iv) \Longrightarrow (v)$. Evident (from the fact that $\lambda \in \triangle(T) \Longleftrightarrow \lambda \in \Pi(T)$).

$(v) \Longrightarrow (iii)$. If $\lambda \notin \sigma_{BW}(T)$, then $T - \lambda$ has finite ascent, hence SVEP.

(b). The implication that $T$ satisfies $gBt \Longrightarrow T$ satisfies $Bt$ is proved in [10]. In view of the equivalence of conditions (i) and (iii) of part (a), to prove the reverse implication it would suffice to prove that SVEP at points $\lambda \notin \sigma_W(T)$ implies SVEP at points $\lambda \notin \sigma_{BW}(T)$. Choose a large enough integer $n \in \mathbf{N}$. Let $\lambda \notin \sigma_{BW}(T)$. Then it follows from [24, theorem 4.7] that $V_n = T - \lambda - \frac{1}{n} \in \Phi(\mathcal{X})$ with $\text{ind}(V_n) = 0$ (equivalently, $\lambda - \frac{1}{n} \notin \sigma_W(T)$). Evidently, if $T$ has SVEP at points $\mu \notin \sigma_W(T)$, then $V_n$ has SVEP (at 0). Since $V_n$ commutes with $T - \lambda$, and since $V_n$ converges to $T - \lambda$ in the uniform topology, $T - \lambda$ has SVEP at 0 [1, theorem 2.11]. (This also follows from an application of [30, proposition 3.4.11].) Hence $T$ has SVEP at $\lambda$. ■

Proposition 3.9(b) answers a question of [10, problem 1], and was first proved in [6, theorem 2.1] using a slightly different argument. Since $\sigma(T) = \sigma(T^*)$, $\sigma_{BW}(T) = \sigma_{BW}(T^*)$ and $\Pi(T) = \Pi(T^*)$, $T^*$ satisfies $gBt$ if and only if $T^*$ satisfies $Bt$ if and only if $T$ satisfies $Bt$.

Recall from [1, p. 156] that $T$ satisfies $a - Bt$ if the Weyl approximate point spectrum $\sigma_{Wa}(T)$ of $T$ equals the upper semi-Browder spectrum of $T$. Equivalently, $T$ satisfies $a - Bt$ if $\sigma_a(T) \setminus \sigma_{Wa}(T) = \Pi_0^a(T)$: a necessary and sufficient condition for $T$ to satisfy $a - Bt$ is that $T$ has SVEP at points $\lambda \notin \sigma_{Wa}(T)$ [20, lemma 2.18].

**Proposition 3.10.** ($a$) *The following statements are equivalent.*

(i) $T$ *satisfies* $a - gBt$*;*

(ii) $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^a(T)$*;*

(iii) $T$ *has SVEP at points* $\lambda \notin \sigma_{SBF_+^-}(T)$*;*

(iv) $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$*;*

(v) $\triangle^a(T) = \Pi^a(T)$*; and*

(vi) $H_0(T-\lambda) = (T-\lambda)^{-d}(0)$*, for some positive integer* $d$*, at points* $\lambda \in \triangle^a(T)$*.*

($b$) $T$ *satisfies* $a - gBt$ *if and only if* $T$ *satisfies* $a - Bt$*.*

PROOF. (a). $(i) \Longleftrightarrow (ii)$ is evident.

$(ii) \Longrightarrow (iii)$. If (ii) is satisfied, then $\lambda \notin \sigma_{SBF_+^-}(T) \Longleftrightarrow \lambda \in \rho(T)$ or $\lambda \in \Pi^a(T)$. In either case, $T$ has SVEP at $\lambda$.

$(iii) \Longrightarrow (iv)$. Since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T)$, we prove the reverse inclusion. If $(iii)$ holds, then SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T) \Longrightarrow T - \lambda \in \Phi_{SBF_+^-}(\mathcal{X})$ and $T$ has SVEP at $\lambda \Longleftrightarrow T - \lambda \in \Phi_{SBF_+^-}(\mathcal{X})$ and $\text{asc}(T - \lambda) < \infty$ (Lemma 3.4) $\Longrightarrow \lambda \in \Pi^a(T) \Longrightarrow \sigma_{LD}(T) \subseteq \sigma_{SBF_+^-}(T)$.

$(iv) \Longrightarrow (v)$. If $(iv)$ is satisfied, then $\lambda \in \Pi^a(T) \Longleftrightarrow \lambda \notin \sigma_{LD}(T) = \sigma_{SBF_+^-}(T) \Longrightarrow \text{asc}(T-\lambda) = d < \infty$ for some positive integer $d$ and $\lambda \notin \sigma_{SBF_+^-}(T)$. Since $\alpha(T-\lambda) = 0 \Longrightarrow \text{asc}(T-\lambda) = 0$, $\lambda \in \triangle^a(T)$. Hence $\Pi^a(T) \subseteq \triangle^a(T)$. For the reverse inclusion, $\lambda \in \triangle^a(T) \Longleftrightarrow \lambda \notin \sigma_{SBF_+^-}(T)$ and $0 < \alpha(T-\lambda)$. But then $\lambda \notin \sigma_{LD}(T) \Longrightarrow \lambda \in \Pi^a(T)$.

$(v) \Longrightarrow (vi)$. Assume without loss of generality that $\lambda = 0 \in \triangle^a(T) = \Pi^a(T)$. Then $T \in \Phi_{SBF_+^-}(\mathcal{X})$ and $\text{asc}(T) = d < \infty$. Let $\hat{T} \in B(\mathcal{X}/T^{d^{-1}}(0))$ denote the mapping induced by $T$ on the quotient space $\hat{\mathcal{X}} = \mathcal{X}/T^{d^{-1}}(0)$, and let $x + T^{d^{-1}}(0) = \hat{x}$. Then $\hat{T}$ is injective and upper semi-$B$-Fredholm. Hence $\hat{T} \in \Phi_{SBF_+^-}(\hat{\mathcal{X}})$ with $\text{asc}(\hat{T}) = 0$, which implies that $H_0(\hat{T}) = \{\hat{0}\}$, and hence that $H_0(T) \subseteq T^{d^{-1}}(0)$. Since $T^{n^{-1}}(0) \subseteq H_0(T)$ for every positive integer $n$, $H_0(T) = T^{d^{-1}}(0)$.

$(vi) \Longrightarrow (ii)$. If $(vi)$ is satisfied, then $\text{asc}(T-\lambda) = d < \infty$ at points $\lambda \notin \sigma_{SBF_+^-}(T) \Longrightarrow \lambda \in \Pi^a(T)$. Hence $\sigma(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \Pi^a(T)$. Conversely, $\lambda \in \Pi^a(T) \Longrightarrow \lambda \notin \sigma_{LD}(T)$. Since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T)$, $\lambda \notin \sigma_{SBF_+^-}(T)$.

(b). The proof here is similar to that of Proposition 3.9 (b). The implication that $T$ satisfies $a-gBt \Longrightarrow T$ satisfies $a-Bt$ is proved in [10]. In view of the equivalence of conditions (i) and (iii) of part (a), to prove the reverse implication it would suffice to prove that SVEP at points $\lambda \in \Phi_{SBF_+^-}(T)$ implies SVEP at points $\lambda \in \Phi_{SF_+^-}(T)$. Choose a large enough integer $n \in \mathbf{N}$. Let $\lambda \in \Phi_{SBF_+^-}(T)$. Then it follows from [24, theorem 4.7] that $V_n = T - \lambda - \frac{1}{n} \in \Phi_{SF_+^-}(\mathcal{X})$. Evidently, if $T$ has SVEP at points $\mu \in \Phi_{SF_+^-}(T)$, then $V_n$ has SVEP (at 0). Since $V_n$ commutes with $T-\lambda$, and since $V_n$ converges to $T-\lambda$ in the uniform topology, $T-\lambda$ has SVEP at 0 [1, theorem 2.11]. Hence $T$ has SVEP at $\lambda \in \Phi_{SBF_+^-}(T)$. ■

*Remark 3.11.* Proposition 3.10(b) answers [10, problem 2], and was first proved in [6, theorem 2.2] using a slightly different argument. Parts of Proposition 3.9(a) are to be found in [4], and parts of Proposition 3.10(a) are to be found in [5].

Applying Banach space duality to statement (ii) of Proposition 3.10, we say that $T \in B(\mathcal{X})$ *satisfies* $s - gBt$ *if* $\sigma_s(T) \setminus \sigma_{SBF_-^+}(T) = \Pi^s(T)$. Evidently, $T$ satisfies $s-gBt$ if and only if $T^*$ satisfies $a-gBt$. (Thus, in the following, instead of discussing $s-gBt$ ( or $s-gWt$) we shall concentrate on $a-gBt$ (resp, $a-gWt$) for $T^*$.) Either of the conditions $a-gBt$ and $s-gBt$ implies $Bt$: this is obvious for $a-gBt$ ($\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T) \subseteq \sigma_W(T)$), and it follows for $s-gBt$ from the implications $T$ satisfies $s-gBt \Longrightarrow T^*$ satisfies $a-gBt \Longrightarrow T^*$ satisfies $gBt \Longrightarrow T^*$ satisfies $Bt \Longleftrightarrow T$ satisfies $Bt$. The following corollary, which gives a sufficient condition for $T$ to satisfy (both) $a-gBt$ and $s-gBt$, has the interesting consequence that $T^*$ has SVEP implies $T$ has SVEP at points $\lambda \in \Phi_{SBF_+^-}(T)$.

**Corollary 3.12.** *If $T^*$ has SVEP, then both $T$ and $T^*$ satisfy $a-gBt$.*

The proof of the corollary is an immediate consequence of the well known fact that $T^*$ has SVEP implies both $T$ and $T^*$ satisfy $a$–Bt, hence also $a$–gBt.

Recall from Lemma 3.5 that if $T^*$ has SVEP (at points $\lambda \notin \sigma_{SBF_+^-}(T)$), then $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. In particular, if $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T - \lambda$ is B-Fredholm, $\text{ind}(T-\lambda) = 0$ and $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$ (see Lemma 3.4). Combining this with *(iv)* of Proposition 3.10, we have the following corollary (see [5, theorem 2.9] for a similar result).

**Corollary 3.13.** *If $T^*$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$, then $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T) = \sigma_{BW}(T) = \sigma_D(T)$, where $\sigma_D(T) = \{\lambda \in \sigma(T) : \lambda \notin \Pi(T)\}$ is the Drazin spectrum of $T$.*

In the following we say that $T$ is polaroid (resp. left polaroid) on a subset $S$ of iso $\sigma(T)$ (resp. iso $\sigma_a(T)$) if the points of $S$ are poles (resp. left poles) of $T$. We are now ready for our main result.

**Theorem 3.14.**

(i) *$T$ satisfies $gWt$ if and only if $T$ has SVEP at points $\lambda \notin \sigma_{Bw}(T)$ and $T$ is polaroid at points $\lambda \in E(T)$.*

(ii) *$T$ satisfies $a - gWt$ if and only if $T$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$ and $T$ is left polaroid at points $\lambda \in E^a(T)$.*

(iii) *If $T$ has SVEP at points $\lambda \in \Phi_{BF}(T)$ and is polaroid, then $f(T)$ satisfies $gWt$ for all $f \in H(\sigma(T))$.*

(iv) *If $T$ has SVEP at points $\lambda \in \Phi_{SBF_+}(T)$ and is left polaroid, then $f(T)$ satisfies $a - gWt$ for all $f \in H_c(\sigma(T))$.*

(v) *Let $p(.)$ be some non-constant polynomial. If $p(T)$ has SVEP at points $\lambda \notin \sigma_{BW}(p(T))$ (resp. $p(T)$ has SVEP) and $p(T)$ is polaroid, then $f(T)$ satisfies $gWt$ for all $f \in H(\sigma(T))$ (resp. $f(T^*)$ satisfies $a - gWt$ for all $f \in H_c(\sigma(T))$).*

PROOF.

(i) If $T$ satisfies $gWt$, then $\lambda \in E(T) \Longrightarrow \lambda \in \text{iso}\,\sigma(T), T - \lambda \in \Phi_{BW}(\mathcal{X}) \Longrightarrow \lambda \in \Pi(T)$. Since $\Pi(T) \subseteq E(T)$ for every $T$, $gWt \Longrightarrow \Pi(T) = E(T) = \sigma(T) \setminus \sigma_{BW}(T) \Longrightarrow T$ satisfies $gBt$ and $E(T) = \Pi(T)$. Hence $T$ has SVEP at points $\lambda \notin \sigma_{BW}(T)$ (by Proposition 3.9) and $T$ is polaroid at points $\lambda \in E(T)$. Conversely, $T$ has SVEP at points $\lambda \notin \sigma_{BW}(T)$ implies $T$ satisfies $gBt$ (by Proposition 3.9), and $T$ is polaroid at points $\lambda \in E(T)$ implies $E(T) = \Pi(T)$. Hence $T$ satisfies $gWt$.

(ii) If $T$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$, then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^a(T)$ (Proposition 3.10) $\subseteq E^a(T)$. Thus, if $E^a(T) \subseteq \Pi^a(T)$, then $T$ satisfies $a - gWt$. Conversely, if $T$ satisfies $a - gWt$, then $T$ satisfies $a - gBt \Longrightarrow T$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$ (Proposition 3.10). Furthermore, since $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^a(T) = E^a(T)$, $\lambda \in E^a(T) \Longleftrightarrow \lambda \in \Pi^a(T)$.

(iii) The polaroid property implies that $T$ is *isoloid*, i.e., $\lambda \in \text{iso}\,\sigma(T) \Longrightarrow \lambda \in$

$E(T)$. A familiar argument, see for example the proof of [1, lemma 3.89], then shows that $f(\sigma(T) \setminus E(T)) = \sigma(f(T)) \setminus E(f(T))$. (We remark here that the hypothesis in [1, Lemma 3.89] that the isolated eigenvalues $E(T)$ have finite multiplicity is immaterial to our case.) Recall from Lemma 3.3 that $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$. Since $T$ satisfies $gWt$ by part (i), it follows that $\sigma(f(T)) \setminus E(f(T)) = f(\sigma(T) \setminus E(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$, i.e., $f(T)$ satisfies $gWt$.

(iv) The proof in this case is similar to that for part (iii). We note that $T$ is left polaroid implies $T$ is $a$–isoloid (i.e., $\lambda \in \text{iso}\,\sigma_a(T) \Longrightarrow \lambda \in E^a(T)$). Evidently, $\sigma_a(T)$ is compact, $f(\sigma_a(T)) = \sigma_a(f(T))$, and $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$ for every $f \in H_c(\sigma(T))$.

(v) The hypothesis $p(T)$ is polaroid implies that $T$ is polaroid (Lemma 3.8), and the hypothesis $p(T)$ has SVEP at points $\mu \notin \sigma_{BW}(p(T))$ implies $T$ has SVEP at points $\lambda \notin \sigma_{BW}(T)$ (see [30, theorem 3.3.9] or [1, theorem 2.40]). Hence $T$ satisfies $gWt$ and $T$ is polaroid. This, as in (iii) above, implies that $f(T)$ satisfies $gWt$ for every $f \in H(\sigma(T))$.

To prove the remaining half of (v), we start by observing that the hypothesis $p(T)$ has SVEP implies that $T$ has SVEP [30, theorem 3.3.9]. Hence $\sigma(T) = \sigma(T^*) = \sigma_a(T^*)$ [30, proposition 1.3.2], $\Pi(T) = \Pi(T^*) = \Pi^a(T^*)$ and $E(T^*) = E^a(T^*)$. The hypothesis $p(T)$ is polaroid implies that $T$ (and so also $T^*$) is polaroid (Lemma 3.8). Evidently, $\lambda \in \text{iso}\,\sigma(T) \Longleftrightarrow \lambda \in \text{iso}\,\sigma(T^*) = \text{iso}\,\sigma_a(T^*)$; hence $T^*$ is left polaroid (i.e., $\lambda \in \text{iso}\,\sigma_a(T) \Longrightarrow \lambda$ is a pole). We prove next that $T^*$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T^*)$; this would then imply, by $(iv)$, that $f(T^*)$ satisfies $a - gWt$ for every $f \in H_c(\sigma(T))$. Recall from Lemma 3.5 that if $T$ has SVEP, then $\sigma_{BW}(T) = \sigma_{SBF_-^+}(T) = \sigma_{SBF_+^-}(T^*)$. Evidently, $T$ satisfies $gBt$, i.e., $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. Hence $\sigma_a(T^*) \setminus \sigma_{SBF_+^-}(T^*) = \Pi^a(T^*)$, i.e., $T^*$ satisfies $a - gBc \Longleftrightarrow T^*$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T^*)$ (by Proposition 3.10). ■

**Corollary 3.15.** *If $T^*$ has SVEP, then $f(T)$ satisfies $a - gWt$ for every $f \in H(\sigma(T))$ if and only if $E(T) = \Pi(T)$.*

Proof. $T^*$ has SVEP implies $\sigma(T) = \sigma_a(T)$ [30, proposition 1.3.2]; hence $E^a(T) = E(T)$ and $\Pi^a(T) = \Pi(T)$. Observe that if $\lambda \notin \sigma_{SBF_+^-}(T)$, then there exists a positive integer $n$ such that the induced operator $T_n = (T - \lambda)|_{(T-\lambda)^n \mathcal{X}}$ is upper semi-Fredholm and $\text{ind}(T_n) = \text{ind}(T-\lambda) \leq 0$. Since $T^*$ has SVEP implies $\text{dsc}(T-\lambda) < \infty$ [9, corollary 3.5], $\text{ind}(T-\lambda) \geq 0$; we conclude that $\text{ind}(T-\lambda) = 0$ and $\lambda \notin \sigma_{BW}(T)$. This, since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T)$ for every $T$, implies that $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Hence $T$ satisfies $gWt$ implies $T$ satisfies $a - gWt$. The proof then follows from an application of Theorem 3.14 (i) and (iii). ■

Recall from [24, corollary 4.9] that points $\lambda$ in the boundary of the spectrum of an operator $T$ such that $T - \lambda$ has topological uniform descent are poles of the resolvent of $T$. Evidently, $E(T) = \Pi(T)$ if and only if $T - \lambda$ has uniform topological descent at every $\lambda \in E(T)$. Hence:

**Corollary 3.16.** *$T \in B(\mathcal{X})$ satisfying $gBt$ satisfies $gWt$ if and only if $T - \lambda$ has uniform topological descent at points $\lambda \in E(T)$.*

The one-way implications $gWt \Longrightarrow Wt$ and $a - gWt \Longrightarrow a - Wt$ hold [10]. The reverse implications, however, are likely to fail, as the following example shows. Let $Q \in B(\ell^2)$ be the quasi-nilpotent $Q(x_0, x_1, x_2, ...) = (\frac{1}{2}x_1, \frac{1}{3}x_2, ...)$ and $N \in B(\ell^2)$ be a nilpotent. Let $T = Q \oplus N$. Then $\sigma_W(T) = \sigma_{BW}(T) = \sigma_{SBF_+^-}(T) = \{0\} = \sigma(T) = \sigma_a(T) = E(T) = E^a(T)$ and $E_0(T) = E_0^a(T) = \emptyset$, which implies that $T$ satisfies $a - Wt$ (so also $Wt$) but fails to satisfy $gWt$ (so also $a - gWt$). The situation for polaroid operators is different.

**Theorem 3.17.** *If $T$ is polaroid (resp. left polaroid), then $T$ satisfies $gWT$ (resp. $a - gWt$) if and only if $T$ satisfies $Wt$ (resp. $a - Wt$).*

PROOF. We have to prove the implications $Wt \Longrightarrow gWt$ and $a - Wt \Longrightarrow a - gWt$. Evidently, if $T$ satisfies $Wt$ (resp. $a - Wt$), then $T$ satisfies $Bt$ (resp. $a - Bt$). Applying Proposition 3.9 (b) (resp. Proposition 3.10 (b)), it follows that $T$ satisfies $gBt$ (resp. $a - gBt$). Thus $\sigma(T) \backslash \sigma_{BW}(T) = \Pi(T) \subseteq E(T)$ (resp. $\sigma_a(T) \backslash \sigma_{SBF_+^-}(T) = \Pi^a(T) \subseteq E^a(T)$). Since the polaroid (resp. left polaroid) assumption implies that $E(T) \subseteq \Pi(T)$ (resp. $E^a(T) \subseteq \Pi^a(T)$), $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$ (resp. $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$). ■

Observe that for polaroid operators $T$ satisfying $gWt$, $E(T) = \Pi(T) = \Pi(T^*) = E(T^*)$. Hence, for a polaroid operator $T$, $T^*$ satisfies $gWt$ if and only if $T$ satisfies $gWt$ if and only if $T$ satisfies $Wt$.

## 4. Examples and applications

The hypotheses of Theorem 3.14 are satisfied by a large number of the commonly considered classes of operators. In this section we list some, by no means all, of these classes of operators. (Note that $\mathcal{H}$ shall denote a Hilbert space.)

The class $H(p)$ of operators $T \in B(\mathcal{X})$ for which there exists a positive integer $p_\lambda$ for every $\lambda \in \mathbf{C}$ such that $H_0(T - \lambda) = (T - \lambda)^{-p_\lambda}(0)$ is (obviously) polaroid and operators $T \in H(p)$ have SVEP. $H(p)$ contains, among other classes, the classes consisting of $p$-hyponormal operators ($T \in B(\mathcal{H}) : |T^*|^{2p} \leq |T|^{2p}$ for some $0 < p \leq 1$), $M$-hyponormal operators ($T \in B(\mathcal{H}) : |(T - \lambda)^* x|^2 \leq M|(T - \lambda)x|^2$ for some $M \geq 1$, all $\lambda \in \mathbf{C}$ and $x \in \mathcal{H}$), totally $*$-paranormal operators ($T \in B(\mathcal{H}) : ||(T - \lambda)^* x||^2 \leq ||(T - \lambda)^2 x||$ for every unit vector $x \in \mathcal{H}$ and all $\lambda \in \mathbf{C}$), totally paranormal operators ($T \in B(\mathcal{X}) : ||(T - \lambda)x||^2 \leq ||(T - \lambda)^2 x||$ for every unit vector $x \in \mathcal{H}$ and all $\lambda \in \mathbf{C}$), transaloid operators ($T \in B(\mathcal{X}) : ||T - \lambda||$ equals the spectral radius $r(T - \lambda)$ for all $\lambda \in \mathbf{C}$), generalized scalar and subscalar operators, and multipliers of commutative semi-simple Banach algebras (see [1, pp 170–1 and

175–6] and [20; 25; 31]). The restriction of an $H(p)$ operator to an invariant subspace is again $H(p)$; see [1, theorem 3.99] and [32].

**Theorem 4.1.** *If $T \in H(p)$, then $f(T)$ satisfies $gWt$ for all $f \in H(\sigma(T))$ and $f(T^*)$ satisfies $a - gWt$ for all $f \in H_c(\sigma(T))$.*

Proof. $f(T) \in H(p)$ for all $f \in H(\sigma(T))$ [1, theorem 3.102]. Hence $f(T)$ is polaroid and has SVEP. Now apply Theorem 3.14. ■

Theorem 4.1 generalizes and in many cases extends to $gWt$ and $a - gWt$, [8, theorem 2.4 and corollary 2.11; 2, theorem 4.4; 13, theorems 2.5 and 3.5; 14, theorems 2.4 and 3.2; 16, theorem 3.1; 24, theorem 2; 25, theorems 2.4 and 2.10; 26, theorem 2.7; 31, proposition 3.3 and corollary 3.6].

Another class of operators, independent of $H(p)$ (but with a non-empty intersection with $H(p)$), is the class of *completely hereditarily normaloid* or $\mathcal{CHN}$ operators [20]. Recall that an operator $T \in B(\mathcal{X})$ is *hereditarily normaloid*, $T \in HN$, if every part of $T$ (i.e., the restriction of $T$ to each of its invariant subspaces) is normaloid (i.e., $||T||$ equals the spectral radius $r(T)$); $T \in HN$ is *totally hereditarily normaloid* if also the inverse of every invertible part of $T$ is normaloid, and $T$ is a $\mathcal{CHN}$ operator if either $T$ is totally hereditarily normaloid or $T - \lambda \in HN$ for every $\lambda \in \mathbf{C}$. Operators $T \in \mathcal{CHN}$ are polaroid [20, proposition 2.1]; furthermore, both $T$ and $T^*$ have SVEP at points $\lambda \in \Phi_{SF}(T)$ [20, theorem 2.9].

**Theorem 4.2.** *If $T \in \mathcal{CHN}$, then $T$ and $T^*$ satisfy $gWt$. If also $\lambda \in iso\ \sigma_s(T) \Longrightarrow \lambda \in iso\ \sigma(T)$, then $T^*$ satisfes $a - gWt$.*

Proof. That $T$ and $T^*$ satisfy $gWt$ is a direct consequence of [19, corollary 2.16, proposition 2.1] and Theorem 3.17. Assume now that $\lambda \in \text{iso } \sigma_s(T) = \text{iso } \sigma_a(T^*) \Longrightarrow \lambda \in \text{iso } \sigma(T) = \text{iso } \sigma(T^*)$.

Then $T$ polaroid $\Longrightarrow T^*$ polaroid $\Longrightarrow T^*$ left polaroid. In particular, $E^a(T^*) \subseteq \Pi^a(T^*)$. Thus to prove that $T^*$ satisfies $a - gWt$, it would suffice to prove that $T^*$ satisfies $a - gBt$. To this end, we prove that $T^*$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T^*) = \sigma_{SBF_-^+}(T)$. If $\lambda \in \Phi_{SBF_-^+}(T)$, then there exists an $\epsilon > 0$ such that $T-\mu \in \Phi_{SF_-^+}(\mathcal{X})$ and $\text{ind}(T-\lambda) = \text{ind}(T-\mu)$ for all $0 < |\mu-\lambda| < \epsilon$ [7, corollary 3.2]. Since $T$ has SVEP at points $\mu \in \Phi_{SF}(T)$ [20, theorem 2.9], $\text{asc}(T-\mu) < \infty$ (Lemma 3.4) $\Longrightarrow \text{ind}(T-\mu) \leq 0$, which (since $\text{ind}(T-\mu) \geq 0$) implies that $\text{ind}(T-\mu) = 0$. By the continuity of the index, this then implies that $\text{ind}(T - \lambda) = 0$, and hence that $\lambda \notin \sigma_{BW}(T)$. Thus $\sigma_{BW}(T) \subseteq \sigma_{SBF_-^+}(T)$. Since the reverse inclusion holds for every $T^*$, we conclude that $\sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T)$. It is apparent from the first part of the proof that if $\lambda \notin \sigma_{BW}(T)$, then $\lambda \in \Pi(T)$ or $\lambda$ is in the resolvent set of $T$. Since, in either case, both $T$ and $T^*$ have SVEP at $\lambda$, we conclude that $T^*$ has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T^*)$. ■

An important class of operators in $\mathcal{CHN}$ that has been studied by a number of authors over the years, is that of paranormal operators: $T \in B(\mathcal{X}) : ||Tx||^2 \leq ||T^2x||$ for each unit vector $x \in \mathcal{X}$ (see, for example, [3; 12; 14] and [30, pp 269–70]). Paranormal operators are not $H(p)$ operators [3]. If $\mathcal{X}$ is a Hilbert space or a separable Banach space, then paranormal operators are known to have SVEP; see [3] and [13, corollary 2.10]. Observe that if a polaroid operator $T$ has SVEP, then $\sigma(T) = \sigma_a(T^*) = \sigma_s(T)$ and $E(T^*) = E^a(T^*) = \Pi(T^*) = \Pi^a(T^*)$. Our Theorems 3.14 and 4.2 generalize [12, theorem 2.12; 17, theorem 3.1; 18, theorems 3.9 and 3.11; 19, theorems 2.17 and 2.19].

A Hilbert space operator $T \in B(\mathcal{H})$ is $(p,k)$-quasihyponormal, $T \in (p,k)-Q$, if $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$ for some positive integer $k$ and $0 < p \leq 1$ [28; 33]. (The class $(p,k)-Q$ does not fit into either of the classes $H(p)$ and $\mathcal{CHN}$.) Evidently, a $(1,1)-Q$ operator is quasihyponormal, a $(1,k)-Q$ operator is $k$-quasihyponormal, and (if we momentarily allow $k = 0$, then) a $(p,0)-Q$ operator is $p$-hyponormal. $(p,k)-Q$ operators are polaroid [34, theorem 6], and have finite ascent (hence, SVEP)[21]. Hence $(p,k)-Q$ operators are polaroid. Applying Theorem 3.14, we have:

**Theorem 4.3.** *If $T \in (p,k)-Q$ or $q(T) \in (p,k)-Q$ for some non-constant polynomial $q(.)$, then $f(T)$ satisfies $gWt$ for every $f \in H(\sigma(T))$ and $f(T^*)$ satisfies $a-gWt$ for every $f \in H_c(\sigma(T))$.*

Theorem 4.3 generalizes [20, corollary 3.6; 28, theorem 10, corollary 11 and corollary 13].

Yet another class of polaroid operators, which does not seem to have attracted much attention in the recent past, is that of operators which satisfy *a local growth condition.* An operator $T \in B(\mathcal{X})$ is said to satisfy a *growth condition of order* $m$, or to be a $(G_m)$-operator, if there exists a constant $K > 0$ such that

$$||(T-\lambda)^{-1}|| \leq \frac{K}{[\text{dist}(\lambda, \sigma(T))]^m}$$

for all $\lambda \notin \sigma(T)$. For an arbitrary closed subset $F$ of the set $\mathbf{C}$ of complex numbers and $T \in B(\mathcal{X})$, let $X_T(F) = \{x \in \mathcal{X} : (T-\lambda)f_x(\lambda) \equiv x$ for some analytic function $f_x : \mathbf{C} \setminus F \longrightarrow \mathcal{X}\}$. The *glocal spectral subspace* $X_T(F)$ is a hyper-invariant linear manifold of $T$ [30, p. 220]. Let $m$ be a positive integer. We say that $T \in \text{loc}(G_m)$ (or, $T \in B(\mathcal{X})$ satisfies a local growth condition of order $m$) if for every closed set $F \subset \mathbf{C}$ and every $x \in X_T(F)$ there exists an analytic function $f : \mathbf{C} \setminus F \longrightarrow \mathcal{X}$ such that $(T-\lambda)f(\lambda) \equiv x$ and $||f(\lambda)|| \leq K[\text{dist}(\lambda, F)]^{-m}||x||$ for some $K > 0$ (independent of $F$ and $x$). Hyponormal operators are $\text{loc}(G_1)$ [33, 28] and spectral operators of type $m-1$ are $\text{loc}(G_m)$ [23, proof of theorem XV.6.7]. Evidently, $T \in \text{loc}(G_m) \Longrightarrow T \in (G_m)$; since $(G_m)$ operators are polaroid, see for example [16, Llemma 3], $\text{loc}(G_m)$ operators are polaroid. It is known, [28, proposition 2], that if the Banach space $\mathcal{X}$ is reflexive (in particular, a Hilbert space), then operators $T \in \text{loc}(G_m)$ satisfy Dunford's condition $(C)$, hence have SVEP [30, proposition 1.2.19]. Appealing to Theorem 3.14, we have the following.

**Theorem 4.4.** *Assume that the Banach space $\mathcal{X}$ is reflexive. If $T \in loc(G_m)$ or $q(T) \in loc(G_m)$ for some non-constant polynomial $q(.)$, then $f(T)$ satisfies $gWt$ for every $f \in H(\sigma(T))$ and $f(T^*)$ satisfies $a - gWt$ for every $f \in H_c(\sigma(T))$.*


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## References


[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer, 2004

[2] P. Aiena, Classes of operators satisfying $a$-Weyl's theorem, *Studia Mathematica* **169** (2005), 137–51.

[3] P. Aiena and J.R. Guillen, Weyl's theorem for perturbations of paranormal operators, *Proceedings American Mathematical Society* **135** (2007), 2443–51.

[4] P. Aiena and O. Garcia, Generalized Browder's theorem and SVEP, *Mediterranean Journal of Mathematics* **4** (2007), 215–28.

[5] P. Aiena and T.L. Miller, On generalized $a$-Browder's theorem, *Studia Mathematica* **180** (2007), 285–300.

[6] M. Amouch and H. Zguitti, 2006 On the equivalence of Browder's and generalized Browder's theorem, *Glasgow Mathematical Journal* **48**, 179–85.

[7] M. Berkani, On a class of quasi-Fredholm operators, *Integral equatations operator theory* **34** (1999), 244–9.

[8] M. Berkani and A. Arroud, =Generalized Weyl's theorem and hyponormal operators, *Journal of the Australian Mathematical Society* **76** (2004), 291–302.

[9] M. Berkani, G.N. Castro-Gonzalez and S. V. Djordjević, The single valued extension property, quasiaffinity and the generalised Weyl's theorem, *Mathematica Bohemica* **131** (2006), 29–38.

[10] M. Berkani and J.J. Koliha, Weyl type theorems for bounded linear operators, *Acta Scientiarum Mathmaticarum (Szeged)* **69** (2003), 359–76.

[11] S.R. Caradus, W.E. Pfaffenberger and Y. Bertram, *Calkin algebras and algebras of operators on Banach spaces*, New York, Marcel-Dekker, 1974

[12] M. Berkani and M. Sarih, On semi B-Fredholm operators, *Glasgow Mathematical Journal* **43** (2001), 457–65.

[13] N.N. Chourasia and P.B. Ramanujan, Paranormal operators on Banach spaces, *Bulletin Australian Mathematical Society* **21** (1980), 161–8.

[14] R.E. Curto and Y.M. Han, Weyl's theorem, $a$-Weyl's theorem and local spectral theory, *Journal of the London Mathematical Society* **67** (2003), 499–509.

[15] R.E. Curto and Y.M. Han, Weyl's theorem for algebraically paranormal operators, *Integral equation operator theory* **47** (2003), 307–14.

[16] S.V. Djordjević, Weyl's theorem through local spectral theory, Glasgow Mathematical Journal **44** (1999), 323–7.

[17] B.P. Duggal and S.V. Djordjević, Weyl's theorem in the class of algebraically $p$-hyponormal operators, *Commentationes Mathematicae Prace Matematyczne* **40** (2000), 49–56.

[18] B. P. Duggal, Weyl's theorem for algebraically totally hereditarily normaloid operators, *Journal of Mathematical Analysis and Applications* **308** (2005), 578–87.

[19] B.P. Duggal, Weyl's theorem for totally hereditarily normaloid operator, *Rendiconti. Circolo Matematico di Palermo* **LIII** (2004), 237–49.

[20] B.P. Duggal, Hereditarily normaloid operators, *Extracta Mathematica***20** (2005), 203–17.

[21] B.P. Duggal, Riesz projections for a class of Hilbert space operators, *Linear Algebra Applications* **407** (2005), 140–8.

[22] B.P. Duggal, Polaroid operators satisfying Weyl's theorem, *Linear Algebra Applications* **414** (2006), 271–7.

[23] N. Dunford and J.T. Schwartz, *Linear operators, Part III*, Interscience, New York, 1971.

[24] S. Grabiner, Uniform ascent and descent of bounded operators, *Journal of the Mathematical Society of Japan* **34** (1982), 317–37.

[25] Y.M. Han and W.Y. Lee, Weyl's theorem holds for algebraically hyponormal operators, *Proceedings American Mathematical Society* **128** (2000), 2291–6.

[26] Young Min Han and An-Hyun Kim, A note on $*$-paranormal operators, *Integral Equations Operator Theory* **49** (2004), 435–44.

[27] Young Min Han, Jun Ik Lee and Derming Wang, Riesz idempotents and Weyl's theorem for $w$-hyponormal operators, *Integral Equations Operator Theory* **53** (2005), 51–60.

[28] A.A. Jafarian and M. Radjabalipour, Transitive algebra problem and local resolvent techniques, *Journal Operator Theory* **1** (1979), 273–85.

[29] In Hyoun Kim, On $(p, k)$-quasihyponormal operators, *Mathematical Inequalities and their Applications* **7** (2004), 629–38.

[30] K.B. Laursen and M.M. Neumann, *Introduction to local spectral theory*. Clarendon Press, Oxford, 2000.

[31] M. Mbekhta and V. Möller, On the axiomatic theorey of the spectrum II, *Studia Mathematica* **119** (1996), 129–47.

[32] M. Oudghiri, Weyl's and Browder's theorem for operators satisfying the SVEP, *Studia Mathematica* **163** (2004), 85–101.

[33] J.G. Stampfli, A local spectral theory for operators: Spectral subspaces for hyponormal operators, *Transactions American Mathematical Society* **217** (1976), 359–65.

[34] K. Tanahashi, A. Uchiyama and M. Chō, Isolated points of spectrum of $(p.k)$-quasihyponormal operators, *Linear Algebra Applications* **382** (2004), 221–9.