ON THE AUTOMORPHISM GROUP OF A FINITE P-GROUP WITH CYCLIC FRATTINI SUBGROUP

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Abstract

It is conjectured that if G is a finite non-cyclic p-group of order greater than p^2, then $|G|$ divides $|\mathrm{Aut}(G)|$. In this paper we characterize the finite non-abelian p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}(G)|_p = |G|$.

1. Introduction

Let G be a finite non-cyclic p-group of order p^n, where $n \geq 3$. We let $|\mathrm{Aut}(G)|_p$ denote the order of a Sylow p-subgroup of $\mathrm{Aut}(G)$, the group of automorphisms of G. It is conjectured that $|G| \leq |\mathrm{Aut}(G)|_p$. The conjecture has been established for certain families of finite p-groups, for example see [7] and the references therein. In particular, it is well known that if the Frattini subgroup $\Phi(G)$ of G is cyclic then $|G|$ divides $|\mathrm{Aut}(G)|$, for example see [6]. In this paper we will prove the following result.

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Theorem 1.1. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Then $|G| = |\mathrm{Aut}(G)|_p$ if and only if either $G \cong S_{16}$ or $Z(G)$ is cyclic and $|G/Z(G)| = p^2$. In particular, if $p = 2$ then G has one of the following types: S_{16}, D_8, Q_8, M_{2^n},or $L_{2^{n+2}}$, where*

$$L_{2^{n+2}} = \langle x, y, z | x^2 = (xy)^2 = 1, z^{2^n} = 1, [x, z] = [y, z] = 1, z^{2^{n-1}} = y^2\rangle \ \ (n > 1).$$

A similar description has been given by Malinowska in [10] for the p-groups of maximal class, in response to a problem posed by Berkovich in [3]. So our investigation solves the problem for a new class of finite non-abelian p-groups, namely the class of finite non-abelian p-groups with cyclic Frattini subgroup.

Throughout the paper all groups are assumed to be finite groups. Our notation is standard, and can be found in [8], for example. In particular, we will use the notation D_{2^n}, Q_{2^n}, and S_{2^n} for the dihedral, generalized quaternion, and semidihedral group of order 2^n; the group M_{2^n} is defined by

$$\langle x, y | x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{1+2^{n-2}}\rangle \qquad (n \geq 4).$$

The group $G = M_{2^n}$ is of class 2 with $G' \cong \mathbb{Z}_2$ and $Z(G) \cong \mathbb{Z}_{2^{n-2}}$. The terms of the lower and the upper central series of G are denoted by $\Gamma_i(G)$ and $Z_i(G)$, respectively. A group G is called a central product of its subgroups A and B if A and B commute elementwise and together generate G. In this situation, we write $G = A * B$. Also a non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian.

2. Some basic results

Let G be a group. An automorphism σ is said to be central if $g^{-1}g^{\sigma} \in Z(G)$ for all $g \in G$. The set of all central automorphisms of G, denoted by $\mathrm{Aut}_c(G)$, form a normal subgroup of $\mathrm{Aut}(G)$. In [1] Adney and Yen proved the following result.

Theorem 2.1. [1, theorem 1]. *For a finite purely non-abelian group G, there is a 1-1 correspondence between* $\mathrm{Hom}(G, Z(G))$ *and* $\mathrm{Aut}_c(G)$*, whence*

$$|\mathrm{Hom}(G/G', Z(G))| = |\mathrm{Aut}_c(G)|.$$

For finite groups having a non-trivial abelian direct factor, we have the following result.

Lemma 2.2. [6, lemma 1]. *Let $G = H \times K$, where H is abelian and K is purely non-abelian. Then* $|\mathrm{Aut}_c(G)|$=$|\mathrm{Aut}(H)||\mathrm{Aut}_c(K)||\mathrm{Hom}(K, H)||\mathrm{Hom}(H, Z(K))|$.

Let G be a purely non-abelian p-group of class 2, where p is an odd prime. Suppose that $Z(G)$ is cyclic and $G' = \langle u \rangle \cong \mathbb{Z}_p$. We may write $u = [g, h]$ with g,$h \in G$ and $|h| = p$. On setting $H = \langle g, h \rangle$, we have the following lemma which is a special case of [1, lemma 1].

Lemma 2.3. *With the above notation and assumption, $G = H\mathcal{C}_G(H)$ and the map α defined by $g \mapsto gh$, $h \mapsto h$ and $x \mapsto x$, $x \in \mathcal{C}_G(H)$ is an automorphism of G of order p.*

We now turn to the finite p-groups with cyclic Frattini subgroup and state the following result.

Theorem 2.4. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup $\Phi(G)$.*

(i) *If either $p > 2$ or $\mathrm{cl}(G) = 2$, then $\Phi(G) \leq Z(G)$ and $|G'| = p$.*

(ii) *If $p = 2$ and $c = \mathrm{cl}(G) > 2$, then $G' = \Phi(G)$, $|G/G'| \geq |Z_2(G)|$. Moreover, if G is purely non-abelian, then $|G| \leq |\mathrm{Aut}_c(G)\mathrm{Inn}(G)|$.*

PROOF. (i) We first assume that $p > 2$. Since G is regular, we have

$$|\Omega_1(G/\Omega_1(G))| = |\Omega_2(G)/\Omega_1(G)| = |\mho_1(G)/\mho_2(G)| \leq p.$$

Therefore G/Ω_1 has at most one subgroup of order p and so G/Ω_1 is cyclic. This implies that $|G'| = p$ and G is of class 2. Whence $[x^p, y] = 1$ for all $x, y \in G$. It follows that $\mho_1(G) \leq Z(G)$, which completes the first part of (i). Next we suppose that $p = 2$. Since $\Phi(G) = \mho_1(G)$, $\Phi(G) = \langle a^2 \rangle$ for some $a \in G$. Let $\mathcal{S}$ be the set of all subgroups H of G containing a such that $\langle a \rangle$ has index 2 in H. For any $x \in G - \langle a \rangle$, we set $H_x = \langle x, a \rangle$ and observe that $|H_x : \langle a \rangle| = 2$ since $\langle a \rangle \lhd G$. This shows that $G = \langle H | H \in \mathcal{S} \rangle$. By [8, theorem 4.4], each H in $\mathcal{S}$ is either abelian, $H/Z(H) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, or is of maximal class. If $\mathcal{S}$ contains no element of maximal class, then $a^2 \in Z(H)$ and hence $\Phi(G) \leq Z(G)$. Next suppose that $\mathcal{S}$ contains an element H of maximal class. Then $|H/H'| = |H/\Phi(G)| = 4$ and $H' \leq G' \leq \Phi(G)$, from which we conclude that $G' = \Phi(G)$. Finally we note that if $\Phi(G) \leq Z(G)$, then G is of class 2 and hence $2 = \exp(G/Z(G)) = \exp(G')$, which implies that $|G'| = 2$.

(ii) It is easily seen that $G' = \Phi(G)$ by the above argument, and so $\exp(G/\Gamma_2(G)) = 2$. Therefore, $\exp(\Gamma_i(G)/\Gamma_{i+1}(G)) = 2$ for $i \geq 2$, by [9, satz III, 2.13], which implies that $|\Gamma_i(G)/\Gamma_{i+1}(G)| = 2$ since Γ_2 is cyclic. Assume that H is a subgroup of G defined by $H/\Gamma_4 = \mathcal{C}_{G/\Gamma_4}(\Gamma_2/\Gamma_4)$. Then $|G : H| = 2$ by [4, theorem 2.5]. Also $Z_{c-1}(G) \leq H$ and $|H : Z_{c-1}(G)| = 2$ by [4, theorem 2.7 and its corollary]. Hence $|G : Z_{c-1}(G)| = 4$. Suppose that $|G/\Gamma_2| = 2^m$, $|Z_2(G)/Z(G)| = 2^r$ and $|Z(G)| = 2^k$. By considering the lower central series of G we deduce that $n = m+(c-1)$. Also the upper central series of G gives $n \geq 2+(c-3)+r+k$. Thus $m \geq r+k$, or equivalently $|G/G'| \geq |Z_2(G)|$. Finally let G be purely non-abelian. Then by Theorem 2.1, $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G/G', Z(G))| \geq |G/G'| = 2^m$ since $G' = \Phi(G)$. On setting $A = \mathrm{Inn}(G)\mathrm{Aut}_c(G)$, we see that $|A| \geq 2^{n-k}2^m2^{-r}$ for $Z(\mathrm{Inn}(G)) \cong Z_2(G)/Z(G)$. This yields $|A| \geq |G|$ by the fact that $m \geq r + k$. ■

Lemma 2.5. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Assume that either $p > 2$ or $\mathrm{cl}(G) = 2$. Then $\exp(G/G') \leq \exp(Z(G))$.*

Proof. Suppose that $\exp(Z(G)) = p^t$. By Theorem 2.4(i), $G/Z(G)$ is elementary abelian, so g^{p^t} is of order at most p for all $g \in G$. However, by the same theorem, $|G'| = p$. It follows that $g^{p^t} \in G'$ because $\Phi(G)$ is cyclic. ■

Lemma 2.6. *Let G be a finite purely non-abelian group of order p^n with cyclic Frattini subgroup. Assume that either $p > 2$ or $\mathrm{cl}(G) = 2$. Then $|\mathrm{Aut}_c(G)| \geq p^{n-1}$. In addition if $Z(G)$ is cyclic, $|\mathrm{Aut}_c(G)| = |G|/p$.*

Proof. This proof follows from Theorem 2.1, Lemma 2.5 and Theorem 2.4(i). ■

3. Proof of theorem

In this section we proceed to prove our main theorem. The following proposition completes the “if ”part of the theorem.

Proposition 3.1. *Let G be a non-abelian p-group of order p^n with cyclic Frattini subgroup. If $Z(G)$ is cyclic and $|G/Z(G)| = p^2$, then $|G| = |\mathrm{Aut}(G)|_p$.*

PROOF. Since $Z(G)$ is cyclic, G is a purely non-abelian group. Also G is of class 2 since $|G/Z| = p^2$. Therefore $|\mathrm{Aut}_c(G)| = p^{n-1}$ by Lemma 2.6. Next on setting $C = \mathrm{Aut}_c(G)$, we observe that $\mathrm{Aut}(G)/C$ is isomorphic to a subgroup of $\mathrm{GL}(2,p)$. Consequently $|\mathrm{Aut}(G)|$ divides $p^n(p-1)^2(p+1)$. Now the result follows at once from the fact that $|G| \leq |\mathrm{Aut}(G)|_p$. ■

Since $L_{2^{n+2}}$ $(n > 1)$ has cyclic center with a central quotient isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, we get the following corollary.

Corollary 3.2. *If G is one of the groups D_8, Q_8, M_{2^n}, or $L_{2^{n+2}}$ with $n > 1$, then $|G| = |\mathrm{Aut}(G)|_2$.*

The rest of the paper is devoted to proving the " only if " part of the theorem.

Lemma 3.3. *Let G be a finite purely non-abelian p-group with cyclic Frattini subgroup. Assume that either $p > 2$ or $\mathrm{cl}(G) = 2$. If $Z(G)$ is non-cyclic, then $|\mathrm{Aut}(G)|_p > |G|$.*

PROOF. Suppose that $Z(G) = A \times H$, where A is a cyclic subgroup of $Z(G)$ with maximum possible order. Since $\exp(G/G') \leq |A|$, by Lemma 2.5, we have

$$|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G/G', Z(G))| = |G/G'||\mathrm{Hom}(G/G', H)| \geq (|G|/p)p^2 = p|G|$$

by Theorem 2.1 and Lemma 2.4(i). ■

The following proposition together with Lemma 3.16 completes the proof of our main theorem when p is an odd prime.

Proposition 3.4. *Let G be a finite purely non-abelian p-group, p odd, with cyclic Frattini subgroup. If $|G| = |\mathrm{Aut}(G)|_p$ then $Z(G)$ is cyclic and $|G/Z(G)| = p^2$.*

PROOF. According to Lemma 3.3, $Z(G)$ is cyclic. Assume by way of contradiction that $|G/Z(G)| > p^2$. Since $G/Z(G)$ is elementary abelian and $Z(G)$ is cyclic, we may write, by [9, III; satz 13.7], $G = G_1 \dots G_k$ where $Z(G_i) = Z(G)$, $|G_i/Z(G_i)| = p^2$ for all i, and $[G_i, G_j] = 1$ whenever $i \neq j$. Since $G_1' = G_2' = G' \cong \mathbb{Z}_p$, we may choose the elements h_1, $g_1 \in G_1$ and h_2, $g_2 \in G_2$ such that ${h_1}^p = {h_2}^p = 1$ and $G_1' = \langle [g_1, h_1] \rangle$, $G_2' = \langle [g_2, h_2] \rangle$. For $i = 1, 2$, we set $H_i = \langle g_i, h_i \rangle$. Using Lemma 2.3, the maps α_i defined by $g_i \mapsto g_i h_i$, $h_i \mapsto h_i$, $x \mapsto x$, $x \in \mathcal{C}_G(H_i)$, are automorphisms of G with $|\alpha_i| = p$. Clearly α_1 and α_2 are both non-central. It is an easy task to see that $\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle$. It follows, by Lemma 2.6, that $|\mathrm{Aut}(G)|_p \geq |\langle \alpha_1, \alpha_2, \mathrm{Aut}_c(G) \rangle| > |G|$, a contradiction. ■

In what follows we consider the case $p = 2$. Before proceeding further we list two families of finite 2-groups introduced by Berger *et al.* in [2].

$$D^+_{2^{n+3}} = \langle a, b, c | a^{2^{n+1}} = b^2 = c^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, [b, c] = 1 \rangle,$$

$$Q^+_{2^{n+3}} = \langle a, b, c | a^{2^{n+1}} = b^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, a^{2^n} = c^2, [b, c] = 1 \rangle,$$

both with $n > 1$.

The following structure theorem is a special case of [2, theorem 2.] which plays an important role in our proof of the main theorem in the case $p = 2$.

Theorem 3.5. [2, theorem 2]. *Let G be a finite purely non-abelian 2-group with cyclic Frattini subgroup. Then*

$$G = G_0 * G_1 * \dots * G_s,$$

where $G_i \cong D_8$ for $i = 1, \dots, s$, $|G_0| > 2$ if $s > 0$, and G_0 has one of the following types: cyclic, non-abelian with a cyclic maximal subgroup, namely D_{2^n}, Q_{2^n}, S_{2^n}, M_{2^n} all with $n \geq 3$; and $D_{2^{n+2}} * \mathbb{Z}_4$, $S_{2^{n+2}} * \mathbb{Z}_4$, $D^+_{2^{n+3}}$, $Q^+_{2^{n+3}}$, $D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n > 1$. Conversely, every such group has cyclic Frattini subgroup.

We begin by stating a number of lemmas that will be used the sequel.

Lemma 3.6. *Let G be one of the groups D_8, Q_8, or M_{2^n}. Then G has a non-central automorphism of order 2 fixing G' elementwise.*

Lemma 3.7.

(i) *If G has one of the following types: D_{2^n}, Q_{2^n} both with $n > 3$, or S_{2^n} with $n > 4$, then $Z(G) \cong \mathbb{Z}_2$, $G' \cong \mathbb{Z}_{2^{n-2}}$, and $|\mathrm{Aut}(G)| = 2^{2n-3}$ when G is either D_{2^n} or Q_{2^n}, and $|\mathrm{Aut}(G)| = 2^{2n-4}$ when $G = S_{2^n}$. Hence $|\mathrm{Aut}(G)|_2 > |G|$.*

(ii) *If G is either $D^+_{2^{n+3}}$ or $Q^+_{2^{n+3}}$ then $|G| = 2^{n+3}$, $\mathrm{cl}(G) = n+1$, $|\mathrm{Aut}(G)| = 2^{2n+2}$ and $\Phi(G)$ is cyclic. Hence $|\mathrm{Aut}(G)|_2 > |G|$.*

Lemma 3.8. *If G has one of the following types: $D_{2^{n+2}} * \mathbb{Z}_4$, $S_{2^{n+2}} * \mathbb{Z}_4$, or $D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n > 1$, then $|\mathrm{Aut}(G)|_2 > |G|$.*

PROOF. By [5, theorem 3.2], we have

$$|\mathrm{Aut}(G)|_2 \geq 2|\mathrm{Aut}(N)|_2|\mathrm{Aut}(G/N)|,$$

where N stands for either $D_{2^{n+2}}$, $S_{2^{n+2}}$, or $D^+_{2^{n+3}}$. However, $|\mathrm{Aut}(N)| = 2^{2n+1}$ when $N = D_{2^{n+2}}$, $|\mathrm{Aut}(N)| = 2^{2n}$ when $N = S_{2^{n+2}}$, $(n > 2)$ and $|\mathrm{Aut}(N)| = 2^{2n+2}$ when $N = D^+_{2^{n+3}}$. By Theorem 3.5, $\Phi(G)$ is cyclic. Hence we are able to determine the order of G. Now comparing $|G|$ and $|\mathrm{Aut}(N)|$ gives $|G| < |\mathrm{Aut}(G)|_2$. So we are left with the case $G = S_{16} * \mathbb{Z}_4$. By using GAP, we find that $|G| = 32$ and $|\mathrm{Aut}(G)| = 64$, as required. ■

From now on we shall suppose throughout that G is a finite purely non-abelian 2-group whose Frattini subgroup is cyclic. For the rest of the paper, we will make use of the notation of Theorem 3.5 without further mention. The following corollary is an immediate consequence of Lemma 3.7, Lemma 3.8 and Corollary 3.2.

Corollary 3.9. *Assume that $s = 0$. If $|\mathrm{Aut}(G)|_2 = |G|$ then G has one of the following types: D_8, Q_8, S_{16}, or M_{2^n}.*

In what follows we consider the case $s > 0$.

We note that $G_i \cap G_j \neq 1$ for $0 \leq i < j \leq s$; otherwise $G_i \cap G_j = 1$ for some i and j, and hence $\Phi(G_i) \times \Phi(G_j)$ would be a subgroup of $\Phi(G)$. Hence either of G_i or G_j would be elementary abelian, a contradiction.

Lemma 3.10. *Let* $L = G_1 * \ldots * G_s$ *with* $s \geq 2$. *Then* $Z(L) = Z(G_i) \cong \mathbb{Z}_2$ *for* $i = 1, \ldots, s$. *Moreover,* L *has two distinct non-central automorphisms* α *and* β *of order* 2 *fixing* $Z(L)$ *elementwise such that* $[\alpha, \beta] = 1$ *and* $\mathrm{Aut}_c(L) \cap \langle \alpha, \beta \rangle = 1$.

Proof. We have $1 \neq G_i \cap G_j \leq Z(G_i) \cap Z(G_j)$, $|Z(G_i)| = 2$, and $Z(L) = Z(G_1) \ldots Z(G_s)$. It follows that $Z(G_i) = Z(G_j) = G_i \cap G_j = Z(L)$. For the second part of the lemma, first we assume that $s = 2$. Let $L = G_1 * G_2$. Now by Lemma 3.6, we let α and β be non-central automorphisms of G_1 and G_2, respectively, with $|\alpha| = |\beta| = 2$. We define α^*, β^*:$L \to L$, by setting $(hk)^{\alpha^*} = h^\alpha k$ and $(hk)^{\beta^*} = hk^\beta$ for $h \in G_1$ and $k \in G_2$. Since both α and β fix $G_1 \cap G_2$ elementwise, α^* and β^* are non-central automorphisms of L satisfying the conditions of the lemma. Now without loss of generality, we may assume that $s = 3$. Let $L = H * K$ where $H = G_1 * G_2$ and $K = G_3$. We know that H has two non-central automorphisms α, β of order 2 fixing $Z(H)$ elementwise such that $[\alpha, \beta] = 1$ and $\mathrm{Aut}_c(H) \cap \langle \alpha, \beta \rangle = 1$. Now since $H \cap K = Z(H) = Z(K) = Z(L)$, we can extend these automorphisms to non-central automorphisms of L satisfying the conditions of the lemma. ■

Proposition 3.11. *If* $s > 0$ *and* $\mathrm{cl}(G_0) > 2$, *then* $|\mathrm{Aut}(G)|_2 > |G|$.

Proof. We may write G as a central product of two subgroups H and K with $\mathrm{cl}(H) > 2$ and $K \cong D_8$. We have $|G| \leq |\mathrm{Inn}(G)\mathrm{Aut}_c(G)|_2$ by Theorem 2.4(ii). By Lemma 3.6, we let σ be a non-central automorphism of K having order 2. Since $H \cap K \leq Z(K)$, the restriction of σ to $H \cap K$ is identity. Hence σ can be extended to a non-central automorphism σ^* of G defined by $(hk)^{\sigma^*} = hk^\sigma$ for $h \in H$ and $k \in K$. We claim that σ^* being of order 2 is not in $\mathrm{Inn}(G)\mathrm{Aut}_c(G)$. To see this, suppose to the contrary $\sigma^* = i_g \tau$, where i_g is the inner automorphism of G induced by $g \in G$, and τ is a central automorphism of G. Writing $g = h_0 k_0$, where $h_0 \in H$ and $k_0 \in K$, gives

$$k^{-1}k^\sigma = k^{-1}k^{\sigma^*} = k^{-1}(g^{-1}kg)^\tau = [k, k_0](k_0^{-1}kk_0)^{-1}(k_0^{-1}kk_0)^\tau,$$

for all $k \in K$. As $\tau \in \mathrm{Aut}_c(G)$ and $K' = Z(K)$, we conclude that $k^{-1}k^\sigma \in Z(K)$ for all k in K, a contradiction. Now since $\mathrm{Aut}_c(G)$ is a 2-subgroup of $\mathrm{Aut}(G)$, we observe that $\langle \sigma^* \rangle \mathrm{Inn}(G)\mathrm{Aut}_c(G)$ is a 2-subgroup of $\mathrm{Aut}(G)$ whose order exceeds that of G. ■

Proposition 3.12. *If* $s \geq 2$ *and* $\mathrm{cl}(G_0) \leq 2$*, then* $|\mathrm{Aut}(G)|_2 > |G|$.

PROOF. By the hypothesis G_0 has one of the following types: $\mathbb{Z}_{2^n}$, M_{2^n}, D_8 or Q_8. We write $G = G_0 * H$, where $H = G_1 * \ldots * G_s$. By Lemma 3.10, it is easily seen that $Z(G) = Z(G_0)$ is cyclic, whence according to Lemma 2.6, $|\mathrm{Aut}_c(G)| = |G|/2$. By Lemma 3.10, H has two distinct non-central automorphisms α and β of order 2 fixing $Z(H)$ elementwise such that $[\alpha, \beta] = 1$ and $\mathrm{Aut}_c(H) \cap \langle \alpha, \beta \rangle = 1$. By extending these automorphisms to non-central automorphisms α^* and β^* of G we have

$$|\mathrm{Aut}(G)| \geq |\mathrm{Aut}_c(G)\langle \alpha^*, \beta^* \rangle| = |\mathrm{Aut}_c(G)||\langle \alpha^*, \beta^* \rangle| = |G|/2 \times 4 = 2|G|,$$

as required. ■

Proposition 3.13. *If* $s = 1$ *and* $\mathrm{cl}(G_0) = 2$*, then* $|\mathrm{Aut}(G)|_2 > |G|$.

PROOF. By our assumption, $G = G_0 * G_1$, where $G_1 \cong D_8$ and G_0 has one of the following types: D_8, Q_8, or M_{2^n}. The case $G = D_8 * D_8$ is treated by using Lemma 3.10 and Lemma 2.6. A similar argument can be applied for the case $G_0 \cong Q_8$. Now let $G_0 \cong M_{2^n}$. Since $G_0 \cap G_1 = G_0'$, by Lemma 3.6, G has a non-central automorphism α^* of order 2 induced by a non-central automorphism α of G_0 which was defined earlier. Again since G_1 has a non-central automorphism β of order 2, we may consider the corresponding non-central automorphism β^* of G with $|\langle \alpha^*, \beta^* \rangle| = 4$. However, $Z(G)$ is cyclic, so we have

$$|\langle \alpha^*, \beta^* \rangle \mathrm{Aut}_c(G)| = |G|/2 \times 4 = 2|G|.$$

■

Corollary 3.14. *Assume that* $s > 0$*. If* $|\mathrm{Aut}(G)|_2 = |G|$ *then* $G \cong L_{2^{n+2}}$.

PROOF. By the above propositions, we are just left with the case $s = 1$ and $\mathrm{cl}(G_0) = 1$. Therefore $G \cong \mathbb{Z}_{2^n} * D_8 \cong L_{2^{n+2}}$, which together with Corollary 3.2 completes the proof. ■

Corollary 3.15. *Let* G *be a purely non-abelian* 2*-group with cyclic Frattini subgroup. If* $\mathrm{cl}(G) > 2$ *and* $G \neq S_{16}$ *then* $|\mathrm{Aut}(G)|_2 > |G|$.

PROOF. This is evident by Corollary 3.9 and Corollary 3.14. ■

Lemma 3.16. *Let $G = H \times K$, where H is abelian and K is purely non-abelian. If $\Phi(G)$ is cyclic and $H \neq 1$, then $|\mathrm{Aut}(G)|_p > |G|$.*

PROOF. Obviously $\Phi(H) = 1$, so that H is an elementary abelian group. We let $|H| = p^m$ and $|K| = p^n$. If $p > 2$ or $\mathrm{cl}(G) = 2$, then

$$|\mathrm{Aut}_c(G)| \geq |\mathrm{Aut}(H)|p^{n-1}|H|^2|H| \geq p^{3m+n-1} > p^{n+m} = |G|$$

by Lemma 2.2 and Lemma 2.6. If $p = 2$ and $\mathrm{cl}(G) > 2$, then $\mathrm{cl}(K) > 2$. Hence, according to Theorem 2.4(ii), $K' = \Phi(K)$ and $|K/K'| \geq |Z_2(K)|$. Therefore $|\mathrm{Aut}_c(K)| \geq |K/K'|$ by Theorem 2.1. Also $|\mathrm{Hom}(K/K', H)| \geq |K/K'|$ since K/K' is elementary abelian. Hence $|\mathrm{Aut}_c(G)|_2 \geq |K/K'|^2|H|$ by Lemma 2.2. Now we have $|G/Z(G)| = |K/Z(K)|$ and $|Z_2(G)/Z(G)| = |Z_2(K)/Z(K)|$, which imply that

$$\begin{aligned} |\mathrm{Aut}_c(G)\mathrm{Inn}(G)|_2 &= |\mathrm{Aut}_c(G)|_2|G/Z(G)|/|Z_2(G)/Z(G)| \\ &\geq |K/K'|^2|H||K/Z(K)|/|Z_2(K)/Z(K)|. \end{aligned}$$

So $|\mathrm{Aut}_c(G)\mathrm{Inn}(G)|_2 \geq |K/K'||H||K| > |G|$, completing the proof. ■

Corollary 3.17. *Let G be a non-abelian 2-group with cyclic Frattini subgroup. If $|\mathrm{Aut}(G)|_2 = |G|$ then G has one of the following types: Q_8, D_8, M_{2^n}, $L_{2^{n+2}}$, or S_{16}.*

Corollary 3.18. *Let G be an extra-special p-group. Then $|G| = |\mathrm{Aut}(G)|_p$ if and only if $|G| = p^3$.*

Remark. (i) We note that the group $G = S_{16}$ does not satisfy the condition $|G/Z(G)| = 4$, although $|\mathrm{Aut}(G)|_2 = |G|$.
(ii) Let G be a non-cyclic abelian group of order p^n with $n \geq 3$ whose Frattini subgroup is cyclic. Then $|\mathrm{Aut}(G)|_p = |G|$ if and only if G is either $G \cong \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. This is immediate from [10, theorem 2.3].

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