ASYMPTOTIC REPRESENTATIONS OF STAR-LIKE FUNCTIONS VIA CONTINUOUS SEMIGROUPS OF HOLOMORPHIC MAPPINGS

By

Alexander Goldvard
Department of Mathematics, ORT Braude College, 21982 Karmiel, Israel

Simeon Reich*
Department of Mathematics, The Technion — Israel Institute of Technology, 32000 Haifa, Israel

and

David Shoikhet
Department of Mathematics, ORT Braude College, 21982 Karmiel, Israel

[Accepted 21 January 2008. Published 28 November 2008.]

Abstract

We first use linearizations of continuous semigroups of holomorphic mappings to study asymptotic representations of star-like *l*-analytic functions defined on the open unit ball of a J^*-algebra and then apply these representations to obtain a distortion theorem for such functions. We also consider some aspects of these representations in general Banach spaces.

1. Introduction

In this paper we shall be concerned with continuous semigroups of holomorphic self-mappings defined on the open unit ball of a complex Banach space and their connections with star-like mappings defined on this ball. Our approach is based on the following fundamental fact (see, for example, [4; 5]):

A biholomorphic mapping h on a domain D of a complex Banach space is star-like if and only if it satisfies the differential equation

$$h(x) = h'(x)f(x), \tag{1}$$

where f is the generator of a continuous one-parameter semigroup of holomorphic self-mappings of D and h' is the Fréchet derivative of the mapping h.

The following asymptotic representation of a star-like mapping h defined on the

*Corresponding author, e-mail: sreich@techunix.technion.ac.il

doi:10.3318/PRIA.2008.108.2.177

Cite as follows: A. Goldvard, S. Reich and D. Shoikhet, Asymptotic representations of star-like functions via continuous semigroups of holomorphic mappings, *Mathematical Proceedings of the Royal Irish Academy* **108**A (2008), 177–197; doi:10.3318/PRIA.2008.108.2.177.

open unit ball of a complex Banach space was given in [6]:

$$h(x) = h'(\tau) \cdot T - \lim_{t\to\infty} e^{Bt} \left[\tau - F_t(x)\right], \tag{2}$$

where $h(\tau) = 0,\ ||\tau|| < 1,\ B = f'(\tau)$, the convergence is locally uniform, and the point τ is the common fixed point of the semigroup $\{F_t\}_{t\geq 0}$ generated by f (for $\tau = 0$ this formula was given in [14]). The study of such a representation becomes more complicated even in the one-dimensional case if $||\tau|| = 1$ (in this case the star-like mapping is called star-like with respect to a boundary point). As shown in [20] for the one-dimensional case, one can write the same formula for a function h which is star-like with respect to a boundary point τ only if the function h is conformal at this point τ. In this paper, we give a new representation formula in the one-dimensional case for functions that are star-like with respect to a boundary point that holds without any condition on the function h. Moreover, we also obtain an analog of this formula for a star-like l-analytic function $\hat{h}$ defined on the open unit ball of a J^*-algebra. This formula can be written in the form

$$\hat{h}(x) = T - \lim_{t\to\infty} A_t \left(\tau I - \hat{F}_t(x)\right), \tag{3}$$

where $\{A_t\}_{t\geq 0}$ is a certain family of linear operators. We give two explicit expressions for the family $\{A_t\}_{t\geq 0}$. We also explore conditions that must be satisfied by the family $\{A_t\}_{t\geq 0}$ in order that the limit

$$h(x) = T - \lim_{t\to\infty} A_t \left(\tau - F_t(x)\right) \tag{4}$$

exists and produces a biholomorphic function. We solve this last problem in a general Banach space. Using the representation formula

$$\hat{h}(x) = \widehat{h'}(\tau I) \cdot T - \lim_{t\to\infty} e^{Bt} \left(\tau I - \hat{F}_t(x)\right) \tag{5}$$

for those functions $\hat{h}$, defined on the open unit ball of a J^*-algebra, which are star-like with respect to the boundary point τI and conformal at this point, we also obtain a distortion theorem for such functions. Note that $\widehat{h'}$ in formula (5) is the l -analytic function corresponding to the function h'.

This paper is organized as follows. In Section 2 we give all relevant definitions and quote some theorems that we use. In Section 3 we consider linearizable semigroups in general Banach spaces. Section 4 is devoted to a study of the case of boundary null points of generators in J^*-algebras. As a consequence of the results obtained in Section 4, in Section 5 we deduce a growth estimate for star-like l -analytic functions.

2. Preliminaries

2.1. Semigroups

Let X and Y be two complex Banach spaces, and let $D_1 \subset X$ and $D_2 \subset Y$ be domains, i.e., nonempty, connected and open subsets of X and Y, respectively.

A mapping $f : D_1 \mapsto D_2$, defined on D_1 with values in D_2, is said to be holomorphic on D_1 if it is Fréchet differentiable at each point in D_1. The set of holomorphic mappings of D_1 into D_2 will be denoted by $\mathrm{Hol}(D_1, D_2)$ and the set of holomorphic mappings of D into D by $\mathrm{Hol}(D)$. We use the following notations:

$$\mathbb{R} := (-\infty, \infty), \quad \mathbb{R}^+ := [0, \infty), \quad \mathbb{N} := \{0, 1, 2, \ldots\}.$$

A family $S = \{F_t\}$, where either $t \in \mathbb{R}^+$ or $t \in \mathbb{N}$, of self-mappings F_t of a domain $D \subset X$ is called a (one-parameter) semigroup if

$$F_{s+t} = F_s \circ F_t, \quad s, t \in \mathbb{R}^+ \quad (s, t \in \mathbb{N}), \tag{6}$$

and

$$F_0 = I_D, \tag{7}$$

where I_D is the identity operator on D.

A semigroup $\mathcal{S} = \{F_t\}_{t \geq 0}$ is said to be (strongly) continuous if for each $x \in D$, the vector valued function $F_t(x) : \mathbb{R}^+ \mapsto X$ is continuous in t. In fact, a semigroup $\{F_t\}_{t \geq 0} \subset \mathrm{Hol}(D)$ is continuous if and only if

$$T - \lim_{t \to 0^+} F_t = I_D. \tag{8}$$

(That is, the family $\{F_t\}_{t \geq 0}$ converges to the identity locally uniformly; see, for example [11, definitions 0.2–0.3]). A continuous semigroup is sometimes called a semiflow on D.

If $t \in \mathbb{N}$, the semigroup $\mathcal{S}$ is called discrete. In other words, a discrete semigroup $\mathcal{S} = \{F_t\}$, $t \in \mathbb{N}$, is the family of iterates of the self-mapping $F = F_1 : D \mapsto D$.

Let $\mathcal{S} = \{F_t\}_{t \geq 0}$ be a continuous semigroup defined on D. If the strong limit

$$f(x) = \lim_{t \to 0^+} \frac{x - F_t(x)}{t} \tag{9}$$

exists for each $x \in D$, then f is called the generator of the (continuous) semigroup S. In this case the vector function $u(t, \cdot) = F_t(\cdot)$, $t \geq 0$, is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + f(u(t,x)) = 0, \ t \geq 0, \\ u(0,x) = x, \quad x \in D. \end{cases} \tag{10}$$

In view of (10), a generator f is also called a semi-complete vector field.

It is not difficult to see that a semigroup $\mathcal{S} = \{F_t\}_{t \geq 0}$ also satisfies the equation

$$\frac{\partial F_t(x)}{\partial t} + \frac{\partial F_t(x)}{\partial x} f(x) = 0. \tag{11}$$

Indeed, we can rewrite (9) as

$$F_s(x) = x - sf(x) + o(s). \tag{12}$$

Therefore, using the semigroup property and the definition of the derivative of $F_t(x)$ with respect to x, we get

$$\begin{aligned} F_{t+s}(x) = F_t\left(F_s(x)\right) &= F_t\left(x - sf(x) + o(s)\right) \\ &= F_t(x) + \frac{\partial}{\partial x}F_t(x)\left[-sf(x) + o(s)\right] + o_1(s) \\ &= F_t(x) - \frac{\partial}{\partial x}F_t(x)sf(x) + o_2(s). \end{aligned} \tag{13}$$

We also note that, by the definition of the derivative of $F_t(x)$ with respect to t,

$$F_{t+s}(x) = F_t(x) + s\frac{\partial}{\partial t}F_t(x) + o_3(s). \tag{14}$$

The desired property (11) now follows from comparing (13) and (14).

Equalities (11) and (10) immediately imply that a semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$ satisfies the equation

$$\frac{\partial F_t(x)}{\partial x}f(x) = f\left[F_t(x)\right]. \tag{15}$$

In what follows the notion of a fixed point $\tau \in \overline{D}$ of a semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$ will be important. For $\tau \in D$, this means that $F_t(\tau) = \tau$ for all $t \geq 0$ and for $\tau \in \partial D$ (the boundary of D), this means that given $\epsilon > 0$, there exists $\delta > 0$ such that $||F_t(x) - \tau|| < \epsilon$ for all $t \geq 0$ whenever $x \in D$ and $||x - \tau|| < \delta$. According to [15, proposition 5.1], the set of fixed points in D of a semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$ of self-mappings F_t of a domain D coincides with the null point set of the generator f of the semigroup $\mathcal{S}$.

2.2. Star-like mappings in Banach spaces

A subset M of a complex Banach space X is called *star-shaped* if for each $w \in M$ and $t \geq 0$, the point $e^{-t}w$ also belongs to M. If D is a domain of X, then a biholomorphic mapping $f : D \to X$ is called a *star-like mapping on D* if its image $\Omega = f(D)$ is a star-shaped set. If $0 \in \Omega$, then the function f is called star-like with respect to an interior point, and if $0 \in \partial\Omega$, the boundary of Ω, then the function f is called star-like with respect to a boundary point.

2.3. J^*-algebras

To proceed we need a multiplication operation in a Banach space. The framework of our considerations will be the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators from a complex Hilbert space $\mathcal{H}$ into $\mathcal{H}$. This algebra is involutive because the mapping $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, given by $A \mapsto A^*$, where the adjoint operator A^* is defined by the equality $(Ax, y) = (x, A^*y)$, satisfies the following properties:

(i) $(A^*)^* = A$

(ii) $(AB)^* = B^*A^*$.

Definition 1 Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. A closed subspace $\mathfrak{A}$ of $\mathcal{L}(\mathcal{H},\mathcal{K})$ is called a J^*-algebra if $AA^*A \in \mathfrak{A}$ whenever $A \in \mathfrak{A}$.

Note that a J^*-algebra is not necessarily an algebra in the usual sense, and that the open unit ball of any J^*-algebra is a natural generalization of the open unit disc in the complex plane $\mathbb{C}$. Any Hilbert space $\mathcal{H}$ may be thought of as a J^*-algebra identified with $\mathcal{L}(\mathcal{H},\mathbb{C})$. Important examples of J^*-algebras are the so-called Cartan factors of type I, II, III and IV, which are the sets $\mathfrak{A}_1 = \mathcal{L}(\mathcal{H},\mathcal{K})$, $\mathfrak{A}_2 = \{A \in \mathcal{L}(\mathcal{H}) : A^t = A\}$, $\mathfrak{A}_3 = \{A \in \mathcal{L}(\mathcal{H}) : A^t = -A\}$ (where $A^t x = \overline{A^*\bar{x}}$ for a given conjugation $x \mapsto \bar{x}$ in $\mathcal{H}$), and $\mathfrak{A}_4$, which is any closed complex subspace $\mathfrak{A}$ of $\mathcal{L}(\mathcal{H})$ such that both $A^* \in \mathfrak{A}$ and $A^2 = zI$ for some complex number z whenever $A \in \mathfrak{A}$. (Cartan factors of type IV are variants of the spin factors.) All four types of the classical Cartan domains and their infinite dimensional analogs are the open unit balls of J^*-algebras, and the same holds for any finite and infinite product of these domains.

A crucial property of a J^*-algebra is that it has a kind of a Jordan triple product structure and that it contains certain symmetrically formed products of its elements. In particular, for all elements A, B, C in a J^*-algebra $\mathfrak{A}$,

$$AB^*C + CB^*A \in \mathfrak{A}. \tag{16}$$

We restrict our considerations to the so-called unital J^*-algebras, that is, J^*-algebras with the same underlying Hilbert spaces that contain the identity operator. Note that a closed subspace of $\mathcal{L}(\mathcal{H})$ that contains the identity operator is a unital J^*-algebra if and only if it contains the squares and adjoints of each of its elements (see the identities (1) in [10]). Thus a unital J^*-algebra contains all polynomials in any of its elements.

2.4. l-analytic generators, l-analytic semigroups, and l-analytic star-like functions on J^*-algebras

Let $\mathfrak{A}$ be a unital J^*-algebra and let Ω be a domain in the complex plane. We denote by $\mathfrak{A}_\Omega$ the set $\{A \in \mathfrak{A} : \sigma(A) \subset \Omega\}$, where $\sigma(A)$ denotes the spectrum of $A \in \mathcal{L}(X)$. Since $\mathfrak{A}$ is a closed subset of $\mathcal{L}(\mathcal{H})$, $\mathfrak{A}_\Omega$ is an open set in the topology of $\mathfrak{A}$ induced by the norm of $\mathcal{L}(\mathcal{H})$. The symbol $\hat{f}(A)$ will denote the operator on $\mathcal{H}$ defined by the Riesz-Dunford integral

$$\hat{f}(A) = \frac{1}{2\pi i}\int_\gamma f(z)(zI - A)^{-1}dz, \tag{17}$$

where $\gamma \subset \Omega$ is a positively oriented simple closed rectifiable contour such that the interior domain of γ contains $\sigma(A)$. The following definition was introduced in [5] (see, also [4; 17, p. 68].

Definition 2 Let D be a domain in a unital J^*-algebra $\mathfrak{A}$. A holomorphic mapping $F : D \mapsto \mathfrak{A}$ is called an *l*-analytic function if:

(i) there is a domain $\Omega \subset \mathbb{C}$ such that $D \subset \mathfrak{A}_\Omega$;

(ii) there is a holomorphic function $f : \Omega \mapsto \mathbb{C}$ such that $F(A) = \hat{f}(A)$ for all $A \in \mathfrak{A}_\Omega$, where $\hat{f}$ is defined by (17).

We call the function f in (ii) the underlying function of $\hat{f}$. It was called the producing function in [4]. Since the composition of *l*-analytic functions corresponds to the composition of the underlying functions, the univalence of $f : \Omega \mapsto \mathbb{C}$ implies that the corresponding *l*-analytic function $\hat{f}$on $\mathfrak{A}_\Omega$ is biholomorphic.

We denote by Δ the open unit disc of the complex plane and by D the open unit ball of a unital J^*-algebra $\mathfrak{A}$. It is clear that $D \subset \mathfrak{A}_\Delta$. The correspondence between semigroups on the open unit disc of the complex plane and semigroups on the open unit ball of a unital J^*-algebra was studied in [4]. We describe here some results from this paper that will be used in the present paper.

Theorem A An *l*-analytic function $\hat{f} : D \mapsto \mathfrak{A}$ is the generator of a one-parameter continuous semigroup (group) $\mathcal{S}_{\hat{f}}$ of holomorphic mappings acting on D if and only if its underlying function $f : \Delta \mapsto \mathbb{C}$ is the generator of a continuous semigroup (group) $\mathcal{S}_f$ of the corresponding underlying holomorphic functions acting on Δ. Moreover, the semigroup (group) $\mathcal{S}_{\hat{f}}$ consists of *l*-analytic functions with underlying functions in $\mathcal{S}_f$.

Theorem B Let $\mathfrak{A}$ be a unital J^*-algebra and let $\hat{f}$ be a univalent *l*-analytic function on D, the open unit ball of $\mathfrak{A}$, with the underlying function $f : \Delta \to \mathbb{C}$. Then the following conditions are equivalent:

(i) $\hat{f}(D)$ is a star-shaped set;

(ii) $f(\Delta)$ is a star-shaped set;

(iii) there exists a holomorphic generator g on Δ (respectively, a holomorphic generator $\hat{g}$ on D) such that $f(z) = f'(z)g(z),\ z \in \Delta$ (respectively, $\hat{f}(A) = \widehat{f'}(A)\hat{g}(A),\ A \in D$, where $\widehat{f'}$ is the *l*-analytic function corresponding to the function f').

Theorem C An *l*-analytic function $\hat{f} : D \mapsto \mathfrak{A}$ is the generator of a one-parameter semigroup (group) $S_{\hat{f}}$ of holomorphic mappings acting on D if and only if there exists $\tau \in \bar{\Delta}$ and an *l*-analytic function $\hat{q}$ with $\mathrm{Re}\hat{q}(A) \geq 0,\ A \in D$, such that $\hat{f}$ admits the representation

$$\hat{f}(A) = (A - \tau I)(I - \bar{\tau} A)\hat{q}(A). \tag{18}$$

It was shown in [4] that the semigroup $S_{\hat{f}}$ has at most one fixed point in $\mathcal{D}$. The semigroup $S_{\hat{f}}$ has indeed a fixed point in $\mathcal{D}$ if and only if $\tau \in \Delta$. If τ belongs to the boundary of Δ, then the semigroup $S_{\hat{f}}$ has no fixed points in $\mathcal{D}$. Moreover, the fixed point τI in (18) is attractive, i.e., the trajectory $\hat{F}_t(A)$ converges to τI as t tends to infinity for all $A \in \mathcal{D}$. The following theorem gives a quantitative description of the convergence of *l*-analytic semigroups with no fixed points in the open unit ball.

Theorem D Let $\{\hat{F}_t\}_{t\geq 0}$ be a one-parameter continuous semigroup of l-analytic proper contractions of D with no stationary point in D. Then there is a unimodular point $\tau \in \partial\Delta$ and a positive number γ such that

$$\Phi\left(\hat{F}_t(A), \tau I\right) \leq e^{-\gamma t}\Phi\left(A, \tau I\right) \tag{19}$$

and

$$||\Phi(\hat{F}_t(A), \tau I)|| \leq e^{-\gamma t}||\Phi(A, \tau I)|| \tag{20}$$

for each $A \in D$ and $t \geq 0$, where $\Phi(A, B) = (B - A)(B^*B - A^*A)(B^* - A^*)$. Moreover, the maximal γ for which (19) (or (20)) holds is the number

$$\beta = -\lim_{r\to 1^-} \left.\frac{\partial^2 u(t, \lambda)}{\partial t \partial \lambda}\right|_{t=0^+, \lambda = r\tau}, \tag{21}$$

where $u(t, \lambda) := F_t(\lambda)$ is the underlying function of $\hat{F}_t$.

We conclude this section with a result that allows us to treat convergence in the holomorphic functional calculus.

Theorem 1 If $\{f_n\}$ is a sequence of functions, holomorphic on Δ and converging locally uniformly on Δ to a function f, then the sequence $\{\hat{f}_n\}$ converges locally uniformly to $\hat{f}$ on $\mathcal{D}$, the open unit ball of a unital J^*-algebra $\mathfrak{A}$.

The proof that the sequence $\{\hat{f}_n\}$ converges pointwise to the function $\hat{f}$ can be found in many books that give an account of the holomorphic functional calculus. The proof of locally uniform convergence requires an additional argument.

3. Linearizable semigroups in general Banach spaces

In this section X is a general complex Banach space. We denote the space of all bounded linear operators on X by $\mathcal{L}(X)$.

Definition 3 Let D be a domain in X. A continuous semigroup $\{F_t\}_{t\geq 0}$ defined on D will be called linearizable if there exists a biholomorphic mapping $h : D \mapsto \Omega \subset X$ such that for all $w \in \Omega$,

$$h\left(F_t\left(h^{-1}(w)\right)\right) = e^{Bt}w, \tag{22}$$

where $B \in \mathcal{L}(X)$.

Note that in this case h is a solution of the generalized functional equation of Schröder

$$h(F_t(z)) = e^{Bt}h(z).$$

Definition 4 We will say that a one-parameter family $\{A_t\}_{t\geq 0} \subset \mathcal{L}(X)$ is asymptotically exponential if

(i) each A_t is invertible;

(ii) there exists $B \in \mathcal{L}(X)$ such that for each $s \geq 0$, the limit

$$\lim_{t\to\infty} A_t A_{t+s}^{-1} = e^{Bs} \tag{23}$$

exists in the norm topology of $\mathcal{L}(X)$.

Definition 5 We will say that the semigroup $\mathcal{S} = \{F_t\}_{t\geq 0}$ is asymptotically linearizableat τ if there exists a family of invertible operators $\{A_t\}_{t\geq 0} \subset \mathcal{L}(X)$ such that the following conditions hold:

(i)

$$||A_t|| \cdot ||F_t(x) - \tau|| \leq M < \infty,\ t \geq 0; \tag{24}$$

(ii) for each $x \in D$, the limit

$$T - \lim_{t\to\infty} A_t \left[\tau - F_t(x)\right] =: h(x) \tag{25}$$

exists in the locally uniform topology on D, and is a biholomorphic mapping on D;

(iii)

$$A_t\Gamma = \Gamma A_t, \tag{26}$$

where $\Gamma = f'(\tau)$.

In this case we will say that the family $\{A_t\}_{t\geq 0}$ is admissible for the semigroup $\{F_t\}_{t\geq 0}$.

We are now ready to formulate and prove a theorem that gives a necessary condition for a family to be admissible in the case where an attractive point τ of the semigroup lies inside the open unit ball.

Theorem 2 Any admissible family $\{A_t\}_{t\geq 0}$ for a semigroup $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(D)$ is asymptotically exponential.

Moreover, the operator B in (23) is exactly $-\Gamma$, where $\Gamma = f'(\tau)$ and f is the generator of $\mathcal{S}$. The semigroup $\mathcal{S}$ is actually linearizable by the mapping h defined in (25).

Proof. The locally uniform convergence of the family $\{A_t(\tau - F_t)\}_{t\geq 0}$ implies (see, for example, [11, theorem 1.5]) that

$$h'(x) = T - \lim_{t\to\infty} A_t \left(-\frac{\partial F_t(x)}{\partial x} \right). \tag{27}$$

Combining (27) and (15), we obtain

$$-h'(x)\left[f(x)\right] = T - \lim_{t\to\infty} A_t \left\{ f\left[F_t(x)\right] \right\}. \tag{28}$$

Let Γ be the derivative (or the angular derivative) of f at τ. Then we can write for t big enough,

$$f\left(F_t(x)\right) = \Gamma\left[F_t(x) - \tau\right] + \epsilon\left(F_t(x)\right)||F_t(x) - \tau||, \tag{29}$$

where $\epsilon\left(F_t(x)\right) \rightarrow 0$ at $t \rightarrow \infty$. Substituting (29) in (28) and taking into account (24) and (25), we get

$$h'(x)\left[f(x)\right] = \Gamma\left[h(x)\right]. \tag{30}$$

Then

$$\Gamma\left[F_t(x)\right] = h'(F_t(x))\left[f(F_t(x))\right] = -h'(F_t(x))\left[\frac{\partial F_t(x)}{\partial t}\right]. \tag{31}$$

We write the last equation as

$$\Gamma\left[G_t(x)\right] = -\frac{\partial G_t(x)}{\partial t}, \tag{32}$$

where $G_t(x) = h(F_t(x)),\ t \geq 0,\ x \in D$, and

$$G_0(x) = h(x). \tag{33}$$

Solving (32) with initial data (33) we obtain (see, for example, [2, p. 125])

$$G_s(x) = e^{-\Gamma s}\left[h(x)\right]$$

and

$$h\left[F_s(x)\right] = e^{-\Gamma s}\left[h(x)\right],\ s \geq 0. \tag{34}$$

On the other hand, by direct calculations we have from (25)

$$h\left[F_s(x)\right] = T - \lim_{t\to\infty} A_t A_{t+s}^{-1} A_{t+s}\left[\tau - F_{t+s}(x)\right] = K(s)\left[h(x)\right], \tag{35}$$

where

$$K(s) = e^{Bs} = \lim_{t\to\infty} A_t A_{t+s}^{-1}. \tag{36}$$

Comparing (34) and (35), we see that

$$h(x) \in Ker\left[e^{-\Gamma s} - K(s)\right]. \tag{37}$$

Since h is a biholomorphic mapping, its image is an open set in X. Thus we see that this open set is contained in $Ker\left[e^{-\Gamma s} - K(s)\right]$. Hence, in fact,

$$Ker\left[e^{-\Gamma s} - K(s)\right] = X$$

and

$$K(s) = e^{-\Gamma s}. \tag{38}$$

The proof is complete. ∎

In [14] an explicit construction of an admissible family for a semigroup defined on a domain D in the case where the generator of this semigroup has an interior null point was given. We recall this result here in the following form.

Theorem E Let D be a bounded domain and $\tau = 0 \in D$ be a null point of $f \in \mathrm{Hol}(D, X)$, that is, $f(\tau) = 0$. Denote $\Gamma := f'(\tau)$. Assume that

(i) $\mathrm{Re}\,[x^*\,(f(x))] > 0$ for all $x \in D \setminus \{0\}$ and $x^* \in X^*$, the dual space of X;

(ii) $m(\Gamma) = \inf \{\mathrm{Re}\,[x^*(\Gamma(x))] : \,||x|| = 1\} > 0$;

(iii) the integer part of $||\Gamma||/m(\Gamma)$ is equal to 1.

Then $A_t = e^{\Gamma t}$ is an admissible family for the semigroup $\{F_t\}_{t\geq 0}$. Hence $\{F_t\}_{t\geq 0}$ is linearizable by

$$h(x) = T - \lim_{t\to\infty} e^{\Gamma t}\,[\tau - F_t(x)]\,. \tag{39}$$

Some improvements of this result can be found in [6], where the conditions imposed on the generator f are relaxed.

4. Boundary null points in J^*-algebras

In the case of a boundary null point of a generator $f : D \to X$, the formula

$$\lim_{t\to\infty} e^{\Gamma t}\,[\tau - F_t(x)] \tag{40}$$

does not always yield a univalent function as the following example shows.

Example 1 Let $\mathbb{C}$ be the complex plane and let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Define the mapping $f : \Delta \mapsto \mathbb{C}$ by $f(z) = -(1-z)^2$. The unique boundary null point of this mapping is $z = 1$. It follows from a theorem of Berkson and Porta [1] that the mapping f is a generator on Δ. By direct calculations we see that the semigroup $\{F_t\}_{t\geq 0}$ generated by f is given by

$$F_t(z) = \frac{(1-t)z + t}{-tz + 1 + t}\,, \quad z \in \Delta,\ t \geq 0. \tag{41}$$

It is also not difficult to see that the limit

$$T - \lim_{t\to\infty} e^{\Gamma t}\,[1 - F_t(z)] \tag{42}$$

is equal to 0 for $\Gamma \leq 0$ and doesn't exist finitely for $\Gamma > 0$.

This fact points to the need for another method for constructing an admissible family. To address this issue we use Valiron's [21] construction of solutions to Schröder's equation in the boundary null point case. We obtain our results for l -analytic functions defined on J^*-algebras. For the reader's convenience we cite here some important results about the boundary behavior of generators in the one-dimensional case (see [7]).

Theorem F A point $\tau \in \partial\Delta$ is the sink point of the semigroup $\{F_t\}_{t\geq 0}$, i.e.,

$$\lim_{t\to\infty} F_t(z) = \tau \text{ and } \angle \lim_{z\to\tau} F_t(z) = \tau \tag{43}$$

if and only if the generator f of the semigroup vanishes at this point and the angular derivative of f at τ is a nonnegative real number, i.e.,

$$f(\tau) := \angle \lim_{z\to\tau} f(z) = 0 \tag{44}$$

and

$$f'(\tau) := \angle \lim_{z\to\tau} \frac{f(z)}{z-\tau} = \angle \lim_{z\to\tau} f'(z) = \beta \geq 0. \tag{45}$$

In our next theorem we present a method for constructing an admissible family.

Theorem 3 Let $\mathfrak{A}$ be a unital J^*-algebra, let $\hat{f}$ be the univalent l-analytic generator of $\{\hat{F}_t\}_{t\geq 0}$, a semigroup of l-analytic self-mappings on D, the open unit ball of $\mathfrak{A}$, and suppose that $\lim_{r\to 1^-} \hat{f}(rI) = 0$ and $\angle \lim_{z\to 1} f'(z) = \beta > 0$. Then the limit

$$h_1(A) := T - \lim_{t\to\infty} (I - \hat{F}_t(A))(I - \hat{F}_t(0))^{-1} \tag{46}$$

(i) exists finitely and is not constant;
(ii) is a biholomorphic function;
(iii) is a star-like function with respect to the boundary point I;
(iv) the semigroup $\{\hat{F}_t\}_{t\geq 0}$ is linearizable by the l-analytic function $\hat{h}$, with the underlying function

$$h(z) = T - \lim_{t\to\infty} \frac{1 - F_t(z)}{1 - F_t(0)}\,.$$

PROOF. For the sake of clarity the proof is divided into six short steps.
(a) By using the simple fact that $\hat{f}(zI) = f(z)I$, we get

$$\left(I - \hat{F}_t(0)\right)^{-1} = \frac{1}{1 - F_t(0)} I. \tag{47}$$

It follows that if the limit

$$h(z) := T - \lim_{t\to\infty} \frac{1 - F_t(z)}{1 - F_t(0)} \tag{48}$$

exists finitely, then $h_1(A) = \hat{h}(A)$, where h_1 is defined by (46).

(*b*) Denote

$$h_t(z) := \frac{1 - F_t(z)}{1 - F_t(0)} \,. \tag{49}$$

Consider also the functions $\{k_t\}_{t\geq 0}$ defined by

$$k_t(z) := \frac{1 - F_t(z)}{|1 - F_t(0)|} \,. \tag{50}$$

Clearly, $\operatorname{Re}[k_t(z)] > 0$ and $h_t(z) = e^{is_t}k_t(z)$, where $s_t = -\arg(1 - F_t(0))$. The domains $h_t(\Delta)$, $t \geq 0$, lie entirely in some half-plane such that $z = 0$ belongs to its boundary, and they contain the point $z = 1$ because of the obvious equality $h_t(0) = 1$. Thus the negative part of the real axis does not intersect the domains $h_t(\Delta)$. This implies that the family $\{h_t\}_{t\geq 0}$ is normal. Since $h_t(0) = 1$ for all $t \geq 0$, the family $\{h_t\}_{t\geq 0}$ is, in fact, compact.

(*c*) Differentiating h_t with respect to z, we get

$$\frac{d}{dz}h_t(z) = \frac{1}{F_t(0) - 1} \cdot \frac{d}{dz}F_t(z). \tag{51}$$

Since for each $z \in \Delta$, the semigroup $\{F_t\}_{t\geq 0}$ satisfies the equations

$$\frac{d}{dt}F_t(z) + f\left(F_t(z)\right) = 0 \tag{52}$$

and

$$\frac{d}{dt}F_t(z) + \frac{d}{dz}F_t(z) \cdot f(z) = 0, \tag{53}$$

we also get

$$\frac{\frac{d}{dz}h_t(z)}{h_t(z)} = \frac{f\left(F_t(z)\right)}{F_t(z) - 1} \cdot \frac{1}{f(z)} \,. \tag{54}$$

(*d*) As the family $\{h_t\}_{t\geq 0}$ is compact, there is a sequence $t_k \to \infty$ such that the sequence $\{h_{t_k}\}$ converges to some nonzero holomorphic function $h : \Delta \to \mathbb{C}$ (since $h_t(0) = 1$ for all t), and by the Weierstrass theorem,

$$\frac{d}{dz}h_{t_k}(z) \to h'(z). \tag{55}$$

On the other hand,

$$\frac{h'(z)}{h(z)} = T - \lim_{k\to\infty}\left[\frac{f\left(F_{t_k}(z)\right)}{F_{t_k}(z) - 1} \cdot \frac{1}{f(z)}\right] = \frac{\beta}{f(z)} \,. \tag{56}$$

Thus the function h is not constant and satisfies the differential equation

$$\beta h(z) = h'(z)f(z) \tag{57}$$

with the initial condition $h(0) = 1$ (since $h_t(0) = 1$ for all t). Therefore h is unique and does not depend on the choice of the sequence $\{t_k\}$. So $\lim_{t\to\infty} h_t(z)$ exists finitely.

(e) It follows from a consequence of the Hurwitz theorem (see, for example, [22, theorem 3.3]) that the function h is univalent on Δ. Hence the function $\hat{h}$ is biholomorphic.

(f) Since the function h satisfies equation (57), it is star-like with respect to the boundary point $h(1) = 0$ and, moreover, the equality

$$h(F_t(z)) = e^{-\beta t} h(z) \tag{58}$$

holds. Therefore the function $\hat{h}$ is star-like with respect to the boundary point $\hat{h}(I) = 0$ and the equality

$$\hat{h}(\hat{F}_t(A)) = e^{-\beta t} \hat{h}(z). \tag{59}$$

holds too. ■

Remark 1 We can get more exact information on the domain $h(\Delta)$. Indeed,

$$\angle \lim_{z\to 1} \frac{(z-1)h'(z)}{h(z)} = \angle \lim_{z\to 1} \frac{(z-1)\beta}{f(z)} = 1, \tag{60}$$

and it follows from [19, remark 5.6.1], that the smallest wedge that contains $h(\Delta)$ is precisely a half-plane such that the point $z = 0$ belongs to its boundary.

The following theorem shows that a slightly different construction also works.

Theorem 4 Let $\mathfrak{A}$ be a unital J^*-algebra, let $\hat{f}$ be the univalent l-analytic generator of $\{\hat{F}_t\}_{t\geq 0}$, a semigroup of l-analytic self-mappings on D, the open unit ball of $\mathfrak{A}$, and suppose that $\lim_{r\to 1^-} \hat{f}(rI) = 0$ and $\angle \lim_{z\to 1} f'(z) = \beta > 0$. Then the limit

$$\hat{k}(A) := T - \lim_{t\to\infty} \frac{(I - \hat{F}_t(A))}{||I - \hat{F}_t(0)||} \tag{61}$$

(i) exists finitely and is not constant;
(ii) is a biholomorphic function;
(iii) is a star-like function with respect to the boundary point I;
(iv) the semigroup $\{\hat{F}_t\}_{t\geq 0}$ is linearizable by the l-analytic function $\hat{k}$ with the underlying function

$$k(z) = T - \lim_{t\to\infty} \frac{1 - F_t(z)}{|1 - F_t(0)|}.$$

Proof. Since $\hat{f}(zI) = f(z)I$, it follows that

$$\frac{1}{||I - \hat{F}_t(0)||} \cdot \Big(I - \hat{F}_t(A)\Big) = \frac{I - \hat{F}_t(A)}{|1 - F_t(0)|}. \tag{62}$$

Thus the underlying function of the l-analytic function

$$\hat{k}_t(A) = \frac{I - \hat{F}_t(A)}{||I - \hat{F}_t(0)||} \tag{63}$$

is the function

$$k_t(z) = \frac{1 - F_t(z)}{|1 - F_t(0)|}. \tag{64}$$

Since $\mathrm{Re}\,[k_t(z)] \geq 0$ and $|k_t(0)| = 1$ for all $t \geq 0$, the family $\{k_t\}_{t\geq 0}$ is compact. It follows that there is a sequence $t_j \to \infty$ such that the sequence $\{k_{t_j}\}$ converges to some holomorphic function $k : \Delta \to \mathbb{C}$. By following the same steps as in the proof of the preceding theorem, we can see that the function k is not constant and satisfies the same linear differential equation (57) as the function h from the previous theorem. Therefore we must have

$$k(z) = \alpha h(z) \tag{65}$$

for some $0 \neq \alpha \in \mathbb{C}$. Since we already know from the proof of the previous theorem that $h(0) = 1$ and that $|k(0)| = 1$, we get that $|\alpha = 1$. It follows from Remark 1 that the smallest wedge which contains $k(\Delta)$ is precisely the right half-plane of the complex plane. Therefore, we can obtain the domain $k(\Delta)$ by rotating the domain $h(\Delta)$ and the angle of this rotation does not depend on the choice of the sequence $\{k_{t_j}\}$. Hence the uniqueness of the function h implies that the limit function k of $\{k_t\}_{t\geq 0}$ is unique.

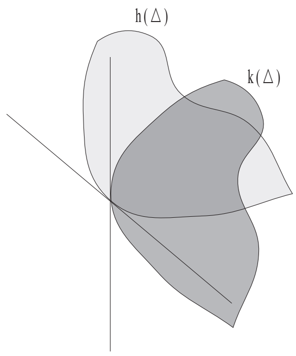

Fig. 1—The domains $h(\Delta)$ and $k(\Delta)$

Therefore the limit

$$\hat{k}(A) := T - \lim_{t\to\infty} \frac{(I - \hat{F}_t(A))}{||I - \hat{F}_t(0)||} \tag{66}$$

exists finitely and is not a constant. The remaining assertions of the theorem follow from equality (65). ■

Our next theorem provides a necessary and sufficient condition for the limit

$$T - \lim_{t\to\infty} e^{t\beta}\left[\tau I - \hat{F}_t(A)\right]$$

to exist finitely and yield a biholomorphic mapping.

Theorem 5 Let $\mathfrak{A}$ be a unital J^*-algebra, let $\hat{f}$ be the univalent l-analytic generator of $\{\hat{F}_t\}_{t\geq 0}$, a semigroup of l-analytic self-mappings on D, the open unit ball of $\mathfrak{A}$, and suppose that $\lim_{r\to 1^-} \hat{f}(rI) = 0$ and $\angle \lim_{z\to 1} f'(z) = \beta > 0$. Then the limit

$$\hat{\varphi}(A) := T - \lim_{t\to\infty} e^{t\beta}\left[I - \hat{F}_t(A)\right],\ A \in D, \tag{67}$$

exists finitely and is an invertible operator on D if and only if the limit

$$B := \lim_{r\to 1^-} \widehat{h'}(rI) \tag{68}$$

exists and is equal to $\alpha I,\ \alpha \neq 0$, where the function $\hat{h} : D \mapsto \mathfrak{A}$ is the solution of the differential equation

$$\beta\hat{h}(A) = \widehat{h'}(A)\hat{f}(A) \tag{69}$$

with $\hat{h}(0) = I$. Moreover, the function $\hat{\varphi}$ is a biholomorphic star-like function with respect to the boundary point I. It satisfies the same equation as the function $\hat{h}$ and linearizes the semigroup $\hat{F}_t$.

PROOF. Necessity: Suppose that the limit (67) exists finitely and is an invertible operator. As we have already done in the proof of Theorem 3, steps (c) and (d), differentiating φ, using the Weierstrass theorem and the semigroup property of F_t, we see that the underlying function φ satisfies the equation

$$\beta\varphi(z) = \varphi'(z)f(z) \tag{70}$$

and is not a constant. Hence the function $\hat{\varphi}$ satisfies the equation

$$\beta\hat{\varphi}(A) = \widehat{\varphi'}(A)\hat{f}(A) \tag{71}$$

and is also not a constant. Take $s \geq 0$ and substitute $\hat{F}_s(A)$ in (67):

$$\begin{aligned} \hat{\varphi}\left(\hat{F}_s(A)\right) &= T - \lim\nolimits_{t\to\infty} e^{t\beta}\left[I - \hat{F}_t\left(\hat{F}_s(A)\right)\right] \\ &= e^{-\beta s}\cdot T - \lim\nolimits_{t\to\infty} e^{(t+s)\beta}\left[I - \hat{F}_{t+s}(A)\right] = e^{-\beta s}\hat{\varphi}(A). \end{aligned} \tag{72}$$

It follows that the function $\hat{\varphi}$ is a solution of the Schröder equation

$$\hat{\varphi}\left(\hat{F}_s(A)\right) = e^{-\beta s}\hat{\varphi}(A). \tag{73}$$

Rewriting the last equation as

$$\hat{\varphi}(A) = e^{t\beta}\hat{\varphi}\left(\hat{F}_t(A)\right) \tag{74}$$

and substituting it in (67), we get

$$T - \lim_{t\to\infty} e^{t\beta}\left[\hat{\varphi}\left(\hat{F}_t(A)\right) - \left(I - \hat{F}_t(A)\right)\right] = 0. \tag{75}$$

By using the representation of generators given in equation (18) and equation (71), we see that

$$\hat{\varphi}(A) = -\frac{1}{\beta}\widehat{\varphi}'(A)(I-A)^2\hat{p}(A). \tag{76}$$

We now use the last formula for $\hat{\varphi}\left(\hat{F}_t(A)\right)$ and substitute it together with (67) in (75) to obtain

$$\frac{-\hat{\varphi}(A)}{\beta}\cdot T - \lim_{t\to\infty}\left[\widehat{\varphi}'\left(\hat{F}_t(A)\right)\left(I - \hat{F}_t(A)\right)\hat{p}\left(\hat{F}_t(A)\right) + \beta I\right] = 0. \tag{77}$$

By the Berkson–Porta representation formula, we have

$$\angle\lim_{z\to 1}(1-z)p(z) = \angle\lim_{z\to 1}\frac{f(z)}{z-1} = f'(1) = \beta. \tag{78}$$

Since, for each $t > 0$,

$$\frac{dF_t}{dz}(1) := \angle\lim_{z\to 1}\frac{dF_t}{dz} = e^{-t\beta} < 1 \tag{79}$$

(see [19]), it follows from Lemma 2.6 in [3] that the semigroup $\{F_t\}_{t\geq 0}$ converges to 1 nontangentially. It follows that

$$T - \lim_{t\to\infty}\left(I - \hat{F}_t(A)\right)\hat{p}\left(\hat{F}_t(A)\right) = \beta I. \tag{80}$$

Substituting (80) in equality (77), we get

$$-\hat{\varphi}(A)\cdot T - \lim_{t\to\infty}\left[\widehat{\varphi}'\left(\hat{F}_t(A)\right) + I\right] = 0. \tag{81}$$

The equality $\hat{g}(zI) = g(z)I$ and (81) imply that

$$-\varphi(z)\cdot T - \lim_{t\to\infty}\left[\varphi'\left(F_t(z)\right) + 1\right] = 0. \tag{82}$$

Since the operator $\hat{\varphi}(A)$ is invertible, $\varphi(z) \neq 0$ for all $|z| < 1$ (see [18, theorem

10.28]). Therefore

$$T-\lim_{t\to\infty}\varphi'\left(F_t(z)\right)=-1. \tag{83}$$

As we saw in the beginning of this proof, the function φ is not a constant. For each $t>0$, the self-mapping F_t is conformal on the unit disc. It follows from a consequence of the Hurwitz theorem (see, for example, [22, corollary 3.1]) that the function φ is conformal on Δ. This implies by [13, proposition 4.9], that φ has an angular derivative at the boundary point $z=1$, i.e.,

$$\angle\lim_{z\to1}\varphi'(z)=-1. \tag{84}$$

Since both the functions φ and h satisfy the same first order linear ordinary differential equation, we conclude that

$$h(z)=k\varphi(z). \tag{85}$$

It follows that

$$\angle\lim_{z\to1}h'(z)=-k. \tag{86}$$

Hence

$$\lim_{r\to1^-}\widehat{h'}(rI)=\lim_{r\to1^-}h'(r)I=-kI. \tag{87}$$

Sufficiency: Suppose that the limit

$$\lim_{r\to1^-}\widehat{h'}(rI) \tag{88}$$

exists and is equal to $\alpha I,\ \alpha\neq0$. Since the function $\hat{h}$ satisfies equation (69), it follows from Theorem B that

$$\hat{h}^{-1}\left(e^{-\beta t}\hat{h}(A)\right):=\hat{G}_t(A)\in D \tag{89}$$

for all $t\geq0$. It is easy to see that the family $\{\hat{G}_t\}_{t\geq0}$ is a continuous semigroup on D and

$$\frac{\partial}{\partial A}\hat{G}_t(A)\bigg|_{t=0}=-\beta\hat{h}(A)\left(\widehat{h'}(A)\right)^{-1}=-\hat{f}(A). \tag{90}$$

Now it follows from the uniqueness of the solution to the Cauchy problem that

$$\frac{\partial\hat{u}(t,A)}{\partial t}+\hat{f}\left(\hat{u}(t,A)\right)=0,\ \hat{u}(0,A)=A, \tag{91}$$

where $\hat{u}(t,A)=\hat{F}_t(A)=\hat{G}_t(A)$ for all $A\in D,\ t\geq0$.

Rewrite (89) as

$$e^{\beta t}\hat{h}\left(\hat{F}_t(A)\right)=\hat{h}(A). \tag{92}$$

We claim that

$$\hat{h}(A) = -\alpha \cdot T - \lim_{t\to\infty} e^{\beta t}\left[I - \hat{F}_t(A)\right]. \tag{93}$$

Indeed, since the family of underlying functions

$$h_t(z) = -\alpha e^{\beta t}\left[1 - F_t(z)\right] \tag{94}$$

is normal ($\mathrm{Re}\,[h_t(z)] < 0$ for all $t \geq 0$), there is a subsequence $\{h_{t_j}\}_{j=1}^{\infty}$ that converges to a holomorphic function b on Δ. It follows that the subsequence $\left\{\hat{h}_{t_j}\right\}$ converges to the l-analytic function $\hat{b}$ on D. Consider the difference $\hat{h} - \hat{b}$. We have

$$\hat{h}(A) - \hat{b}(A) = T - \lim_{t_j\to\infty} e^{\beta t_j}\left[\hat{h}\left(\hat{F}_{t_j}(A)\right) + \alpha\left(I - \hat{F}_{t_j}(A)\right)\right]. \tag{95}$$

By using (69) and (18) with $\tau = 1$, we get

$$\hat{h}(A)-\hat{b}(A) = T-\lim_{t_j\to\infty} e^{\beta t_j}\left(I - \hat{F}_{t_j}(A)\right)\left[-\frac{1}{\beta}\widehat{h'}\left(\hat{F}_{t_j}(A)\right)\left(I - \hat{F}_{t_j}(A)\right)\hat{p}\left(\hat{F}_{t_j}(A)\right) + \alpha I\right]. \tag{96}$$

Formula (80) and the definition of the function $\hat{b}$ imply that

$$\hat{h}(A) - \hat{b}(A) = -\frac{1}{\alpha}\hat{b}(A)\cdot T - \lim_{t_j\to\infty}\left[\alpha I - \widehat{h'}\left(\hat{F}_{t_j}(A)\right)\right] = 0. \tag{97}$$

Thus the limit (67) exists and $\hat{\varphi}(A) = -\dfrac{1}{\alpha}\hat{h}(A)$. ■

Remark 2 Note that if $\hat{g}$ is a solution of equation (69), then any other solution of this equation is $k\hat{g}$ for some $k \in \mathbb{C}$. So, if the operator $B = \lim_{r\to 1^-}\hat{h}'(rI)$ is equal to αI, $\alpha \in \mathbb{C}$ (borrowing the terminology from the one-dimensional case, we say that it is conformal at the point I), then every other nonzero solution is also conformal at the point I.

Thus we have the following assertion.

Corollary 1 Under the conditions of Theorem 5, the limit (67) exists finitely and is an invertible operator if and only if the function $\hat{h}$ defined by

$$\hat{h}(A) := T - \lim_{t\to\infty}(I - \hat{F}_t(A))(I - \hat{F}_t(0))^{-1} \tag{98}$$

satisfies the condition

$$\lim_{r\to 1^-}\widehat{h'}(rI) = \alpha I. \tag{99}$$

Moreover, in this case the functions $\hat{h}$ and $\hat{\varphi}$ are connected by the equality

$$\hat{\varphi}(A) = \hat{\varphi}(0)\hat{h}(A), \tag{100}$$

where the function $\hat{\varphi}$ is defined by

$$\hat{\varphi}(A) := T - \lim_{t\to\infty} e^{t\beta}\left[I - \hat{F}_t(A)\right]. \tag{101}$$

Remark 3 One can see that $\hat{\varphi}$, the limit obtained in (67), exists finitely and that $\hat{\varphi}(A)$ is an invertible operator for each $A \in D$ if and only if the limit

$$\hat{\varphi}(0) := T - \lim_{t\to\infty} e^{t\beta}\left[I - \hat{F}_t(0)\right] \tag{102}$$

exists finitely and is an invertible operator.

5. A growth estimate for *l*-analytic star-like functions

In the previous section we have proved that the limit (if it exists)

$$T - \lim_{t\to\infty} e^{t\beta}\left[I - \hat{F}_t(A)\right],\ A \in D, \tag{103}$$

yields a star-like *l*-analytic function with respect to the boundary point I. On the other hand, each *l*-analytic function $\hat{h} : D \mapsto \mathfrak{A}$, star-like with respect to the boundary point I, that satisfies the condition $\lim_{r\to 1^-} \widehat{h'}(rI) = \alpha I$, can be written as such a limit. Indeed, since $\hat{h}$ is star-like, it follows from Theorem B that the function $\hat{h}$ satisfies the equation

$$\hat{h}(A) = \widehat{h'}(A)\hat{f}(A). \tag{104}$$

Let $\{\hat{F}_t\}_{t\geq 0}$ be the semigroup generated by the generator $\hat{f}$. It follows from Theorem 5 that the operator

$$\hat{\varphi}(A) = T - \lim_{t\to\infty} e^{t}\left[I - \hat{F}_t(A)\right] \tag{105}$$

is well defined for all $A \in D$. Moreover,

$$\hat{\varphi}(A) = -\frac{1}{\alpha}\left[\hat{h}(0)\right]^{-1}\hat{h}(A). \tag{106}$$

Thus

$$\hat{h}(A) = -\alpha\hat{h}(0)\cdot T - \lim_{t\to\infty} e^{t}\left[I - \hat{F}_t(A)\right],\ A \in D. \tag{107}$$

In the following theorem we use this asymptotic representation to get a growth estimate of functions which are star-like with respect to a boundary point.

Theorem 6 Let $\mathfrak{A}$ be a unital J^*-algebra, $\alpha \in \mathbb{C}$ and $\hat{h} : D \mapsto \mathfrak{A}$ an *l*-analytic that is star-like with respect to the boundary point I such that $\lim_{r\to 1^-} \widehat{h'}(rI)$ is equal to αI. If q is a positive number such that the inequality

$$||A - \frac{1}{1+q}I|| \leq \frac{q}{1+q} \tag{108}$$

holds for all $A \in D$, then

$$||\hat{h}(A)|| \leq 2q\alpha||\hat{h}(0)||. \tag{109}$$

Proof. We use the growth estimate of the semigroup $\left\{\hat{F}_t\right\}$ given in Theorem D, but in a slightly different form. By [8, lemma 3], condition (20) is equivalent to the following one:

$$||\hat{F}_t(A) - \frac{\tau}{1+\exp(-t\gamma)q}I|| \leq \frac{q}{q+\exp(t\gamma)} \tag{110}$$

for each $q > 0$, whenever

$$||A - \frac{\tau}{1+q}I|| \leq \frac{q}{1+q}\,. \tag{111}$$

Since the function $\hat{h}$ satisfies equation (104), we can take $\beta = 1$ in Theorem D. So, the maximal γ is equal to 1. Now we estimate the left-hand side of inequality (110) with $\tau = 1$:

$$\begin{aligned} &||\hat{F}_t(A) - \tfrac{1}{1+\exp(-t)q}I|| \geq ||\hat{F}_t(A) - I|| - ||\tfrac{1}{1+\exp(-t)q}I - I|| \\ &= ||\hat{F}_t(A) - I|| - \tfrac{\exp(-t)q}{1+\exp(-t)q}\,. \end{aligned} \tag{112}$$

Hence

$$||\hat{F}_t(A) - I|| \leq ||\hat{F}_t(A) - \frac{1}{1+\exp(-t)q}I|| + \frac{\exp(-t)q}{1+\exp(-t)q}\,. \tag{113}$$

By applying inequality (110), we conclude that the inequality

$$\exp(t)||\hat{F}_t(A) - I|| \leq \frac{2q}{1+q\exp(-t)} \tag{114}$$

holds whenever

$$||A - \frac{1}{1+q}I|| \leq \frac{q}{1+q}\,. \tag{115}$$

This fact and equality (107) imply the assertion of our theorem. ■

Acknowledgement

The second author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund.

References

[1] E. Berkson and H. Porta, Semigroups of analytic functions and composition operators, *Michigan Mathematical Journal* **25** (1978), 101–15.
[2] H. Cartan, *Differential calculus,* Kershaw Publishing Company, London, 1971.
[3] C.C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, FL, 1995.

[4] M. Elin, L.A. Harris, S. Reich and D. Shoikhet, Evolution equations and geometric function theory in J^*-algebras, *Journal of Nonlinear and Convex Analysis* **3** (2002), 81–121.
[5] M. Elin, S. Reich and D. Shoikhet, Holomorphically accretive mappings and spiral-shaped functions of proper contractions, *Nonlinear Analysis Forum* **5** (2000), 149–61.
[6] M. Elin, S. Reich and D. Shoikhet, Complex dynamical systems and the geometry of domains in Banach spaces, *Dissertationes Mathematicae* **427** (2004).
[7] M. Elin and D. Shoikhet, Dynamic extension of the Julia–Wolf–Carathéodory theorem, *Dynamic Systems and Applications* **10** (2001), 421–37.
[8] K. Fan, Julia's lemma for operators, *Mathematische Annalen* **239** (1979), 241–5.
[9] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances,* North Holland, Amsterdam, 1980.
[10] L.A. Harris, Bounded symmetric homogeneous domains in infinite-dimensional spaces, *Lecture Notes in Mathematics* **364** (1974), 13–40.
[11] J.M. Isidro and L.L. Stachó, *Holomorphic automorphism groups in Banach spaces,* North-Holland, Amsterdam, 1984.
[12] V. Khatskevich and D. Shoikhet, *Differentiable operators and nonlinear equations,* Birkhäuser, Basel, 1994.
[13] C. Pommerenke, *Boundary behaviour of conformal maps,* Springer, Berlin, 1992.
[14] T. Poreda, On generalized differential equations in Banach spaces, *Dissertationes Mathematicae* **310** (1991).
[15] S. Reich and D. Shoikhet, Generation theory for semigroups of holomorphic mappings in Banach spaces, *Abstract and Applied Analysis* **1** (1996), 1–44.
[16] S. Reich and D. Shoikhet, Metric domains, holomorphic mappings and nonlinear semigroups, *Abstract and Applied Analysis* **3** (1998), 203–28.
[17] S. Reich and D. Shoikhet, *Nonlinear semigroups, fixed points, and geometry of domains in Banach spaces*, Imperial College Press, London, 2005.
[18] W. Rudin, *Functional analysis*, McGraw-Hill Book Company, New York, 1973.
[19] D. Shoikhet, *Semigroups in geometrical function theory*, Kluwer, Dordrecht, 2001.
[20] D. Shoikhet, A note on the Koenigs function of a one–parameter semigroup and a distortion theorem for starlike functions with respect to a boundary point, preprint, 2003.
[21] G. Valiron, *Fonctions analytiques*, Paris, Presses Universitaires de France, 1954.
[22] W.A. Veech, *A second course in complex analysis,* W.A. Benjamin, New York, 1967.