ON MANIFOLD STRUCTURE OF CLOSED CURVES IN EUCLIDEAN SPACE

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Abstract

The space of all closed curves in Euclidean 3-space constitutes a Banach manifold. The existence of parallel mates to a given closed space curve depends on its total normal twist α. As a submanifold we consider the space of curves which has total normal twist an integer multiple of 2π. Also construction of special curves, like Bertrand, and helical curves with invariant total normal twist from a spherical curve will be described, and moreover, it constitutes a Banach submanifold.

1. Introduction

The exterior parallelism of two regular, smooth, closed curves $c, \bar{c} : S^1 \longrightarrow E^3$ is defined by the following condition: for every parameter $t \in S^1$ the affine spaces normal to c at $c(t)$ and $\bar{c}$ at $\bar{c}(t)$ coincide in the Euclidean 3-space. Parallel transfer of the normal plane along one period of c with respect to the normal connection leads to a rotation of the normal plane which is characterized (up to integer multiples of 2π) by an oriented angle $\alpha(c)$ which we call the *total normal twist* of c [1; 7; 8].

Let $H(t, r)$ be an isotopy of smooth regular closed curves c_r such that all curves are parametrized over the segment [0,1]. Then the derivative of the total normal twist $\alpha(r)$ of c_r as a function of the deformation parameter r is given by the following expression [5]:

$$\frac{d\alpha}{dr} = \int_0^1 \frac{1}{\|\frac{\partial H}{\partial t}\|^3} \det(\frac{\partial^2 H}{\partial t \partial r}, \frac{\partial^2 H}{\partial t^2}, \frac{\partial H}{\partial t})dt. \tag{1.1}$$

We used the results above to detect isotopes leaving the total normal twist invariant, and in particular, to provide examples for deformations of curves remaining in the class of curves of global parallel rank 2(for more details see[4]). Denoting to this space of all smooth regular closed curves which has parallel rank 2 by $I_2^k(S^1, E^3)$. It is clear that

$$I_2^k(S^1, E^3) \subset I^k(S^1, E^3) \subset C^k(S^1, E^3),$$

where $I^k(S^1, E^3)$ is the space of all smooth immersions of S^1 into Euclidean 3-space

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E^3, and $C^k(S^1, E^3)$ is the space of all C^k functions of S^1 into E^3. It is clear also that for any integer number m,

$$I_2^k(S^1, E^3) = \alpha^{-1}(0) \cup \ldots \cup \alpha^{-1}(m.2\pi).$$

2. Manifold structure on $I_2^k(S^1, E^3)$

A neighbourhood of a curve c in $C^k(S^1, E^3)$ in the topology consists of all curves $\bar{c}$ which are close to c together with their derivatives of order k. The degree of closeness is specified by arbitrary positive numbers controlling the closeness of local representations of c and $\bar{c}$. Choosing cartesian coordinates in E^3. It is a simple procedure to equip the space $C^k(S^1, E^3)$ of C^k-curves of S^1into E^3 with the structure of a Banach space taking into account uniform convergence for all derivatives up to order k.

Proposition 2.1. ([4], proposition 4.2.1). *$I^k(S^1, E^3)$ is an open subset of $C^k(S^1, E^3)$ for all $k \geq 1$.*

The above proposition assures that the space $I^k(S^1, E^3)$ is an open submanifold of the Banach manifold $C^k(S^1, E^3)$. Let us concentrate on the space $I_2^k(S^1, E^3)$ of all curves which has parallel rank 2. Now we introduce our theorem.

Theorem 2.1. *$I_2^k(S^1, E^3)$ is a submanifold of $I^k(S^1, E^3)$ for all $k \geq 1$.*

PROOF. Let c be a curve in $I_2^k(S^1, E^3)$. Then $\alpha(c) = 0$ (integer multiples of 2π). Let

$$\varphi : S^3(1) - \{x_0\} \longrightarrow E^3 \subset E^4$$

be the stereographic projection from the north pole x_0 of the unit sphere $S^3(1)$ into the hyperplane $E^3 \subset E^4$. Define the curve

$$\bar{c} = \varphi^{-1} oc : S^1 \longrightarrow S^3(1) \subset E^4.$$

Define

$$\|.\|_k : I_2^k(S^1, E^3) \longrightarrow \Re$$

by

$$\|c\|_k = \max\{\sum_{i=0}^{k} c^i(t) : t \in S^1\}.$$

Since c and φ^{-1} are of class C^k, then $\bar{c}$ is a spherical curve of class C^k, and $\alpha(\bar{c}) = 0$ (integer multiples of 2π). Since $S^3(1) - \{x_0\}$ is dense in $S^3(1)$. Then for given $\epsilon > 0$, there exists $\bar{x} \in S^3(1)$, $\bar{x} \neq x_0$ such that $\|\bar{x} - x_0\| < \epsilon$.

As we used in [4] denoting by R the unique rotation of $\bar{c}$ into $\bar{c}_R$ in $S^3(1)$,

$\bar{c}_R = Ro\bar{c}$ such that R mapping $\bar{x}$ to the north pole x_0. Applying the stereographic projection φ to the curve $\bar{c}_R$. This yields a curve c_R in E^3, $\varphi o\bar{c}_R = c_R$. Since the stereographic projection φ is differentiable of class C^∞, there exists for given $\delta > 0$ and c an $\epsilon > 0$ such that

$$\|\varphi o\bar{c} - \varphi o\bar{c}_R\|_k < \delta,$$

whenever

$$\|\bar{x} - x_0\| < \epsilon,$$

and this means that

$$\|c - c_R\|_k < \delta.$$

The spherical curve $\bar{c}_R$ has total normal twist 0 (integer multiples of 2π) and φ is a conformal mapping (preserves the total normal twist), then $c_R = \varphi o\bar{c}_R$ has total normal twist 0 (integer multiples of 2π). Hence $c_R \in I_2^k(S^1, E^3)$, i.e., for each curve c in $I_2^k(S^1, E^3)$, there exists an open neighbourhood containing c and remaining a subset of $I_2^k(S^1, E^3)$. As an open subset of the manifold $I^k(S^1, E^3)$, $I_2^k(S^1, E^3)$ automatically is provided with a global chart and the manifold structure will be defined by this chart. ■

Example. Let c be a spherical curve parametrised by arc length s and have length 1. Assume that c is symmetric with respect to reflection of the center of the sphere. The homotopy H given by

$$H(s, r) = c(s) + \frac{r}{2}(\cos 2ns\pi)T(s),$$

where n is positive integer, is an immersion for all $r \in [0, 1]$. By using Formula (1.1) we can see that $\alpha(r) = 0$ for all r, and consequently $H(s, r) \subset I_2^k(S^1, E^3)$.

3. Construction of special curves with constant α

It is well known that spherical curves has total normal twist 0. let $c : S^1 \longrightarrow S^2(1)$ be a unit speed closed spherical curve parametrized by arc length s. We denote $T(s) = \frac{dc}{ds}$ the unit tangent vector of c at s. We put the vector

$$A(s) = c(s) \times T(s).$$

Then we have an orthonormal frame

$$\{c(s), T(s), A(s)\}$$

along c. This frame is called the Sabban frame of c [2; 3]. Hence we have the following spherical Frenent-Serret formula of c:

$$c'(s) = T(s),$$

$$T'(s) = c(s) + \kappa_g(s)A(s)$$

$$A'(s) = -\kappa_g(s)T(s),$$

where $\kappa_g(s)$ is the geodesic curvature of the curve c in S^2, which is given by

$$\kappa_g(s) = \det(c(s), T(s), T'(s)).$$

Now we define the space curve

$$\tilde{c}(s) = \int_{s_0}^{s} c(t)dt + b\int_{s_0}^{s} A(t)dt,$$

where b is a constant number. It is clear that $\tilde{c}(s)$ is a closed curve.

Define an isotopy $H : S^1 \times (-\epsilon, \epsilon) \longrightarrow E^3$,

$$H(s, r) = \tilde{c}_r(s) = \phi(r)[\int_{s_0}^{s} c(t)dt + b\int_{s_0}^{s} A(t)dt], \tag{3.1}$$

where $\phi(r)$ is continuous real valued function, $\phi(0) = 1$.

It is clear that $H(s, 0) = \tilde{c}(s)$. Taking partial derivatives of H , and using spherical Frenent-Serret formulas gives us

$$\frac{\partial H}{\partial s} = \phi(r)[c(s) + bA(s)],$$

$$\frac{\partial^2 H}{\partial s^2} = \phi(r)[1 - b\kappa_g(s)]T(s),$$

and

$$\frac{\partial^2 H}{\partial r \partial s} = \phi'(r)[c(s) + bA(s)].$$

Then we can see that

$$\det(\frac{\partial^2 H}{\partial r \partial s}, \frac{\partial^2 H}{\partial s^2}, \frac{\partial H}{\partial s}) =$$

$$\phi'(r)\phi^2(r)\det(c(s) + bA(s), [1 - b\kappa_g(s)]T(s), c(s) + bA(s)) = 0,$$

and consequently by using Formula (1.1) we obtain

$$\frac{d\alpha}{dr} = \int_0^1 \frac{1}{\|\frac{\partial H}{\partial s}\|^3} \det(\frac{\partial^2 H}{\partial s \partial r}, \frac{\partial^2 H}{\partial s^2}, \frac{\partial H}{\partial s})dt = 0.$$

This implies that α is constant along the isotopy $H(s, r)$.

Furthermore, to see that the family of curves $H(s, r)$ are Bertrand curves, at $r = r_0$,

$$\frac{\partial^3 H}{\partial s^3} = \phi(r_0)\{(1 - b\kappa_g(s))(c(s) + \kappa_g(s)A(s)) - b\kappa_g'(s)T(s)\}.$$

Straightforward calculations gives us

$$\det(\frac{\partial H}{\partial s}, \frac{\partial^2 H}{\partial s^2}, \frac{\partial^3 H}{\partial s^3}) = \phi^3(r_0)(1 - b\kappa_g(s))^2(\kappa_g(s) - b),$$

$$|\frac{\partial H}{\partial s}| = \phi(r_0)(1 + b^2)^{1/2}$$

and

$$|\frac{\partial H}{\partial s} \times \frac{\partial^2 H}{\partial s^2}| = \phi^2(r_0)(1 - b\kappa_g(s))(1 + b^2)^{1/2}.$$

Substituting in the formulas of the curvature κ and torsion τ we obtain

$$\kappa(s) = \frac{1}{\phi(r_0)(1 + b^2)}(1 - b\kappa_g(s)).$$

and

$$\tau(s) = \frac{1}{\phi(r_0)(1 + b^2)}(\kappa_g(s) - b)$$

Hence it follows that

$$\bar{a}\kappa(s) + \bar{b}\tau(s) = 1,$$

where

$$\bar{a} = \frac{\phi(r_0)(1 + b^2)}{1 - b^2},$$

and

$$\bar{b} = \frac{b\phi(r_0)(1 + b^2)}{1 - b^2},$$

It follows that for every $r \in (-\epsilon, \epsilon)$, all curves of the family $H(s, r)$ are Bertrand curves with the same total normal twist zero (integer multiples of 2π), and consequently has parallel rank 2. From the above construction and Theorem 2.1, we have the following corollary:

Corollary 3.1. *The family of Bertrand curves $H(s, r)$ constitutes a Banach submanifold in $C^k(S^1, E^3)$.*

Special Case. Suppose that $c(s)$ be a geodesic on $S^2(1)$. This implies that $\kappa_g(s) = 0$, and consequently

$$\tau(s) = \frac{-b}{\phi(r_0)(1 + b^2)},$$

and

$$\kappa(s) = \frac{1}{\phi(r_0)(1 + b^2)}.$$

Hence

$$\frac{\tau}{\kappa} = -b = \text{constant}.$$

It follows that the family $H(s,r)$ are general helices. Then we have:

Corollary 3.2. *The family of helical curves $H(s,r)$ constitutes a Banach submanifold in $C^k(S^1,E^3)$.*

Looking into the formula of grad α, it is obvious that grad α vanishes if $\frac{\partial^3 H}{\partial s^3} \times \frac{\partial H}{\partial s} = 0$. From the equations above, we have

$$\frac{\partial^3 H}{\partial s^3} \times \frac{\partial H}{\partial s} =$$

$$\phi^2(r)(-b^2\kappa_g'(s)c(s) + (1-b\kappa_g)(\kappa_g - b)T(s) + b\kappa_g'(s))A(s).$$

It is easy to see that $\frac{\partial^3 H}{\partial s^3} \times \frac{\partial H}{\partial s} = 0$ implies that $\kappa_g(s)$ is constant, $\kappa_g(s) = b$ or $\frac{1}{b}$. Taking into account that c is a spherical curve. Hence c is a circle [6].

Conversely, let c be a circle in the isotopy (3.1) with geodesic curvature $\kappa_g(s) = b$, Then $\kappa_g'(s) = 0$, and consequently,

$$\frac{\partial^3 H}{\partial s^3} \times \frac{\partial H}{\partial s} = 0.$$

Hence we have the following proposition.

Proposition 3.1. *The spherical curve c of the isotopy (3.1) is a circle with geodesic curvature $\kappa_g(s) = b$ if and only if $\frac{\partial^3 H}{\partial s^3} \times \frac{\partial H}{\partial s} = 0$.*

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