ASYMPTOTIC FORMULAS FOR EIGENVALUES OF A HILL'S EQUATION WITH PIECEWISE CONSTANT COEFFICIENT

By

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Abstract

In this paper, we obtain the asymptotic formulas for eigenvalues of a Hill's equation with piecewise constant coefficient.

1. Introduction

Let us consider the differential equation

$$-y'' + q(x)y = \lambda r(x)y, \quad (-\infty < x < \infty), \tag{1.1}$$

where λ is a complex parameter, $q(x)$ is real valued function with period $\omega > 0$, and for positive constants α, β and fixed point $a \in (0, \omega)$ $r(x)$ is defined by

$$r(x) = \begin{cases} \alpha^2, & \text{if } 0 < x \leq a, \\ \beta^2, & \text{if } a < x \leq \omega. \end{cases}$$

Following [1; 2; 5; 6] we first present some necessary facts about Hill's equation. We consider the periodic

$$\begin{cases} -y'' + q(x)y = \lambda r(x)y & 0 \leq x \leq \omega, \\ y(0) = y(\omega), \quad y'(0) = y'(\omega) \end{cases} \tag{1.2}$$

and the semi-periodic (or anti-periodic)

$$\begin{cases} -y'' + q(x)y = \lambda r(x)y & 0 \leq x \leq \omega, \\ y(0) = -y(\omega), \quad y'(0) = -y'(\omega) \end{cases} \tag{1.3}$$

boundary value problems associated with the equation (1.1).

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Let $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ be solutions of the equation (1.1) satisfying the initial conditions

$$\theta(0,\lambda)=1, \quad \theta'(0,\lambda)=0; \quad \textit{and} \quad \varphi(0,\lambda)=0, \quad \varphi'(0,\lambda)=1 \tag{1.4}$$

respectively. We define the Hill discriminant of the equation (1.1) by the function

$$F(\lambda)=\theta(\omega,\lambda)+\varphi'(\omega,\lambda). \tag{1.5}$$

Thus the eigenvalues of the periodic boundary value problem (1.2) coincide with the roots of $F(\lambda)=2$ and also the eigenvalues of the anti-periodic boundary value problem (1.3) coincide with the roots of $F(\lambda)=-2$.

Each of the problems (1.2) and (1.3) has a countably infinite number of real eigenvalues with the points accumulate at $+\infty$. The eigenvalues $\mu_{2k}^{\pm}$ and $\mu_{2k+1}^{\pm}$ $(k=0,\pm1,\pm2,\dots)$ of the problem (1.2) and (1.3) respectively occur in the order

$$\cdots<\mu_{-2}^{-}\leq\mu_{-2}^{+}<\mu_{-1}^{-}\leq\mu_{-1}^{+}<\mu_{0}^{-}\leq\mu_{0}^{+}<\mu_{1}^{-}\leq\mu_{1}^{+}<\mu_{2}^{-}\leq\mu_{2}^{+}<\cdots.$$

In the case of an even piecewise constant function $\rho(x)$ the equation (1.1) when $q(x)=0$ was studied earlier in [4]. Guseinov and Karaca [3] have investigated the asymptotic formulas for eigenvalues of the Hill's equation (1.1) with coefficients $q(x)$ and $r(x)$ where $q(x)=0$ and $r(x)$ is piecewise constant.

In this paper we are concerned with the cases when $q(x)$ is first arbitrary constant and later any real valued.

2. Preliminaries

We consider $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ solutions of the equation (1.1) satisfying the initial conditions

$$\theta(0,\lambda)=1, \quad \theta'(0,\lambda)=0; \quad \varphi(0,\lambda)=0, \quad \varphi'(0,\lambda)=1;$$

$\theta_0(x,\lambda)$ and $\varphi_0(x,\lambda)$ solutions of the equation

$$-y''=\lambda r(x)y \quad (0\leq x\leq\omega) \tag{2.1}$$

satisfying the initial conditions

$$\theta_0(0,\lambda)=1, \quad \theta_0'(0,\lambda)=0; \quad \varphi_0(0,\lambda)=0, \quad \varphi_0'(0,\lambda)=1.$$

Using the method of parameter variable, the solutions of equation (1.1) can be investigated as linear combination of the solutions of equation (1.2).

These solutions verify the following integral equations;

$$\theta(x,\lambda)=\theta_0(x,\lambda)+\int_0^x[\varphi_0(x,\lambda)\theta_0(t,\lambda)-\theta_0(x,\lambda)\varphi_0(t,\lambda)]q(t)\theta(t,\lambda)dt,$$

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x[\varphi_0(x,\lambda)\theta_0(t,\lambda)-\theta_0(x,\lambda)\varphi_0(t,\lambda)]q(t)\varphi(t,\lambda)dt.$$

If we take $\lambda = s^2$, then we get

$$\theta(x,\lambda)=\cos s\alpha x+\int_0^x\frac{\sin s\alpha(x-t)}{s\alpha}q(t)\theta(t,\lambda)dt \quad x\in(0,a], \tag{2.2}$$

$$\theta(x,\lambda)=\cos s\alpha a\cos s\beta(x-a)-\frac{\alpha}{\beta}\sin s\alpha a\sin s\beta(x-a)$$

$$+\int_0^a[\frac{\cos s\beta(x-a)\sin s\alpha(a-t)}{s\alpha}+\frac{\sin s\beta(x-a)\cos s\alpha(a-t)}{s\beta}]q(t)\theta(t,\lambda)dt$$

$$+\int_a^x\frac{\sin s\beta(x-t)}{s\beta}q(t)\theta(t,\lambda)dt \quad x\in(a,\omega], \tag{2.3}$$

and

$$\varphi(x,\lambda)=\frac{\sin s\alpha x}{s\alpha}+\int_0^x\frac{\sin s\alpha(x-t)}{s\alpha}q(t)\varphi(t,\lambda)dt \quad x\in(0,a], \tag{2.4}$$

$$\varphi(x,\lambda)=\frac{\sin s\alpha a}{s\alpha}\cos s\beta(x-a)+\cos s\alpha a\frac{\sin s\beta(x-a)}{s\beta}$$

$$+\int_0^a[\frac{\cos s\beta(x-a)\sin s\alpha(a-t)}{s\alpha}+\frac{\sin s\beta(x-a)\cos s\alpha(a-t)}{s\beta}]q(t)\varphi(t,\lambda)dt$$

$$+\int_a^x\frac{\sin s\beta(x-t)}{s\beta}q(t)\varphi(t,\lambda)dt \quad x\in(a,\omega]. \tag{2.5}$$

Using integral equations (2.2), (2.3), (2.4), (2.5) we have for $x\in(0,a]$,

$$\theta(x,\lambda)=\cos s\alpha x+O(\frac{e^{|\tau|\alpha x}}{|s|}) \tag{2.6}$$

and

$$\varphi(x,\lambda)=\frac{\sin s\alpha x}{s\alpha}+O(\frac{e^{|\tau|\alpha x}}{|s|}) \tag{2.7}$$

and for $x\in(a,\omega]$,

$$\theta(x,\lambda)=\cos s\alpha a\cos s\beta(x-a)-\frac{\alpha}{\beta}\sin s\alpha a\sin s\beta(x-a)+O(\frac{e^{|\tau|(\alpha a+\beta(\omega-a))}}{|s|}), \tag{2.8}$$

$$\varphi(x,\lambda)=\frac{\sin s\alpha a}{s\alpha}\cos s\beta(x-a)+\cos s\alpha a\frac{\sin s\beta(x-a)}{s\beta}+O(\frac{e^{|\tau|(\alpha a+\beta(\omega-a))}}{|s|}), \tag{2.9}$$

and

$$\varphi'(x,\lambda) = -\frac{\beta}{\alpha}\sin s\alpha a \sin s\beta(x-a) + \cos s\alpha a \cos s\beta(x-a) + O(\frac{e^{|\tau|(\alpha a+\beta(\omega-a))}}{|s|}). \tag{2.10}$$

Therefore, according to (2.8) and (2.10) we find the explicit formula for the Hill discriminant:

$$F(\lambda) = A\cos s\delta + B\cos s\gamma + O(\frac{e^{|\tau|(\alpha a+\beta(\omega-a))}}{|s|}), \tag{2.11}$$

where $A = 1 + \frac{1}{2}(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})$, $B = 1 - \frac{1}{2}(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})$, $\delta = \alpha a + \beta(\omega - a)$, and $\gamma = \alpha a - \beta(\omega - a)$.

From (2.11), we have the formula

$$\Phi^+(\lambda) = F(\lambda) - 2 = -2A\sin^2\frac{s\delta}{2} - 2B\sin^2\frac{s\gamma}{2} + O(\frac{e^{|\tau|(\alpha a+\beta(\omega-a))}}{|s|}).$$

Take

$$s = \frac{2\pi}{\delta}z. \tag{2.12}$$

Then we get

$$\Phi^+(\lambda) = \Phi^+(\frac{4\pi^2}{\delta^2}z^2) = \Phi_1^+(z), \tag{2.13}$$

where

$$\Phi_1^+(z) = -2A\sin^2\pi z - 2B\sin^2\frac{\gamma}{\delta}\pi z + O(\frac{e^{2\pi|Imz|}}{|z|}). \tag{2.14}$$

Further, for any natural number n define the square contour

$$\Gamma_n = \{z \in \mathbf{C} : |Rez| = n + \frac{1}{2}, \quad |Imz| = n + \frac{1}{2}\}.$$

We will use the following well-known theorem;

Theorem. [**Rouché**] *If $f(z)$ and $g(z)$ are analytic functions inside and on a closed contour Γ, and $|g(z)| < |f(z)|$ on Γ, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside Γ.*

We apply the Rouché theorem by putting

$$\Gamma = \Gamma_n, \quad f(z) = -2A\sin^2\pi z - 2B\sin^2\frac{\gamma}{\delta}\pi z, \quad g(z) = O(\frac{e^{2\pi|Imz|}}{|z|}).$$

We need the following Lemmas proved in [3].

Lemma 1. *There is a positive number C such that*

$$|\sin \pi z| \geq Ce^{|Imz|\pi}, \qquad \forall z \in \Gamma_n,$$

where C does not depend on z and n.

Lemma 2. *There is a natural number n_1 such that*

$$|\frac{\sin \frac{\gamma}{\delta}\pi z}{\sin \pi z}| \leq 1, \quad \forall z \in \Gamma_n, \quad \forall n \geq n_1.$$

Using $|\frac{B}{A}| < 1$ and Lemma 2, we obtain

$$|\frac{B}{A}||\frac{\sin^2 \frac{\gamma}{\delta}\pi z}{\sin^2 \pi z}| \leq |\frac{B}{A}| = 1 - r_0 \quad (0 < r_0 < 1). \tag{2.15}$$

Since $g(z) = O(\frac{e^{2\pi|Imz|}}{|z|})$, there exists a positive number C_1 such that

$$|g(z)| \leq C_1 \frac{e^{2\pi|Imz|}}{|z|}. \tag{2.16}$$

By (2.16), Lemma 1 and inequality (2.15),

$$|\frac{g(z)}{f(z)}| \leq \frac{C_2}{|z|}$$

occurs, where $C_2 = \frac{C_1}{2C^2|A|r_0}$. Since for all $z \in \Gamma_n$,

$$|z| \geq n + \frac{1}{2},$$

and there exists a natural number n_2 such that for all $n \geq n_2$,

$$C_2 < n + \frac{1}{2},$$

we have $|\frac{g(z)}{f(z)}| < 1$ for all $z \in \Gamma_n$, $n \geq n_0$, where $n_0 = max\{n_1, n_2\}$. By Rouché's Theorem $f(z)$ and $\Phi^+(z)$ have same number of zeros on the square contour Γ_n. In [3], it has been shown that $f(z)$ has $4n+2$ zeros on the square contour Γ_n. So $\Phi^+(z)$ has $4n + 2$ zeros on the same contour. Since by (2.12), $\Phi_1^+(z)$ is an even function, we can denote the zeros of $\Phi_1^+(z)$ lying inside Γ_n by

$$\pm z_{-2n}^-, \pm z_{-2n}^+, ..., \pm z_{-2}^-, \pm z_{-2}^+, \pm z_0^-, \pm z_0^+, \pm z_2^-, \pm z_2^+, ..., \pm z_{2n}^-, \pm z_{2n}^+.$$

We note that the zeros of $\Phi_1^+(z)$ are real in virtue of (2.12) and (2.13), since the eigenvalues of the periodic boundary value problem (1.2) are non-negative.

Similarly for $n \geq n_0 + 1$ the function $\Phi_1^+(z)$ has $4n - 2$ zeros on the square contour Γ_{n-1}. There are four zeros between Γ_{n-1} and Γ_n. Two of them lie in the region

$$D_n = \{z \in \mathbf{C} : n - \frac{1}{2} < Rez < n + \frac{1}{2}, \quad |Imz| < n + \frac{1}{2}\}.$$

Therefore, the zeros z_{2n}^{-} and z_{2n}^{+} of $\Phi_1^{+}(z)$ lie in the interval $(n - \frac{1}{2}, n + \frac{1}{2})$ for $n \geq n_0 + 1$.

Now let's consider circles $C_\rho = \{z \in \mathbf{C} : |z - \dot{z}_{2n}^{\mp}| = \rho\}$ where ρ is any number such that $0 < \rho < \frac{1}{4}$ and $\dot{z}_{2n}^{-}$, $\dot{z}_{2n}^{+}$ are zeros of $f(z)$. Functions $f(z)$ and $g(z)$ are analytic on the circles and the region bounded by these circles. In order to show the inequality $|f(z)| > |g(z)|$ on these circles we need to prove that there exists a positive number M such that for all $z \in C_\rho$

$$|f(z)| \geq M\rho^2 e^{2\pi|Imz|}. \tag{2.17}$$

It can be easily seen that for all $z \in C_\rho$

$$|f(z)| \geq 2|A||\sin \pi z|^2 |1 - |\frac{B}{A}||\frac{\sin^2 \frac{\gamma}{\delta}\pi z}{\sin^2 \pi z}||. \tag{2.18}$$

So it remains to show that there exists a positive number N such that for all $z \in C_\rho$

$$|\sin \pi z|^2 \geq N\rho^2 e^{2\pi|Imz|} \tag{2.19}$$

and

$$|1 - |\frac{B}{A}||\frac{\sin^2 \frac{\gamma}{\delta}\pi z}{\sin \pi z}|| \geq \varepsilon_0, \tag{2.20}$$

where ε_0 is an arbitrary positive number.

First let's show inequality (2.19). It is clear that the equality

$$|\sin \pi z|^2 = \frac{1}{4}[e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi x].$$

holds. Let $F(y) = \frac{1}{4}[e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi(\dot{z}_{2n}^{\mp} \mp \sqrt{\rho^2 - y^2})]$. Since F is even, it is enough to consider $F(y)$ for all $y \in [0, \rho]$. We must show that there is a positive number K such that $F(y) \geq K\rho^2$.

Case 1: Let $x = \dot{z}_{2n}^{\mp} - \sqrt{\rho^2 - y^2}$. Then we have

$$F(y) = \frac{1}{4}[e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi\dot{z}_{2n}^{\mp} \cos 2\pi\sqrt{\rho^2 - y^2} - 2\sin 2\pi\dot{z}_{2n}^{\mp} \sin 2\pi\sqrt{\rho^2 - y^2}].$$

Case 1.1: Let $\cos 2\pi\dot{z}_{2n}^{\mp} > 0$ and $\sin 2\pi\dot{z}_{2n}^{\mp} > 0$. Then $F'(y) > 0$. Therefore F is increasing for $y \in [0, \rho]$ and then we can find a positive number K_1 such that

$$F(y) \geq F(0) = \sin^2 \pi(\dot{z}_{2n}^{\mp} - \rho) \geq K_1\rho^2.$$

Case 1.2: Let $\cos 2\pi\dot{z}_{2n}^{\mp} < 0$ and $\sin 2\pi\dot{z}_{2n}^{\mp} > 0$. Then we get

$$F(y) \geq \frac{1}{4}[e^{2\pi y} + e^{-2\pi y} - 2\sin 2\pi\dot{z}_{2n}^{\mp} \sin 2\pi\sqrt{\rho^2 - y^2}].$$

Let

$$G(y) = \frac{1}{4}[e^{2\pi y} + e^{-2\pi y} - 2\sin 2\pi\dot{z}_{2n}^{\mp} \sin 2\pi\sqrt{\rho^2 - y^2}].$$

Since $G'(y) > 0$, the inequality $G(0) \leq G(y)$ holds. On the other hand there exists a positive number K_2 such that

$$G(0) \geq \frac{1}{2}[1 - \sin 2\pi\rho] \geq K_2\rho^2 4,$$

and hence we have $F(y) \geq K_2\rho^2$.

Case 1.3: Let $\cos 2\pi\dot{z}_{2n}^{\mp} < 0$ and $\sin 2\pi\dot{z}_{2n}^{\mp} < 0$. Then we can find a positive number K_3 such that

$$F(y) \geq \frac{1}{2}\cosh y \geq \frac{1}{2} \geq K_3\rho^2.$$

Case 1.4: Let $\cos 2\pi\dot{z}_{2n}^{\mp} > 0$ and $\sin 2\pi\dot{z}_{2n}^{\mp} < 0$. Then the inequality

$$F(y) \geq \frac{1}{4}[e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi\dot{z}_{2n}^{\mp} \cos 2\pi\sqrt{\rho^2 - y^2}]$$

holds. Let

$$H(y) = \frac{1}{4}[e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi\dot{z}_{2n}^{\mp} \cos 2\pi\sqrt{\rho^2 - y^2}].$$

It is easy to see that $H'(y) > 0$. Thus $H(0) \leq H(y)$. From this observation a positive number K_4 can be found such that

$$F(y) \geq H(0) \geq \sin^2 \pi\rho \geq K_4\rho^2.$$

Take $K = min\{K_1, K_2, K_3, K_4\}$. Then $|\sin \pi z|^2 \geq K\rho^2$.

Case 2: Let $x = \dot{z}_{2n}^{\mp} + \sqrt{\rho^2 - y^2}$.

Similarly one can easily show that the inequality

$$|\sin \pi z|^2 \geq K\rho^2.$$

So we get $|\sin \pi z|^2 \geq K\rho^2$ for all cases. Since

$$\frac{e^{2\pi|Imz|}}{|\sin \pi z|^2} \leq \frac{e^{\pi/2}}{K\rho^2},$$

we get the inequality

$$|\sin \pi z|^2 \geq N\rho^2 e^{2\pi|Imz|},$$

where $N = \frac{K}{e^{\pi/2}}$.

Let $k = \frac{\gamma}{\delta}$. Then we have

$$|\frac{\sin k\pi z}{\sin \pi z}|^2_{x=\dot{z}^{\mp}_{2n}\mp\sqrt{\rho^2-y^2}} = \frac{e^{2|k|\pi y} + e^{-2|k|\pi y} - 2\cos 2|k|\pi(\dot{z}^{\mp}_{2n} \mp \sqrt{\rho^2 - y^2})}{e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi(\dot{z}^{\mp}_{2n} \mp \sqrt{\rho^2 - y^2})}.$$

Substitute

$$\frac{e^{2|k|\pi y} + e^{-2|k|\pi y} - 2\cos 2|k|\pi(\dot{z}^{\mp}_{2n} \mp \sqrt{\rho^2 - y^2})}{e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi(\dot{z}^{\mp}_{2n} \mp \sqrt{\rho^2 - y^2})}$$

for

$$\frac{e^{2\pi y\varepsilon} + e^{-2\pi y\varepsilon} - 2\cos 2\pi\varepsilon(\dot{z}^{\mp}_{2n} \mp \sqrt{\rho^2 - y^2})}{e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi(\dot{z}^{\mp}_{2n} \mp \sqrt{\rho^2 - y^2})} \quad (0 \leq y < \infty, \quad 0 \leq \varepsilon < 1).$$

Let $f(\varepsilon) = e^{2\pi y\varepsilon} + e^{-2\pi y\varepsilon} - 2\cos 2\pi\varepsilon(\dot{z}^{\mp}_{2n} \mp \sqrt{\rho^2 - y^2})$. It is clear that the function $f(\varepsilon)$ is non-decreasing. Hence for $\varepsilon \in [0, 1), f(\varepsilon) \leq f(1)$. Thus for $z \in C_\rho$, we have the inequality

$$|\frac{\sin^2 \frac{\gamma}{\delta}\pi z}{\sin^2 \pi z}| \leq 1.$$

Since the inequality

$$|\frac{B}{A}||\frac{\sin^2 \frac{\gamma}{\delta}\pi z}{\sin^2 \pi z}| \leq |\frac{B}{A}| = 1 - \varepsilon_0 \quad (0 < \varepsilon_0 < 1)$$

holds we get the inequality (2.20). Therefore by the equations (2.18), (2.19), (2.20), the inequality

$$|f(z)| \geq 2|A|\varepsilon_0 N\rho^2 e^{2\pi|Imz|}$$

holds for $z \in C_\rho$. So we obtain

$$|f(z)| \geq M\rho^2 e^{2\pi|Imz|},$$

where $M = 2|A|\varepsilon_0 N$.

Choose

$$\rho = \sqrt{\frac{2C_1}{M(n - \frac{1}{2})}}.$$

There exists a positive number m_0 such that for all $n \geq m_0$, $\rho < \frac{1}{4}$. Hence $|\frac{g(z)}{f(z)}| < 1$, for all z on these circles, $n \geq m_0$. From the Rouché theorem, $f(z)$ and $\Phi^+(z)$ have same number of zeros inside these circles. Thus the function $\Phi^+(z)$ has one zero inside the circle

$$|z - \dot{z}^{-}_{2n}| = \rho$$

and also has one zero inside the circle

$$|z - \dot{z}_{2n}^{-}| = \rho.$$

Denote these zeros by z_{2n}^{-}, z_{2n}^{+} respectively. Let $z_{2n}^{\mp} - \dot{z}_{2n}^{\mp} = \frac{r_{2n}^{\mp}}{\sqrt{n}}$. Then inequality

$$|r_{2n}^{\mp}| < \sqrt{\frac{2C_1 n}{K(n-\frac{1}{2})}}$$

holds. Therefore $Sup|r_{2n}^{\mp}|$ is bounded above i.e. the sequence $r_{2n}^{\mp}$ is bounded. Thus we have $z_{2n}^{\mp} = \dot{z}_{2n}^{\mp} + O(\frac{1}{\sqrt{n}})$. Since $z_{2n}^{\mp} = \frac{s_{2n}^{\mp}\delta}{2\pi}$, the equality

$$s_{2n}^{\mp} = \dot{s}_{2n}^{\mp} + O(\frac{1}{\sqrt{n}})$$

holds. So we get an asymptotic formula for eigenvalues of periodic problem.

Similarly, for antiperiodic boundary value problem, we have

$$s_{2n+1}^{\mp} = \dot{s}_{2n+1}^{\mp} + O(\frac{1}{\sqrt{n}}).$$

Combining these two results, we get

$$s_{n}^{\mp} = \dot{s}_{n}^{\mp} + O(\frac{1}{\sqrt{n}}).$$

Our aim is to find better asymptotic formula.

3. Special case

Let $q(x) = c$. In this case, we investigate asymptotic formula for eigenvalues of equation (1.1).

Theorem 1. *If we have $q(x) = c$, then*
$\mu_n^{\mp} = (\dot{s}_n^{\mp})^2 + \frac{\dot{s}_n^{\mp} c}{n\pi} \frac{1}{A \sin \dot{s}_n^{\mp}\delta + B\frac{\gamma}{\delta} \sin \dot{s}_n^{\mp}\gamma} [A(\frac{a}{\alpha} + \frac{\omega - a}{\beta}) \sin \dot{s}_n^{\mp}\delta + B(\frac{a}{\alpha} - \frac{\omega - a}{\beta}) \sin \dot{s}_n^{\mp}\gamma] + 0(\frac{1}{n^2})$.

PROOF. When its solutions $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ with initial value conditions

$$\theta(0,\lambda) = 1, \quad \theta'(0,\lambda) = 0; \quad \varphi(0,\lambda) = 0, \quad \varphi'(0,\lambda) = 1.$$

are researched, we have

$$\theta(x,\lambda) = \begin{cases} \cos s'x, & \text{if } 0 < x \leq a, \\ \cos s'a \cos s''(x-a) - \frac{s'}{s''} \sin s'a \sin s''(x-a), & \text{if } a < x \leq \omega, \end{cases}$$

$$\varphi(x,\lambda)=\begin{cases}\dfrac{\sin s'x}{s'}, & \text{if } 0<x\le a,\\ \frac{\sin s'a}{s'}\cos s''(x-a)+\cos s'a\frac{\sin s''(x-a)}{s''}, & \text{if } a<x\le\omega,\end{cases}$$

where $\lambda\alpha^2-c=s'^2, \quad \lambda\beta^2-c=s''^2$.

Moreover we get
$F(\lambda)=\theta(\omega,\lambda)+\varphi'(\omega,\lambda)=2\cos(s'a+s''(\omega-a))-\frac{(s'-s'')^2}{s's''}\sin s'a\sin s''(\omega-a)$.
Define

$$\Phi^+(\lambda)=F(\lambda)-2=-4\sin^2(\frac{s'a+s''(\omega-a)}{2})-\frac{(s'-s'')^2}{s's''}\sin s'a\sin s''(\omega-a).$$

Since eigenvalues of periodic problem coincide with $F(\lambda)=2$, the equation

$$-4\sin^2(\frac{s'^{\mp}_{2n}a+s''^{\mp}_{2n}(\omega-a)}{2})-\frac{(s'^{\mp}_{2n}-s''^{\mp}_{2n})^2}{s''^{\mp}_{2n}s''^{\mp}_{2n}}\sin s'^{\mp}_{2n}a\sin s''^{\mp}_{2n}(\omega-a)=0 \qquad (3.1)$$

holds. Since $s^{\mp}_{2n}=\dot{s}^{\mp}_{2n}+\frac{1}{\delta}\delta^{\mp}_{2n}$ and $\delta^{\mp}_{2n}=0(\frac{1}{\sqrt{n}})$, we have

$$s'^{\mp}_{2n}=\sqrt{(s^{\mp}_{2n})^2\alpha^2-c}=s^{\mp}_{2n}\alpha-\frac{c}{2s^{\mp}_{2n}\alpha}+0(\frac{1}{n^3})$$

and

$$s'^{\mp}_{2n}=\sqrt{(s^{\mp}_{2n})^2\beta^2-c}=s^{\mp}_{2n}\beta-\frac{c}{2s^{\mp}_{2n}\beta}+0(\frac{1}{n^3}).$$

Then the following equalities hold;

$$\sin^2(\frac{s'^{\mp}_{2n}+s''^{\mp}_{2n}(\omega-a)}{2})=\sin^2\frac{\dot{s}^{\mp}_{2n}\delta}{2}+\frac{\delta^{\mp}_{2n}}{2}\sin\dot{s}^{\mp}_{2n}\delta+O(\frac{1}{n}), \qquad (3.2)$$

$$\frac{(s'^{\mp}_{2n}-s''^{\mp}_{2n})^2}{s'^{\mp}_{2n}s''^{\mp}_{2n}}=\frac{(\alpha-\beta)^2}{\alpha\beta}+O(\frac{1}{n^2}), \qquad (3.3)$$

and

$$\sin s'^{\mp}_{2n}a\sin s''^{\mp}_{2n}(\omega-a)=-\frac{1}{2}[\cos\dot{s}^{\mp}_{2n}\delta-\cos\dot{s}^{\mp}_{2n}\gamma]+\frac{\delta^{\mp}_{2n}}{2}[\sin\dot{s}^{\mp}_{2n}\delta-\frac{\gamma}{\delta}\sin\dot{s}^{\mp}_{2n}\gamma]+O(\frac{1}{n}). \qquad (3.4)$$

When we put the equalities (3.2), (3.3), (3.4) in the equation (3.1), we get

$$\delta^{\mp}_{2n}[A\sin\dot{s}^{\mp}_{2n}\delta+B\frac{\gamma}{\delta}\sin\dot{s}^{\mp}_{2n}\gamma]=O(\frac{1}{n}).$$

Since $A\sin\dot{s}^{\mp}_{2n}\delta+B\frac{\gamma}{\delta}\sin\dot{s}^{\mp}_{2n}\gamma\neq 0$, we have

$$\delta^{\mp}_{2n}=O(\frac{1}{n}).$$

Thus the equalities

$$\sin^2\left(\frac{s'^{\mp}_{2n} + s''^{\mp}_{2n}(\omega - a)}{2}\right) = \sin^2 \frac{\dot{s}^{\mp}_{2n}\delta}{2} + \left[\frac{\delta^{\mp}_{2n}}{2} - \frac{c\delta}{8n\pi}\left(\frac{a}{\alpha} + \frac{\omega - a}{\beta}\right)\right] \sin \dot{s}^{\mp}_{2n}\delta + O\left(\frac{1}{n^2}\right), \tag{3.5}$$

$$\frac{(s'^{\mp}_{2n} - s''^{\mp}_{2n})^2}{s'^{\mp}_{2n} s''^{\mp}_{2n}} = \frac{(\alpha - \beta)^2}{\alpha\beta} + O\left(\frac{1}{n^2}\right), \tag{3.6}$$

and

$$\sin s'^{\mp}_{2n} a \sin s''^{\mp}_{2n}(\omega - a) = -\tfrac{1}{2}[\cos \dot{s}^{\mp}_{2n}\delta - \cos \dot{s}^{\mp}_{2n}\gamma] + \tfrac{\delta^{\mp}_{2n}}{2}[\sin \dot{s}^{\mp}_{2n}\delta - \tfrac{\gamma}{\delta} \sin \dot{s}^{\mp}_{2n}\gamma]$$
$$-\frac{c\delta}{8n\pi}\left[\left(\frac{a}{\alpha} + \frac{\omega - a}{\beta}\right) \sin \dot{s}^{\mp}_{2n}\delta - \left(\frac{a}{\alpha} - \frac{\omega - a}{\beta}\right) \sin \dot{s}^{\mp}_{2n}\gamma\right] + O\left(\frac{1}{n^2}\right). \tag{3.7}$$

exist. Similarly when the equalities (3.5), (3.6), (3.7) are written in equation (3.1), the equalities

$$\delta^{\mp}_{2n} = \tfrac{c\delta}{4n\pi} \tfrac{1}{A \sin \dot{s}^{\mp}_{2n}\delta + B\frac{\gamma}{\delta} \sin \dot{s}^{\mp}_{2n}\gamma}[A(\tfrac{a}{\alpha} + \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n}\delta + B(\tfrac{a}{\alpha} - \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n}\gamma] + 0(\tfrac{1}{n^2})$$

and

$$s^{\mp}_{2n} = \dot{s}^{\mp}_{2n} + \tfrac{c}{4n\pi} \tfrac{1}{A \sin \dot{s}^{\mp}_{2n}\delta + B\frac{\gamma}{\delta} \sin \dot{s}^{\mp}_{2n}\gamma}[A(\tfrac{a}{\alpha} + \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n}\delta + B(\tfrac{a}{\alpha} - \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n}\gamma] + 0(\tfrac{1}{n^2})$$

hold. Moreover we have

$$\mu^{\mp}_{2n} = (s^{\mp}_{2n})^2 = (\dot{s}^{\mp}_{2n})^2 + \tfrac{\dot{s}^{\mp}_{2n} c}{2n\pi} \tfrac{1}{A \sin \dot{s}^{\mp}_{2n}\delta + B\frac{\gamma}{\delta} \sin \dot{s}^{\mp}_{2n}\gamma}[A(\tfrac{a}{\alpha} + \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n}\delta$$
$$+B(\tfrac{a}{\alpha} - \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n}\gamma] + 0(\tfrac{1}{n^2}).$$

Similarly for $\mu^{\mp}_{2n+1}$, we can get

$$\mu^{\mp}_{2n+1} = (\dot{s}^{\mp}_{2n+1})^2 + \tfrac{\dot{s}^{\mp}_{2n+1} c}{(2n+1)\pi} \tfrac{1}{A \sin \dot{s}^{\mp}_{2n+1}\delta + B\frac{\gamma}{\delta} \sin \dot{s}^{\mp}_{2n+1}\gamma}[A(\tfrac{a}{\alpha} + \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n+1}\delta$$
$$+B(\tfrac{a}{\alpha} - \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{2n+1}\gamma] + 0(\tfrac{1}{n^2}).$$

Combining results of $\mu^{\mp}_{2n}$, $\mu^{\mp}_{2n+1}$ we find the formula

$$\mu^{\mp}_{n} = (\dot{s}^{\mp}_{n})^2 + \tfrac{\dot{s}^{\mp}_{n} c}{n\pi} \tfrac{1}{A \sin \dot{s}^{\mp}_{n}\delta + B\frac{\gamma}{\delta} \sin \dot{s}^{\mp}_{n}\gamma}[A(\tfrac{a}{\alpha} + \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{n}\delta + B(\tfrac{a}{\alpha} - \tfrac{\omega - a}{\beta}) \sin \dot{s}^{\mp}_{n}\gamma] + 0(\tfrac{1}{n^2}). \quad \blacksquare$$

4. General case

In this section, we investigate a asymptotic formula for eigenvalues of equation (1.1).

Theorem 2. *If $q(x)$ is a real valued function with period $\omega > 0$, then*

$$
\begin{aligned}
&\mu_n^{\mp} = \dot{\mu}_n^{\mp} + \tfrac{\dot{s}_n^{\mp}}{n\pi[A\sin\dot{s}_n^{\mp}\delta + B\frac{\gamma}{\delta}\sin\dot{s}_n^{\mp}\gamma]}\ \{A[\tfrac{1}{\alpha}\textstyle\int_0^a q(t)dt + \tfrac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}_n^{\mp}\delta\\
&+B[\tfrac{1}{\alpha}\textstyle\int_0^a q(t)dt - \tfrac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}_n^{\mp}\gamma\\
&+\tfrac{1}{2}(\tfrac{1}{\beta} - \tfrac{\beta}{\alpha^2})\sin\dot{s}_n^{\mp}\beta(\omega - a)\textstyle\int_0^a \cos\dot{s}_n^{\mp}\alpha(a-2t)q(t)dt\\
&+\tfrac{1}{2}(\tfrac{1}{\alpha} - \tfrac{\alpha}{\beta^2})\sin\dot{s}_n^{\mp}\alpha a\textstyle\int_a^\omega \cos\dot{s}_n^{\mp}\beta(\omega - 2t + a)q(t)dt\} + O(\tfrac{1}{n^2}).
\end{aligned}
$$

PROOF. By the equations (2.3), (2.6), (2.8), we have the formula

$$
\begin{aligned}
&\theta(x,\lambda) = \tfrac{1}{s}\textstyle\int_0^a[\tfrac{\cos s\beta(x-a)\sin s\alpha(a-t)\cos s\alpha t}{\alpha} + \tfrac{\sin s\beta(x-a)\cos s\alpha(a-t)\cos s\alpha t}{\beta}]q(t)dt\\
&+\tfrac{1}{s}\textstyle\int_a^x[\tfrac{\sin s\beta(x-t)\cos s\alpha a\cos s\beta(t-a)}{\beta} - \tfrac{\alpha\sin s\beta(x-t)\sin s\alpha a\sin s\beta(t-a)}{\beta^2}]q(t)dt\\
&\qquad + \cos s\alpha a\cos s\beta(x-a) - \frac{\alpha}{\beta}\sin s\alpha a\sin s\beta(x-a) + O(\frac{1}{|s|^2}) \qquad (4.1)
\end{aligned}
$$

where $a < x \leq \omega$. Using the (2.5), (2.7), (2.9), the formula

$$
\begin{aligned}
&\varphi'(x,\lambda) = \tfrac{1}{s}\textstyle\int_0^a[\tfrac{-\beta\sin s\beta(x-a)\sin s\alpha(a-t)\sin s\alpha t}{\alpha^2} + \tfrac{\cos s\beta(x-a)\cos s\alpha(a-t)\sin s\alpha t}{\alpha}]q(t)dt\\
&+\tfrac{1}{s}\textstyle\int_a^x[\tfrac{\cos s\beta(x-t)\sin s\alpha a\cos s\beta(t-a)}{\alpha} + \tfrac{\cos s\beta(x-t)\cos s\alpha a\sin s\beta(t-a)}{\beta}]q(t)dt\\
&\qquad -\frac{\beta}{\alpha}\sin s\alpha a\sin s\beta(x-a) + \cos s\alpha a\cos s\beta(x-a) + O(\frac{1}{|s|^2}) \qquad (4.2)
\end{aligned}
$$

holds for $x \in (a,\omega]$. From the equations (4.1), (4.2), it is easily seen that

$$F(\lambda) = \theta(\omega,\lambda) + \varphi'(\omega,\lambda)$$

$$
\begin{aligned}
&= A\cos s\delta + B\cos s\gamma + \frac{\sin s\alpha a\cos s\beta(\omega-a)}{s\alpha}\int_0^a q(t)dt + \frac{\cos s\alpha a\sin s\beta(\omega-a)}{s\beta}\int_a^\omega q(t)dt\\
&+\frac{\sin s\beta(\omega-a)}{s\beta}\int_0^a[\cos s\alpha(a-t)\cos s\alpha t - \frac{\beta^2}{\alpha^2}\sin s\alpha(a-t)\sin s\alpha t]q(t)dt\\
&+\frac{\sin s\alpha a}{s\alpha}\int_a^\omega[\cos s\beta(\omega-t)\cos s\beta(t-a) - \frac{\alpha^2}{\beta^2}\sin s\beta(\omega-t)\sin s\beta(t-a)]q(t)dt\\
&+O(\frac{1}{|s|^2}).
\end{aligned}
$$

In [3], it has been shown that $F_0(\lambda) = A\cos s\delta + B\cos s\gamma$ when $q(x) = 0$.

Since the eigenvalue of periodic problem (1.2) is in the form

$$s_{2n}^{\mp} = \dot{s}_{2n}^{\mp} + \frac{1}{\delta}\delta_{2n}^{\mp},$$

where $\delta_{2n}^{\mp} = 0(\frac{1}{\sqrt{n}})$, we get

$$F((s_{2n}^{\mp})^2) = 2, \quad and \quad F_0(\dot{s}_{2n}^{\mp})^2) = 2.$$

Thus the following formula holds,

$$F(\mu_{2n}^{\mp}) - 2 = F_0(\dot{\mu}_{2n}^{\mp}) - 2 + \tfrac{\sin s_{2n}^{\mp}\beta(\omega-a)}{s_{2n}^{\mp}\beta} \textstyle\int_0^a [\cos s_{2n}^{\mp}\alpha(a-t)\cos s_{2n}^{\mp}\alpha t - \frac{\beta^2}{\alpha^2}$$

$$\sin s_{2n}^{\mp}\alpha(a-t)\sin s_{2n}^{\mp}\alpha t]q(t)dt + \frac{\sin s_{2n}^{\mp}\alpha a}{s_{2n}^{\mp}\alpha}\int_a^{\omega}[\cos s_{2n}^{\mp}\beta(\omega-t)\cos s_{2n}^{\mp}\beta(t-a)$$

$$-\frac{\alpha^2}{\beta^2}\sin s_{2n}^{\mp}\beta(\omega-t)\sin s_{2n}^{\mp}\beta(t-a)]q(t)dt + \frac{\sin s_{2n}^{\mp}\alpha a\cos s_{2n}^{\mp}\beta(\omega-a)}{s_{2n}^{\mp}\alpha}\int_0^a q(t)dt$$

$$+\frac{\cos s_{2n}^{\mp}\alpha a\sin s_{2n}^{\mp}\beta(\omega-a)}{s_{2n}^{\mp}\beta}\int_a^{\omega} q(t)dt + O(\frac{1}{|s_{2n}^{\mp}|^2}) = 0, \tag{4.3}$$

where $\mu_{2n}^{\mp} = (s_{2n}^{\mp})^2, \quad \dot{\mu}_{2n}^{\mp} = (\dot{s}_{2n}^{\mp})^2$. It is clear that

$$F_0(\dot{\mu}_{2n}^{\mp}) - 2 = -2A\sin\dot{s}_{2n}^{\mp}\delta\sin\frac{\delta_{2n}^{\mp}}{2} - 2B\sin\dot{s}_{2n}^{\mp}\gamma\sin\frac{\gamma}{\delta}\delta_{2n}^{\mp} + O(\frac{1}{n}) \tag{4.4}$$

and

$$\frac{1}{s_{2n}^{\mp}} = O(\frac{1}{n}). \tag{4.5}$$

When we put the equations (4.4) and (4.5) in the equation (4.3), we get

$$\delta_{2n}^{\mp}[A\sin\dot{s}_{2n}^{\mp}\delta + B\frac{\gamma}{\delta}\sin\dot{s}_{2n}^{\mp}\gamma] = O(\frac{1}{n}).$$

Since $A\sin\dot{s}_{2n}^{\mp}\delta + B\frac{\gamma}{\delta}\sin\dot{s}_{2n}^{\mp}\gamma \neq 0$, we have

$$\delta_{2n}^{\mp} = O(\frac{1}{n}). \tag{4.6}$$

Using the equation (4.6) we get the following equalities:

$$\frac{\sin s_{2n}^{\mp}\alpha a\cos s_{2n}^{\mp}\beta(\omega-a)}{s_{2n}^{\mp}} = \frac{\delta}{2n\pi}\sin\dot{s}_{2n}^{\mp}\alpha a\cos\dot{s}_{2n}^{\mp}\beta(\omega-a) + O(\frac{1}{n^2}), \tag{4.7}$$

$$\frac{\cos s_{2n}^{\mp}\alpha a\sin s_{2n}^{\mp}\beta(\omega-a)}{s_{2n}^{\mp}} = \frac{\delta}{2n\pi}\cos\dot{s}_{2n}^{\mp}\alpha a\sin\dot{s}_{2n}^{\mp}\beta(\omega-a) + O(\frac{1}{n^2}), \tag{4.8}$$

$$\frac{\sin s_{2n}^{\mp}\beta(\omega-a)\cos s_{2n}^{\mp}\alpha(a-2t)}{s_{2n}^{\mp}}=\frac{\delta}{2n\pi}\sin \dot{s}_{2n}^{\mp}\beta(\omega-a)\cos \dot{s}_{2n}^{\mp}\alpha(a-2t)+O(\frac{1}{n^2}), \tag{4.9}$$

$$\frac{\sin s_{2n}^{\mp}\alpha a\cos s_{2n}^{\mp}\beta(\omega-2t+a)}{s_{2n}^{\mp}}=\frac{\delta}{2n\pi}\sin \dot{s}_{2n}^{\mp}\alpha a\cos \dot{s}_{2n}^{\mp}\beta(\omega-2t+a)+O(\frac{1}{n^2}), \tag{4.10}$$

$$F_0(\dot{\mu}_{2n}^{\mp})-2=-2A\sin \dot{s}_{2n}^{\mp}\delta\sin\frac{\delta_{2n}^{\mp}}{2}-2B\sin \dot{s}_{2n}^{\mp}\gamma\sin\frac{\gamma}{2\delta}\delta_{2n}^{\mp}+O(\frac{1}{n^2}). \tag{4.11}$$

When the equations (4.7), (4.8), (4.9), (4.10), (4.11) are written in formula (4.3), the equality

$$F(\mu_{2n}^{\mp})-2=-2A\sin \dot{s}_{2n}^{\mp}\delta\sin\frac{\delta_{2n}^{\mp}}{2}-2B\sin \dot{s}_{2n}^{\mp}\gamma\sin\frac{\gamma}{2\delta}\delta_{2n}^{\mp}$$

$$+\frac{\delta}{2n\pi}[\frac{1}{\alpha}\sin \dot{s}_{2n}^{\mp}\alpha a\cos \dot{s}_{2n}^{\mp}\beta(\omega-a)+\frac{1}{2}(\frac{1}{\beta}+\frac{\beta}{\alpha^2})\cos \dot{s}_{2n}^{\mp}\alpha a\sin \dot{s}_{2n}^{\mp}\beta(\omega-a)]\int_0^a q(t)dt$$

$$+\frac{\delta}{4n\pi}(\frac{1}{\beta}-\frac{\beta}{\alpha^2})\sin \dot{s}_{2n}^{\mp}\beta(\omega-a)\int_0^a \cos \dot{s}_{2n}^{\mp}\alpha(a-2t)q(t)dt$$

$$+\frac{\delta}{2n\pi}[\frac{1}{\beta}\cos \dot{s}_{2n}^{\mp}\alpha a\sin \dot{s}_{2n}^{\mp}\beta(\omega-a)+\frac{1}{2}(\frac{1}{\alpha}+\frac{\alpha}{\beta^2})\sin \dot{s}_{2n}^{\mp}\alpha a\cos \dot{s}_{2n}^{\mp}\beta(\omega-a)]\int_a^\omega q(t)dt$$

$$+\frac{\delta}{4n\pi}(\frac{1}{\alpha}-\frac{\alpha}{\beta^2})\sin \dot{s}_{2n}^{\mp}\alpha a\int_a^\omega \cos \dot{s}_{2n}^{\mp}\beta(\omega-2t+a)q(t)dt+O(\frac{1}{n^2})=0$$

exists. Since

$$\sin\frac{\delta_{2n}^{\mp}}{2}=\frac{\delta_{2n}^{\mp}}{2}+O(\frac{1}{n^3})$$

, and

$$\sin\frac{\gamma}{2\delta}\delta_{2n}^{\mp}=\frac{\gamma}{2\delta}\delta_{2n}^{\mp}+O(\frac{1}{n^3})$$

, we have the equality

$$\delta_{2n}^{\mp}=\frac{\delta}{2n\pi[A\sin \dot{s}_{2n}^{\mp}\delta+B\frac{\gamma}{\delta}\sin \dot{s}_{2n}^{\mp}\gamma]}\{[\frac{1}{\alpha}\sin \dot{s}_{2n}^{\mp}\alpha a\cos \dot{s}_{2n}^{\mp}\beta(\omega-a)$$

$$+\frac{1}{2}(\frac{1}{\beta}+\frac{\beta}{\alpha^2})\cos \dot{s}_{2n}^{\mp}\alpha a\sin \dot{s}_{2n}^{\mp}\beta(\omega-a)]\int_0^a q(t)dt$$

$$+\frac{1}{2}(\frac{1}{\beta}-\frac{\beta}{\alpha^2})\sin\dot{s}_{2n}^{\mp}\beta(\omega-a)\int_0^a\cos\dot{s}_{2n}^{\mp}\alpha(a-2t)q(t)dt$$

$$+[\frac{1}{\beta}\cos\dot{s}_{2n}^{\mp}\alpha a\sin\dot{s}_{2n}^{\mp}\beta(\omega-a)+\frac{1}{2}(\frac{1}{\alpha}+\frac{\alpha}{\beta^2})\sin\dot{s}_{2n}^{\mp}\alpha a\cos\dot{s}_{2n}^{\mp}\beta(\omega-a)]\int_a^\omega q(t)dt$$

$$+\frac{1}{2}(\frac{1}{\alpha}-\frac{\alpha}{\beta^2})\sin\dot{s}_{2n}^{\mp}\alpha a\int_a^\omega\cos\dot{s}_{2n}^{\mp}\beta(\omega-2t+a)q(t)dt\}+O(\frac{1}{n^2}).$$

Thus we get the formula

$$s_{2n}^{\mp}=\dot{s}_{2n}^{\mp}+\tfrac{1}{\delta}\delta_{2n}^{\mp}$$

$$=\dot{s}_{2n}^{\mp}+\frac{1}{2n\pi[A\sin\dot{s}_{2n}^{\mp}\delta+B\frac{\gamma}{\delta}\sin\dot{s}_{2n}^{\mp}\gamma]}\{[\frac{1}{\alpha}\sin\dot{s}_{2n}^{\mp}\alpha a\cos\dot{s}_{2n}^{\mp}\beta(\omega-a)$$

$$+\frac{1}{2}(\frac{1}{\beta}+\frac{\beta}{\alpha^2})\cos\dot{s}_{2n}^{\mp}\alpha a\sin\dot{s}_{2n}^{\mp}\beta(\omega-a)]\int_0^a q(t)dt$$

$$+\frac{1}{2}(\frac{1}{\beta}-\frac{\beta}{\alpha^2})\sin\dot{s}_{2n}^{\mp}\beta(\omega-a)\int_0^a\cos\dot{s}_{2n}^{\mp}\alpha(a-2t)q(t)dt$$

$$+[\frac{1}{\beta}\cos\dot{s}_{2n}^{\mp}\alpha a\sin\dot{s}_{2n}^{\mp}\beta(\omega-a)+\frac{1}{2}(\frac{1}{\alpha}+\frac{\alpha}{\beta^2})\sin\dot{s}_{2n}^{\mp}\alpha a\cos\dot{s}_{2n}^{\mp}\beta(\omega-a)]\int_a^\omega q(t)dt$$

$$+\frac{1}{2}(\frac{1}{\alpha}-\frac{\alpha}{\beta^2})\sin\dot{s}_{2n}^{\mp}\alpha a\int_a^\omega\cos\dot{s}_{2n}^{\mp}\beta(\omega-2t+a)q(t)dt\}+O(\frac{1}{n^2}).$$

So we get the equality

$$\mu_{2n}^{\mp}=\dot{\mu}_{2n}^{\mp}+\frac{\dot{s}_{2n}^{\mp}}{2n\pi[A\sin\dot{s}_{2n}^{\mp}\delta+B\frac{\gamma}{\delta}\sin\dot{s}_{2n}^{\mp}\gamma]}\{A[\frac{1}{\alpha}\int_0^a q(t)dt$$

$$+\frac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}_{2n}^{\mp}\delta+B[\frac{1}{\alpha}\int_0^a q(t)dt-\frac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}_{2n}^{\mp}\gamma$$

$$+\frac{1}{2}(\frac{1}{\beta}-\frac{\beta}{\alpha^2})\sin\dot{s}_{2n}^{\mp}\beta(\omega-a)\int_0^a\cos\dot{s}_{2n}^{\mp}\alpha(a-2t)q(t)dt$$

$$+\frac{1}{2}(\frac{1}{\alpha}-\frac{\alpha}{\beta^2})\sin\dot{s}_{2n}^{\mp}\alpha a\int_a^\omega\cos\dot{s}_{2n}^{\mp}\beta(\omega-2t+a)q(t)dt\}+O(\frac{1}{n^2}). \qquad (4.12)$$

Similarly we can obtain

$$\begin{aligned}\mu^{\mp}_{2n+1} = \dot{\mu}^{\mp}_{2n+1} + \frac{\dot{s}^{\mp}_{2n+1}}{2(n+1)\pi[A\sin\dot{s}^{\mp}_{2n+1}\delta + B\frac{\gamma}{\delta}\sin\dot{s}^{\mp}_{2n+1}\gamma]}\{A[\frac{1}{\alpha}\int_0^a q(t)dt \\ +\frac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}^{\mp}_{2n+1}\delta + B[\frac{1}{\alpha}\int_0^a q(t)dt - \frac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}^{\mp}_{2n+1}\gamma \\ +\frac{1}{2}(\frac{1}{\beta} - \frac{\beta}{\alpha^2})\sin\dot{s}^{\mp}_{2n+1}\beta(\omega - a)\int_0^a \cos\dot{s}^{\mp}_{2n+1}\alpha(a-2t)q(t)dt \\ +\frac{1}{2}(\frac{1}{\alpha} - \frac{\alpha}{\beta^2})\sin\dot{s}^{\mp}_{2n+1}\alpha a\int_a^\omega \cos\dot{s}^{\mp}_{2n+1}\beta(\omega - 2t + a)q(t)dt\} + O(\frac{1}{n^2}).\end{aligned} \tag{4.13}$$

From the equations (4.12) and (4.13), we can get

$$\begin{aligned}\mu^{\mp}_{n} = \dot{\mu}^{\mp}_{n} + \frac{\dot{s}^{\mp}_{n}}{n\pi[A\sin\dot{s}^{\mp}_{n}\delta + B\frac{\gamma}{\delta}\sin\dot{s}^{\mp}_{n}\gamma]} \\ \{A[\frac{1}{\alpha}\int_0^a q(t)dt + \frac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}^{\mp}_{n}\delta \\ +B[\frac{1}{\alpha}\int_0^a q(t)dt - \frac{1}{\beta}\int_a^\omega q(t)dt]\sin\dot{s}^{\mp}_{n}\gamma \\ +\frac{1}{2}(\frac{1}{\beta} - \frac{\beta}{\alpha^2})\sin\dot{s}^{\mp}_{n}\beta(\omega - a)\int_0^a \cos\dot{s}^{\mp}_{n}\alpha(a-2t)q(t)dt \\ +\frac{1}{2}(\frac{1}{\alpha} - \frac{\alpha}{\beta^2})\sin\dot{s}^{\mp}_{n}\alpha a\int_a^\omega \cos\dot{s}^{\mp}_{n}\beta(\omega - 2t + a)q(t)dt\} + O(\frac{1}{n^2}).\end{aligned}$$

■

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