UNIVERSAL SPACES, TYCHONOFF AND SPECTRAL SPACES

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Abstract

This paper deals with some universal spaces. The class of morphisms in TOP orthogonal to all Tychonoff spaces is characterized. We also characterize topological spaces X for which the universal Tychonoff space associated to X is a spectral space.

Introduction

Let X be a topological space. The ring of all real valued continuous functions defined on X will be denoted by $C(X)$. Two subsets A and B are said to be *completely separated* in X if there exists a mapping f in $C(X)$ such that $f(a) = 0$ for all a in A and $f(b) = 1$ for all b in B. It will be convenient to say that $x, y \in X$ are *completely separated* if $\{x\}$ and $\{y\}$ are completely separated.

A space X is said to be *completely regular* if every closed set F of X is completely separated from any point x not in F. Recall that a topological space X is called a T_1*-space* if each singleton of X is closed. A completely regular T_1-space is called a *Tychonoff space* [20]. Tychonoff spaces are also called $T_{3\frac{1}{2}}$-spaces because they clearly sit between regular (or T_3-) spaces and normal (or T_4-) spaces.

Now, let us recall the construction of the universal Tychonoff space (or ρ-identification) of a topological space: Let X be a topological space. We define the equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in C(X)$. Let $\rho(X)$ denote the set of equivalence classes and let $\theta : X \longrightarrow \rho(X)$ be the canonical surjection map assigning to each point of X its equivalence class. Since every f in $C(X)$ is constant on each equivalence class, we can define $\rho(f) : \rho(X) \longrightarrow \mathbb{R}$ by $\rho(f)(\theta_X(x)) = f(x)$.

Now equip $\rho(X)$ with the topology whose closed sets are of the form $\cap[\rho(f_\alpha)^{-1}(F_\alpha)$:

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$\alpha \in I]$, where $f_\alpha : X \longrightarrow \mathbb{R}$ (resp. F_α) is a continuous map (resp. a closed set of $\mathbb{R}$). It is well known that, with this topology, $\rho(X)$ is a Tychonoff space (see for instance [22]).

The construction $\rho(X)$ satisfies some categorical properties:

For each Tychonoff space Y and each continuous map $f : X \longrightarrow Y$, there exists a unique continuous map $\widetilde{f} : \rho(X) \longrightarrow Y$ such that $\widetilde{f} \circ \theta_X = f$. We will say that $\rho(X)$ is the *ρ-reflection* (or Tychonoff reflection) of X.

Let $f : X \longrightarrow Y$ be a continuous map. Then there exists a continuous map $\rho(f) : \rho(X) \longrightarrow \rho(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \theta_X \downarrow & \circlearrowleft & \downarrow \theta_Y \\ \rho(X) & \xrightarrow{\rho(f)} & \rho(Y) \end{array}$$

From the above properties, it is clear that ρ is a covariant functor from the category of topological spaces **TOP** into the full subcategory **TYCH** of **TOP** whose objects are Tychonoff spaces.

In [4], the authors have introduced the following separation axioms.

Definition 0.1. Let i, j be two integers such that $0 \leq i < j \leq 2$. Let us denote by $\mathbf{T_i}$ the functor from **TOP** to **TOP** which takes each topological space X to its T_i-identification (the universal T_i-space associated with X). A topological space X is said to be $T_{(i,j)}$-space if $\mathbf{T}_i(X)$ is a T_j-space (thus we have three new types of separation axioms namely; $T_{(0,1)}$, $T_{(0,2)}$ and $T_{(1,2)}$).

Definition 0.2. Let $\mathbf{C}$ be a category and $\mathbf{F}$, $\mathbf{G}$ two (covariant) functors from $\mathbf{C}$ to itself.

(1) An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},\mathbf{G})}$-object if $\mathbf{G}(\mathbf{F}(X))$ is isomorphic with $\mathbf{F}(X)$.
(2) Let P be a topological property on the objects of $\mathbf{C}$. An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},P)}$-object if $\mathbf{F}(X)$ satisfies the property P.

Following Definition 0.2, one may define other new separation axioms namely:

- a space X is said to be a $T_{(i,\rho)}$-space if $\mathbf{T}_i(X)$ is a Tychonoff space; and
- a space X is a $T_{(S,\rho)}$-space if $\mathbf{S}(X)$ is a Tychonoff space, where $\mathbf{S}$ is the soberification functor from **TOP** to itself.

This paper consists of three investigations, the first one deals with $T_{(0,\rho)}$-spaces and $T_{(S,\rho)}$-spaces.

The second investigation deals with some categorical properties of the category **TYCH**; more precisely, a characterization of the class of morphisms in **TOP** rendered invertible by the functor ρ is given.

The third investigation is devoted to the study of topological spaces X such that $\rho(X)$ is a spectral space.

1. $T_{(0,\rho)}$-spaces, $T_{(1,\rho)}$-spaces and $T_{(S,\rho)}$-spaces

Recall that a continuous map $q : X \longrightarrow Y$ is said to be a *quasihomeomorphism* if $U \longmapsto q^{-1}(U)$ defines a bijection $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$, where $\mathcal{O}(X)$ is the set of all open subsets of X [14].

A subset F of a topological space X is said to be *irreducible* if for each open subsets U and V of X such that $F \cap U \neq \emptyset$ and $F \cap V \neq \emptyset$, we have $F \cap U \cap V \neq \emptyset$.

A topological space X is called *sober* if each nonempty irreducible closed set F of X has a unique generic point (i.e., there exists a unique $x \in X$ such that $F = \overline{\{x\}}$).

Let X be a topological space. We define the binary relation $\equiv$ on X by $x \equiv y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Then $\equiv$ is an equivalence relation on X and the resulting quotient space $\mathbf{T}_0(X) := X/\sim$ is the T_0-identification of X (the universal T_0-space associated with X). The canonical map $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is an onto quasihomeomorphism.

If $q : X \longrightarrow Y$ is a continuous map, then there exists a unique continuous map $\mathbf{T}_0(q) : \mathbf{T}_0(X) \to \mathbf{T}_0(Y)$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ \mu_X \downarrow & \circlearrowleft & \downarrow \mu_Y \\ \mathbf{T}_0(X) & \xrightarrow{\mathbf{T}_0(q)} & \mathbf{T}_0(Y) \end{array}$$

Now, we give a characterization of a $T_{(0,\rho)}$-spaces.

Theorem 1.1. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is a $T_{(0,\rho)}$-space;*

(2) *the following properties hold:*

  (*i*) *X is a $T_{(0,1)}$-space;*

  (*ii*) *Every closed subspace F of X is completely separated from any point $x \notin F$.*

PROOF.

(1) $\Longrightarrow$ (2)

(*i*) Straightforward.

(*i*) Let F be a closed set of X and $x \notin F$. Since $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is a quasihomeomorphism, there exists a closed set $F^{'}$ of $\mathbf{T}_0(X)$ such that $F = \mu_X^{-1}(F^{'})$. Clearly, $\mu_X(x) \notin F^{'}$.

On the other hand, there exists a continuous map $g : \mathbf{T}_0(X) \longrightarrow \mathbb{R}$ such that $g(\mu_X(x)) = 1$ and $g(F^{'}) = \{0\}$. Set $f = g \circ \mu_X$; then $f(x) = 1$ and $f(F) = g(\mu_X(F)) = g(F^{'}) = \{0\}$.

(2) $\Longrightarrow$ (1).

It suffices to show that every closed set $F^{'}$ of $\mathbf{T}_0(X)$ is separated from any point $\mu_X(x) \notin F^{'}$. Indeed, $F = \mu_X^{-1}(F^{'})$ is a closed set of X not containing x. By (*ii*), there exists a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f(x)$=1 and $f(F) = \{0\}$.

Since $\mathbb{R}$ is a T_0-space, there exists a unique continuous map making the following diagram is commutative:

$$\begin{array}{ccc} X & \stackrel{\mu_X}{\longrightarrow} & \mathbf{T}_0(X) \\ f\downarrow & \swarrow & \\ \mathbb{R} & g & \end{array}$$

Then g satisfies the following relations:
$g(\mu_X(x)) = g \circ \mu_X(x) = f(x) = 1$ and $g(F^{'}) = f(F) = \{0\}$.

On the other hand, $\mathbf{T}_0(X)$ is a T_1-space (since X is a $T_{(0,1)}$-space). Therefore, X is a $T_{(0,\rho)}$-space. ■

The following result gives a class of topological spaces in which there is equivalence between $T_{(0,1)}$ and $T_{(0,2)}$-spaces.

Clearly, a $T_{(0,2)}$-space is $T_{(0,1)}$; and the converse does not hold.

Let X be a $T_{(0,1)}$-space. According to Theorem 1.1, X is a $T_{(0,\rho)}$-space and thus X is $T_{(0,2)}$-space. This leads to the following result.

Corollary 1.2. *Let X be a topological space such that every closed set F of X is completely separated from any point $x \notin F$. Then, X is a $T_{(0,1)}$-space if and only if X is a $T_{(0,2)}$-space.*

Proposition 1.3. *Let $q : X \longrightarrow Y$ be a quasihomeomorphism. Then the following statements are equivalent:*

(i) X is a $T_{(0,\rho)}$-space;

(ii) Y is a $T_{(0,\rho)}$-space.

Proof.

$(i) \Longrightarrow (ii)$

Of course, $\mathbf{T}_0(q) : \mathbf{T}_0(X) \longrightarrow \mathbf{T}_0(Y)$ is a quasihomeomorphism. On the other hand, since $\mathbf{T}_0(X)$ is a Tychonoff space, it is a T_2-space, so that it is a sober space. Since in addition $\mathbf{T}_0(Y)$ is a $\mathbf{T}_0$-space; then $\mathbf{T}_0(q)$ is a homeomorphism. Therefore, $\mathbf{T}_0(Y)$ is completely regular, which means that Y is a $T_{(0,\rho)}$-space.

$(ii) \Longrightarrow (i)$

The complete regularity of $\mathbf{T}_0(Y)$ implies that it is a T_1-space. Since in addition $\mathbf{T}_0(X)$ is a T_0-space, we conclude that $\mathbf{T}_0(q)$ is a homeomorphism. Therefore, $\mathbf{T}_0(X)$ is a Tychonoff space. This means that X is a $T_{(0,\rho)}$-space. ■

Let us recall the T_1-reflection of a space X (the universal T_1-space associated with X). Let X be a topological space and $\mathbf{R}$ be the intersection of all closed equivalence relations on X (an equivalence relation on X is said to be closed if its equivalence classes are closed in X). The quotient space $X/\mathbf{R}$ is homeomorphic to the T_1-reflection of X. Let $\eta_X : X \longrightarrow \mathbf{T}_1(X)$ be the canonical onto map and $q : X \longrightarrow Y$ a continuous map. Then the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ \eta_X \downarrow & \circlearrowright & \downarrow \eta_Y \\ \mathbf{T}_1(X) & \xrightarrow{\mathbf{T}_1(q)} & \mathbf{T}_1(Y) \end{array}$$

Unfortunately, we have not been able to characterize $T_{(1,\rho)}$-spaces. However, the following result sheds some light on these spaces.

Proposition 1.4. *Let X be a topological space. If every closed subspace F of X is completely separated from any point $x \notin F$, then X is a $T_{(1,\rho)}$-space.*

Proof. Let X be a topological space and $q : X \longrightarrow \mathbf{T}_1(X)$ be the canonical onto map. Let F be a closed set of $\mathbf{T}_1(X)$ and $q(x) \notin F$. It is clear that $G = q^{-1}(F)$ is a closed set of X not containing x. Hence there exists a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f(G) = \{0\}$ and $f(x) = 1$. According to the fact that $\mathbb{R}$ is a T_1-space, there exists a unique continuous map g making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{q} & \mathbf{T}_1(X) \\ f \downarrow & \swarrow & \\ \mathbb{R} & g & \end{array}$$

Thus $g(F) = g(q(q^{-1}(F))) = g \circ q(G) = f(G) = \{0\}$ and $g(q(x)) = f(x) = 1$. Therefore, $\mathbf{T}_1(X)$ is completely regular. ■

The following example shows that the converse of Proposition 1.4 does not hold.

Example 1.5. Let X be a topological space with a generic point α (that is, $X = \overline{\{\alpha\}}$). Suppose that X has a nonempty closed set F not containing α. To construct such a space X, it suffices to take any space Y and $\alpha \notin Y$; set $X := Y \cup \{\alpha\}$. Equip X with the topology whose closed sets are all closed sets of Y and X.

Then X is a $T_{(1,\rho)}$-space; however there is no continuous map $f : X \longrightarrow \mathbb{R}$ that separates F and α.

Proof.

- Of course, $\mathbf{T}_1(X)$ is a one-point space. Hence $\mathbf{T}_1(X)$ is a Tychonoff space; that is X is a $T_{(1,\rho)}$-space.
- Note that X is irreducible. But, it is easily seen that a continuous map from an irreducible space into $\mathbb{R}$ is constant (since the continuous image of an irreducible space is irreducible and each irreducible subset of a Hausdorff space is a one-point set).

It follows that there is no continuous map $f : X \longrightarrow \mathbb{R}$ separating F and α. ■

Question 1.6. *Give a characterization of $T_{(1,\rho)}$-spaces.*

Let X be a topological space and $\mathbf{S}(X)$ be the set of all nonempty irreducible closed set of X. Let F be a closed set of X, set $\widetilde{F} = \{G \in \mathbf{S}(X) : G \subseteq F\}$ then ($\widetilde{F}$, F is a closed set of X) is the collection of closed sets of a topology on $\mathbf{S}(X)$ and the following properties hold ([14]).

(i) The map $\eta_X : X \longrightarrow \mathbf{S}$ that carries $x \in X$ to $\eta_X(x) = \overline{\{x\}}$, is a quasi-homeomorphism.

(ii) $\mathbf{S}(X)$ is a sober space.

(iii) Let $q : X \longrightarrow Y$ be a continuous map. Then the following diagram is commutative (where $\mathbf{S}(q)$ is defined by $\mathbf{S}(q)(G) = \overline{q(G)}$, for each $G \in \mathbf{S}(X)$):

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ \eta_X \downarrow & \circlearrowright & \downarrow \eta_Y \\ \mathbf{S}(X) & \xrightarrow{\mathbf{S}(q)} & \mathbf{S}(Y) \end{array}$$

(iv) The topological space $\mathbf{S}(X)$ is called the *soberification* of X, and the assignment $\mathbf{S}$ defines a functor from the category **TOP** to itself.

(v) $\mathbf{S}(q)$ is a homeomorphism if and only if q is a quasihomeomorphism (see [7, theorem 2.2]).

Now, we give a characterization of $T_{(S,\rho)}$-spaces.

Theorem 1.7. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is a $T_{(S,\rho)}$-space;*

(2) *X satisfies the following properties:*

(i) *X is a $T_{(S,1)}$-space;*

(ii) *Every closed subspace F of X is completely separated from any point $x \notin F$.*

PROOF.

(1) $\Longrightarrow$ (2)

(i) Since $\mathbf{S}(X)$ is a Tychonoff space, it is a T_1-space. Thus X is a $T_{(S,1)}$-space.

(i) Let F be a closed set of X and $x \notin F$. Since $\overline{\{x\}} \notin \widetilde{F}$ and X is a $T_{(S,\rho)}$-space, there exists a continuous map $f : \mathbf{S}(X) \longrightarrow \mathbb{R}$ such that $f(\widetilde{F}) = \{0\}$ and $f(\overline{\{x\}}) = 1$. Set $g = f \circ \eta_X$; then we have

- $g(x) = f(\eta_X(x)) = f(\overline{\{x\}}) = 1$; and
- if $y \in F$, then $\overline{\{y\}} \in \widetilde{F}$; so that $g(y) = f(\overline{\{y\}}) \in f(\widetilde{F}) = \{0\}$, consequently, $g(F) = \{0\}$. Therefore, f separates x and F.

(2) $\Longrightarrow$ (1)

Let $\widetilde{F}$ be a closed subspace of $\mathbf{S}(X)$ and $G \in \mathbf{S}(X)\backslash\widetilde{F}$. Let $x \in G\backslash F$. By (ii), there exists a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f(F) = \{0\}$ and $f(x) = 1$. Let $g = \eta_{\mathbb{R}}^{-1} \circ \mathbf{S}(f) : \mathbf{S}(X) \longrightarrow \mathbb{R}$, where $\eta_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbf{S}(\mathbb{R})$ is the map sending each $x \in \mathbb{R}$ to the one-point set $\{x\}$.

Let $H \in \mathbf{S}(X)$; it is clear that $f(H)$ is irreducible in $\mathbb{R}$ (any continuous image of an irreducible set is irreducible). But, any nonempty irreducible subset of a Hausdorff space is a one-point set. Then, there exists $a \in \mathbb{R}$ such that $f(H) = \{a\}$. Hence $g(H) = \eta_{\mathbb{R}}^{-1}(\mathbf{S}(f)(H)) = \eta_{\mathbb{R}}^{-1}(\overline{f(H)}) = \eta_{\mathbb{R}}^{-1}(\{a\}) = a$.

Now, since $G \in \mathbf{S}(X)$ and $f(G) \supseteq \{f(x)\} = \{1\}$, we have $f(G) = \{1\}$ and thus $g(G) = 1$.

Let $H \in \widetilde{F}$. Since $f(F) = \{0\}$ and $H \subseteq F$, we conclude that $f(H) = \{0\}$; so that $g(H) = 0$; and consequently, $g(\widetilde{F}) = \{0\}$. ■

A similar proof to that of Corollary 1.2 gives the following.

Corollary 1.8. *Let X be a topological space such that every closed set F of X is completely separated from any point $x \notin F$. Then, X is $T_{(S,1)}$ if and only if X is $T_{(S,2)}$.*

2. The class of continuous maps orthogonal to all Tychonoff spaces

It is worth noting that reflective subcategories arise throughout mathematics, via several examples such as the free group and free ring functors in algebra, various compactification functors in topology, and completion functors in analysis: cf. [18, p. 90]. Recall from [18, p. 89] that a subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is called *reflective* (in $\mathbf{C}$) if the inclusion functor $\mathbf{I} : \mathbf{D} \longrightarrow \mathbf{C}$ has a left adjoint functor $\mathbf{F} : \mathbf{C} \longrightarrow \mathbf{D}$; i.e., if, for each object A of $\mathbf{C}$, there exist an object $\mathbf{F}(A)$ of $\mathbf{D}$ and a morphism $\mu_A : A \longrightarrow \mathbf{F}(A)$ in $\mathbf{C}$ such that, for each object X in $\mathbf{D}$ and each morphism $f : A \longrightarrow X$ in $\mathbf{C}$, there exists a unique morphism $\widetilde{f} : F(A) \longrightarrow X$ in $\mathbf{D}$ such that $\widetilde{f} \circ \mu_A = f$.

The concept of reflections in categories has been investigated by several authors (see for example [9; 10; 11; 12; 15; 16; 19]). This concept serves the purpose of unifying various constructions in mathematics.

Historically, the concept of reflections in categories seems to have its origin in the universal extension property of the Stone-Čech compactification of a Tychonoff space.

A morphism $f : A \longrightarrow B$ and an object X in a category $\mathbf{C}$ are called *orthogonal* [13], if the mapping $\hom_{\mathbf{C}}(f, X) : \hom_{\mathbf{C}}(B, X) \longrightarrow \hom_{\mathbf{C}}(A, X)$ that takes g to gf is bijective. For a class of morphisms Σ (resp., a class of objects $\mathbf{D}$), we denote by $\Sigma^{\perp}$ the class of objects orthogonal to every f in Σ (resp., by $\mathbf{D}^{\perp}$ the class of morphisms orthogonal to all X in $\mathbf{D}$) [13].

The orthogonality class of morphisms $\mathbf{D}^{\perp}$ associated with a reflective subcategory $\mathbf{D}$ of a category $\mathbf{C}$ satisfies the following identity $\mathbf{D}^{\perp\perp} = \mathbf{D}$ [1, proposition 2.6]. Thus, it is of interest to give explicitly the class $\mathbf{D}^{\perp}$. Note also that, if $\mathbf{I} : \mathbf{D} \longrightarrow \mathbf{C}$ is the inclusion functor and $\mathbf{F} : \mathbf{C} \longrightarrow \mathbf{D}$ is a left adjoint functor of $\mathbf{I}$, then the class $\mathbf{D}^{\perp}$ is the collection of all morphisms of $\mathbf{C}$ rendered invertible by the functor $\mathbf{F}$ [1, proposition 2.3].

This section is devoted to the study of the orthogonal class $\mathbf{TYCH}^{\perp}$; hence we will give a characterization of morphisms rendered invertible by the functor ρ.

The following concepts are needed.

Definitions 2.1. Let $f : X \longrightarrow Y$ be a continuous map.

(1) f is said to be *ρ-injective* (or *ρ-one-to-one*) if for each $x, y \in X$; x and y are completely separated, then so are $f(x)$ and $f(y)$.

(2) f is said to be *ρ-surjective* (or *ρ-onto*) if for each $y \in Y$, there exists $x \in X$ such that $f(x)$ and y are not completely separated.
(3) f is said to be *ρ-bijective* if it is both ρ-injective and ρ-surjective.

Examples 2.2.

(1) Every onto continuous map is ρ-onto.
(2) A ρ-onto continuous map need not be onto. For, let $X = \{0\}$ and $Y = \{0, 1\}$ equipped with the trivial topologies. Set $f : X \longrightarrow Y$, which takes 0 to 0.
Clearly, f is a ρ-onto continuous map that is not onto.
(3) A ρ-injective continuous map need not be one-to-one: Let X be a topological space that is not a Tychonoff space. Of course, θ_X is a ρ-injective continuous map but it is not one-to-one.
(4) A one-to-one continuous map need not be ρ-injective: Let $X = \{0, 1\}$ equipped with the discrete topology and $Y = \{0, 1\}$ equipped with the trivial topology. Let $f = 1_X : X \longrightarrow Y$. Then, f is a one-to-one continuous map. However, 0 and 1 are completely separated in X but $f(0)$ and $f(1)$ are not completely separated in Y.

Before giving the main result of this section, we need a lemma:

Lemma 2.3. *Let $f : X \longrightarrow Y$ be a continuous map. Then the following properties hold:*

(1) *f is ρ-injective if and only if $\rho(f)$ is injective.*
(2) *f is ρ-surjective if and only if $\rho(f)$ is surjective.*
(3) *f is ρ-bijective if and only if $\rho(f)$ is bijective.*
(4) *Let $g : Y \longrightarrow Z$ be a continuous map. If two among f, g, $g \circ f$ are ρ-bijective, then so is the third one.*

Proof.

(1)
- Suppose that $\rho(f)$ is injective. Let $x, y \in X$ be two completely separated points, that is $\theta_X(x) \neq \theta_X(y)$. Since $\rho(f)$ is one-to-one, $\rho(f)(\theta_X(x)) \neq \rho(f)(\theta_X(y))$. Hence $\theta_Y(f(x)) \neq \theta_Y(f(y))$. Thus, $f(x)$ and $f(y)$ are completely separated. Therefore, f is ρ-injective.
- Conversely, suppose that f is ρ-injective. Let $x, y \in X$ be such that $\rho(f)(\theta_X(x)) = \rho(f)(\theta_X(y))$. Then $\theta_Y(f(x)) = \theta_Y(f(y))$. Hence $f(x)$ and $f(y)$ are not completely separated. Since f is ρ-injective, we conclude that x and y are not completely separated; and thus $\theta_X(x) = \theta_X(y)$. Therefore, $\rho(f)$ is one-to-one.

(2)
- Suppose that $\rho(f)$ is surjective. Let $y \in Y$. Since $\rho(f)$ is onto, there exists $x \in X$ such that $\rho(f)(\theta_X(x)) = \theta_Y(y)$. Hence $\theta_Y(f(x)) = \theta_Y(y)$; and consequently $f(x)$ and y are not completely separated. Therefore, f is ρ-onto.
- Conversely, suppose that f is ρ-onto. Let $y \in Y$. Since f is ρ-onto,

there exists $x \in X$ such that $f(x)$ and y are not completely separated. Hence $\theta_Y(f(x)) = \theta_Y(y) = \rho(f)(\theta_X(x))$. Thus, $\rho(f)$ is onto.

(3) Is a direct consequence of (1) and (2).

(4) This is a direct consequence of (3) and the fact that ρ is a functor.

■

Now, we give a characterization of morphisms rendered invertible by the functor ρ.

Theorem 2.4. *Let $f : X \longrightarrow Y$ be a continuous map. Then the following statements are equivalent:*

(1) *$\rho(f)$ is a homeomorphism;*

(2) *f is ρ-bijective and for each continuous map $g : X \longrightarrow \mathbb{R}$, there exists a unique continuous map $h : Y \longrightarrow \mathbb{R}$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} & Y \\ g\downarrow & \swarrow & \\ \mathbb{R} & h & \end{array}$$

Proof.

(1) $\Longrightarrow$ (2)

According to Lemma 2.3, f is ρ-bijective.

By [1, proposition 2.6] and [1, proposition 2.3], the morphism f is orthogonal to $\mathbb{R}$, since $\mathbb{R}$ is a Tychonoff space. Hence, for any continuous map $g : X \longrightarrow \mathbb{R}$ there exists a unique continuous map $h : Y \longrightarrow \mathbb{R}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} & Y \\ g\downarrow & \swarrow & \\ \mathbb{R} & h & \end{array}$$

(2) $\Longrightarrow$ (1)

• By Lemma 2.3, $\rho(f)$ is bijective.

• *$\rho(f)$ is closed.*

Let $F^{'}$ be a closed set of $\rho(X)$. Then there exists a collection $g_i : X \longrightarrow \mathbb{R}$ of continuous maps and a collection of closed sets F_i of $\mathbb{R}$ such that $F^{'} = \cap[\rho(g_i)^{-1}(F_i) : i \in I]$. Let us prove that $\rho(f)(F')$ is closed in $\rho(Y)$. Without loss of generality, one may suppose that $F^{'} = \rho(g)^{-1}(F)$ where $g : X \longrightarrow \mathbb{R}$ is a continuous map and F is a closed set of $\mathbb{R}$ (because $\rho(f)$ is bijective). By (2), there exists a continuous map $h : X \longrightarrow \mathbb{R}$ such that $g = h \circ f$

The following diagram is commutative:

$$\begin{array}{ccccccccc} & & X & & \stackrel{f}{\rightarrow} & & Y & & \\ & \theta_X \swarrow & \circlearrowleft & g \searrow & \circlearrowleft & \swarrow h & \circlearrowleft & \searrow \theta_Y & \\ \rho(X) & & \underset{\rho(g)}{\longrightarrow} & & \mathbb{R} & & \underset{\rho(h)}{\longleftarrow} & & \rho(Y) \end{array}$$

Then we have

$$\begin{aligned}\rho(f)(\rho(g)^{-1}(F)) &= \rho(f)(\theta_X(g^{-1}(F)) \\ &= (\theta_Y \circ f)(g^{-1}(F)) \\ &= (\theta_Y \circ f)(f^{-1}(h^{-1}(F))) \\ &= (\theta_Y \circ f)[(\theta_Y \circ f)^{-1}(\rho(h)^{-1}(F))].\end{aligned}$$

Of course, $\theta_Y \circ f$ is onto (since f is ρ-onto). Hence $\rho(f)(\rho(g)^{-1}(F)) = \rho(h)^{-1}(F)$; and consequently, $\rho(f)$ is closed.

Now, $\rho(f)$ is a bijective continuous closed map; so that it is a homeomorphism. ∎

3. Tychonoff spectral spaces

The main result of this section is the characterization of topological spaces X such that $\rho(X)$ is a spectral space.

Let us first recall that a topological space X is said to be *spectral* if the following axioms hold [17]:

(i) X is a sober space;
(ii) X is compact and has a basis of compact open sets;
(iii) the family of compact open sets of X is closed under finite intersections.

Let $Spec(R)$ denote the set of prime ideals of a commutative ring R with identity. Recall that, the *Zariski topology* or the *hull-kernel topology* for $Spec(R)$ is defined by letting $C \subseteq Spec(R)$ be closed if and only if there exists an ideal $\mathcal{A}$ of R such that $C = \{\mathcal{P} \in Spec(R) : \mathcal{P} \supseteq \mathcal{A}\}$. Hochster [17] has proved that a topological space is homeomorphic to the prime spectrum of some ring equipped with the Zariski topology if and only if it is spectral. In lattice theory, a spectral space is characterized by the fact that it is homeomorphic to the prime spectrum of a bounded (with a 0 and a 1) distributive lattice.

Note that spectral spaces are of interest not only in (topological) ring and lattice theory, but also in computer science, in particular, in domain theory.

In order to motivate the reader, we give some links between the previous axioms (i) and (ii) functional analysis.

Bratteli and Elliott showed in [8] that a topological space X is homeomorphic to the primitive spectrum of an approximately finite-dimensional C^*-algebra (called $A.F$ C^*-algebra) equipped with the Jacobson topology if and only if it has a countable basis and it satisfies the above axioms (i) and (ii). By the way, C^*-algebras and foliation theory are strongly linked. Thus, there must be some link between spectral spaces and foliation theory; this was done by the authors of [5] and [6].

Recently, some authors (see for example [2] and [3]) have been interested on particular type of spectral spaces constructed from some compactifications (namely, the one point-compactification for [3], and the Walman compactification and the T_0-compactification for [2]).

Pursuing this kind of investigations for spectral spaces, we are interested, here, in topological spaces such that the Tychonoff reflection is a spectral space.

Definition 3.1. A topological space X is said to be *Tychonoff spectral*, if $\rho(X)$ is a spectral space.

First, we need to characterize when $\rho(X)$ is a compact space.
Some concepts have to be introduced.

Definition 3.2. Let X be a topological space, H a subset of $C(X)$.
We say that H has the *finite intersection property* (FIP, for short) if for each finite subset J of H, we have $\cap[f^{-1}(\{0\}) : f \in J] \neq \emptyset$.

Definitions 3.3. Let X be a topological space and U an open subset of X.
(1) U is said to be *zero-closed* (z-closed, for short) if there exists a subset H of $C(X)$ such that $U = \cap[f^{-1}(\{0\}) : f \in H]$.
(2) U is said to be *zero-clopen* (z-clopen, for short) if U and $X \setminus U$ are both z-closed subsets of X.

Let us recall an interesting result that characterizes completely regular spaces in terms of zero-sets. Let X be a topological space and $A \subseteq X$. A is called a *zero-set* if there exists $f \in C(X)$ such that $A = f^{-1}(\{0\})$.

Proposition 3.4. *[22, proposition 1.7] A space is Tychonoff if and only if the family of zero-sets of the space is a base for the closed sets.*

Let us state a useful remark.

Remark 3.5. A closed set of $\rho(X)$ is of the form $\cap[\rho(f)^{-1}(\{0\}) : f \in H]$, where H is a collection of continuous maps $f : X \longrightarrow \mathbb{R}$.

Indeed, $\rho(X)$ is a Tychonoff space. Then the collection $\{g^{-1}\{0\} \mid g : \rho(X) \longrightarrow \mathbb{R}$ continuous$\}$ is a basis of closed sets of $\rho(X)$.
According to the universal property of $\rho(X)$, each continuous map $g : \rho(X) \longrightarrow \mathbb{R}$ may be written as $g = \rho(f)$ with $f = g \circ \theta_X$.

Proposition 3.6. *Let X be a topological space. Then the following statements are equivalent:*
(*i*) *$\rho(X)$ is compact;*
(*ii*) *For each subset H of $C(X)$ satisfying the FIP, $\cap[f^{-1}(\{0\}) : f \in H] \neq \emptyset$.*

PROOF.
$(i) \Longrightarrow (ii)$
Let H be a subset of $C(X)$ satisfying the FIP.
Suppose that $\cap[f^{-1}(\{0\}) : f \in H] = \emptyset$. Then $\cap[\theta_X^{-1}(\rho(f)^{-1}(\{0\})) : f \in H] = \theta_X^{-1}(\cap[(\rho(f)^{-1}(\{0\}) : f \in H]) = \emptyset$. Since θ_X is onto, we have $\cap[\rho(f)^{-1}(\{0\}) : f \in H] = \emptyset$. According to (i), there exists a finite subset J of H such that $\cap[\rho(f)^{-1}(\{0\}) : f \in J] = \emptyset$. Thus, $\cap[f^{-1}(\{0\}) : f \in J] = \emptyset$, which contradicts the FIP. Therefore, $\cap[f^{-1}(\{0\}) : f \in H] \neq \emptyset$.

$(ii) \Longrightarrow (i)$

To prove that $\rho(X)$ is compact, it suffices to show that for each subset H of $C(X)$, with the property $\cap[\rho(f)^{-1}(\{0\}) : f \in H] = \emptyset$, there exists a finite subset J of H such that $\cap[\rho(f)^{-1}(\{0\}) : f \in J] = \emptyset$ (see Remark 3.5).

Indeed, $\cap[\rho(f)^{-1}(\{0\}) : f \in H] = \emptyset$ implies that $\cap[f^{-1}(\{0\}) : f \in H]) = \emptyset$. It follows that H does not satisfy the FIP; which shows that there is a finite subset J of H such that $\cap[f^{-1}(\{0\}) : f \in J] = \emptyset$; and consequently, $\cap[\rho(f)^{-1}(\{0\}) : f \in J] = \emptyset$. Therefore, $\rho(X)$ is compact.

■

Remark 3.7. Clearly if X is a compact space, then $\rho(X)$ is compact.

The converse does not hold, as shown by the following example.

Example 3.8. A non-compact space X such that $\rho(X)$ is compact.

Let Y be a non-compact space and $\alpha \notin Y$. Set $X = Y \cup \{\alpha\}$ endowed with the topology whose closed sets are X and those of Y. It is clear that $\rho(X)$ is a one-point set and X is not compact.

Now, we are in a position to give the main result of this section.

Theorem 3.9. *Let X be a topological space. Then the following statements are equivalent:*

(1) *$\rho(X)$ is spectral;*

(2) *X satisfies the following properties:*

  (i) *for each subset H of $C(X)$ satisfying the FIP, we have $\cap[f^{-1}(\{0\}) : f \in H] \neq \emptyset$;*

  (ii) *for each completely separated points $x, y \in X$, there exists a z-clopen subset U of X containing one of the x, y and not containing the other.*

We need two lemmata.

Lemma 3.10. *[17] Let X be a T_1-space. Then X is spectral if and only if X is compact and totally disconnected. (i.e., X is a Stone space)*

Lemma 3.11. *[23, Lemma 29.6] A compact T_2-space is totally disconnected if and only if whenever $x \neq y$ in X, there is a clopen set in X containing x and not containing y.*

Proof of Theorem 3.9.

$(1) \Longrightarrow (2)$

Since each spectral space is compact follows immediately from Proposition 3.6. Let $x, y \in X$ be completely separated points. Then $\theta_X(x) \neq \theta_X(y)$. Since $\rho(X)$ is a T_1-spectral space, $\rho(X)$ is totally disconnected, by Lemma 3.10. Now according to Lemma 3.11, there exists a clopen set $\widetilde{U}$ of $\rho(X)$ containing $\theta_X(x)$ and not containing $\theta_X(y)$. By Remark 3.5, there exists a subset H of $C(X)$ such that $\widetilde{U} = \cap[\rho(f)^{-1}(\{0\}) : f \in H]$. Let $U := \theta_X^{-1}(\widetilde{U}) = \cap[f^{-1}(\{0\}) : f \in H]$. Thus U is a z-closed subset of X.

On the other hand, we have $X \backslash U = \theta_X^{-1}(\rho(X)\backslash\widetilde{U})$. Since $\rho(X)\backslash\widetilde{U}$ is closed in $\rho(X)$, there exists a subset H_1 of $C(X)$ such that $\rho(X)\backslash\widetilde{U} = \cap[\rho(f)^{-1}(\{0\}) : f \in H_1]$ (see Remark 3.5). Hence, $X\backslash U = \cap[f^{-1}(\{0\}) : f \in H_1])$ is a z-closed subset of X. Therefore, U is a z-clopen subset of X containing x and not containing y.

$(2) \Longrightarrow (1)$

First, let us remark that $\rho(X)$ is compact, by Proposition 3.6.

But, $\rho(X)$ is a T_1-space, thus to prove that it is spectral, it is enough to show that $\rho(X)$ is totally disconnected (see Lemma 3.10). To do this, we use Lemma 3.11 (since $\rho(X)$ is a compact T_2-space).

Let $x, y \in X$ be such that $\theta_X(x) \neq \theta_X(y)$. Then x and y are completely separated. By (ii), there exists a z-clopen subset $U = \cap[f^{-1}(\{0\}) : f \in H]$ of X containing x not containing y. Set $\widetilde{U} = \cap[\rho(f)^{-1}(\{0\}) : f \in H]$. Thus, $\widetilde{U}$ is a closed set of $\rho(X)$ containing $\theta_X(x)$ and not containing $\theta_X(y)$.

On the other hand, we have $\rho(X)\backslash\widetilde{U} = \theta_X(X\backslash U)$; since $X\backslash U$ is a z-closed subset of X, there exists a subset $H^{'}$ of $C(X)$ such that $X\backslash U = \cap[f^{-1}(\{0\}) : f \in H^{'}]$. Consequently, $\rho(X)\backslash\widetilde{U} = \theta_X(\cap[f^{-1}(\{0\}) : f \in H^{'}]) = \theta_X(\cap[\theta_X^{-1}(\rho(f)^{-1}(\{0\}) : f \in H^{'}] = \cap[\rho(f)^{-1}(\{0\}) : f \in H^{'}]$ is a closed set of $\rho(X)$; so that $\widetilde{U}$ is a clopen subset of $\rho(X)$ containing $\theta_X(x)$ and not containing $\theta_X(y)$. Therefore, $\rho(X)$ is totally disconnected. ■

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