A HUA TYPE THEOREM AND LINEAR PRESERVER PROBLEMS

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Abstract

In this paper we establish a Hua type theorem for generalized inverse (respectively, group inverses, Drazin inverses) in the context of Banach algebra. Furthermore, we provide a characterization of all the linear maps from a C^*-algebra of real rank zero onto a prime C^*-algebra that preserve the Moore-Penrose inverse, and illustrate its usefulness with an application to the algebra of bounded operators on a complex Hilbert space.

1. Introduction

A linear map $\phi : \mathcal{A} \to \mathcal{B}$ between two unital complex Banach algebras is called *Jordan homomorphism* if $\phi(a^2) = \phi(a)^2$ for every a in $\mathcal{A}$, or equivalently, $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for all a, b in $\mathcal{A}$. The class of Jordan homomorphisms includes obviously every homomorphism or anti-homomorphism (recall that ϕ is an anti-homomorphism if $\phi(ab) = \phi(b)\phi(a)$ for all a, b in $\mathcal{A}$). We say that ϕ is unital if $\phi(1) = 1$.

We denote by $\mathcal{A}^{-1}$ the set of invertible elements of $\mathcal{A}$.

In the last few decades a lot of work has been done on the so-called linear preservers problem. One of the most famous problems in this direction is Kaplansky's conjecture [18] : Let ϕ be a unital surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$ that preserves invertibility (i.e. $\phi(x) \in \mathcal{B}^{-1}$ whenever $x \in \mathcal{A}^{-1}$). Is it true that ϕ is a Jordan homomorphism?

For more details on this subject, we refer the reader to some survey articles [2; 3; 4; 21; 25; 27] and the references therein.

Recently, M. Mbekhta, L. Rodman and P. Šemrl [21], M. Mbekhta [22] studied bijective linear maps on $\mathcal{B}(H)$ that preserve generalized invertibility in both

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directions, respectively preserve the set of Fredholm operators on a complex Hilbert space.

An element $b \in \mathcal{A}$ is called a *generalized inverse* of $a \in \mathcal{A}$ if b satisfies the following two identities

$$aba = a \quad \text{and} \quad bab = b. \tag{1.1}$$

Let $\mathcal{A}^\wedge$ denote the set of all the elements of $\mathcal{A}$ having a generalized inverse. Then by Kaplansky's theorem [17], *if* $\mathcal{A} = \mathcal{A}^\wedge$ *then* $\mathcal{A}$ *is finite dimensional.* Furthermore, if $\mathcal{A}$ is semi-simple, then $\mathcal{A} = \mathcal{A}^\wedge \iff dim(A) < \infty$.

An element $b \in \mathcal{A}$ is called the *Drazin inverse* of $a \in \mathcal{A}$, if b is a solution of the following equations

$$ab = ba, \quad bab = b \quad \text{and} \quad a^k ba = a^k \quad \text{for some positive integer } k. \tag{1.2}$$

When $k = 1$, then we say that b is the *group inverse* of a. Notice that contrary to the generalized inverse, the Drazin inverse, and consequently the group inverse, is unique whenever it exists. Let $\mathcal{A}^D$ (resp. $\mathcal{A}^G$) denote the set of all the elements of $\mathcal{A}$ having Drazin inverse (resp. group inverse) and let a^D (resp. a^G) denote the Drazin inverse (resp. group inverse) of a for any $a \in \mathcal{A}^D$ (resp. $\mathcal{A}^G$). We have the following inclusions

$$\mathcal{A}^{-1} \subseteq \mathcal{A}^G \subseteq \mathcal{A}^\wedge \cap \mathcal{A}^D.$$

Moreover, in the context C^*-algebras, it is well known that every generalized invertible element a has a unique generalized inverse b for which ab and ba are projections, see [12]; such an element b is called the *Moore-Penrose inverse* of a and denoted by $a^\dagger$. In other words, $a^\dagger$ is the unique element of $\mathcal{A}$ that satisfies

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a. \tag{1.3}$$

Let $\mathcal{A}^\dagger$ denote the set of all elements of $\mathcal{A}$ having a Moore-Penrose inverse. Then we have $\mathcal{A}^{-1} \subseteq \mathcal{A}^\dagger$ and $\mathcal{A}^\dagger = \mathcal{A}^\wedge$.

For more details on the generalized inverse, Drazin inverse, group inverse and Moore-Penrose inverse we refer the reader to [10; 12; 13; 20; 23; 24] and the references therein.

It is the purpose of this paper to study the following :

Problem. *Let F be a given function on $\mathcal{D}_F$, a subset of $\mathcal{A}$. Characterize those linear maps ϕ that satisfy*

$$\phi(F(x)) = F(\phi(x)) \quad (x \in \mathcal{D}_F). \tag{1.4}$$

The motivation for this problem is the Hua theorem ensuring that every unital additive map ϕ between two fields such that $\phi(x^{-1}) = \phi(x)^{-1}$ is an isomorphism or an anti-isomorphism (see [15]). This result has been later extended to the algebra of matrices over some fields (see [11]).

The purpose of Section 2 is to establish that the continuous Jordan homomorphisms are the only linear unital maps between two unital Banach algebras that preserve 'strongly' generalized inverse (respectively, group inverses, Drazin inverses, inverses) (see Theorem 2.2) though some of its implications are purely algebraic (collected in Theorem 2.1).

In Section 3, we characterize the linear maps ϕ from a C^*-algebra of real rank zero onto a prime C^*-algebra $\mathcal{B}$ that give a positive solution to the above Problem when $\mathcal{D}_F = \mathcal{A}^\dagger$ and $F(x) = x^\dagger$, and we illustrate its usefulness with an application to the algebra of bounded operators on a complex Hilbert space.

2. A generalization of Hua theorem

Before formulating our results, we need some definitions.

We will say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$

- *preserves strongly generalized invertibility* if $\phi(y)$ is a generalized inverse of $\phi(x)$ whenever y is a generalized inverse of x.
- *preserves strongly group invertibility* if $\phi(x^G) = \phi(x)^G$ for all $x \in \mathcal{A}^G$.
- *preserves strongly Drazin invertibility* if $\phi(x^D) = \phi(x)^D$ for all $x \in \mathcal{A}^D$.
- *preserves strongly invertibility* if $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in \mathcal{A}^{-1}$.

The first result points out the purely algebraic part of the main result of this Section, which is Theorem 2.2

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be Jordan homomorphism map. The following assertions hold:*

(i) *ϕ preserves strongly generalized invertibility;*

(ii) *ϕ preserves strongly group invertibility;*

(iii) *ϕ preserves strongly Drazin invertibility;*
if, further, $\mathcal{A}$ and $\mathcal{B}$ have identity and ϕ is unital, then

(iv) *ϕ preserves strongly invertibility.*

Proof. It is well known that, if ϕ is a Jordan homomorphism, then for any x, y in $\mathcal{A}$ we have,

$$\phi(xyx) = \phi(x)\phi(y)\phi(x). \tag{2.1}$$

Now (i) follows from (2.1).

We claim that if $x = xyx$ and $xy = yx$, then

$$\phi(x)\phi(y) = \phi(y)\phi(x). \tag{2.2}$$

Indeed, since ϕ is a Jordan homomorphism, by (2.1), $\phi(x) = \phi(x)\phi(y)\phi(x)$. Denote $e = \phi(x)\phi(y)$ and $f = \phi(y)\phi(x)$. Then e and f are idempotents in $\mathcal{B}$. We have

$$\begin{aligned} ef &= \phi(x)\phi(y)\phi(y)\phi(x) = \phi(x)\phi(y)^2\phi(x) = \phi(x)\phi(y^2)\phi(x) = \phi(xy^2x) \\ &= \phi(yxyx) = \phi(yx), \end{aligned}$$

also, we have

$$\begin{aligned} fe &= \phi(y)\phi(x)\phi(x)\phi(y) = \phi(y)\phi(x)^2\phi(y) = \phi(y)\phi(x^2)\phi(y) = \phi(yx^2y) \\ &= \phi(yxyx) = \phi(yx). \end{aligned}$$

Thus $\quad ef = fe = \phi(yx)$.

On the other hand, since ϕ is a Jordan homomorphism, and y commutes with x, we have

$$\phi(x)\phi(xy) + \phi(xy)\phi(x) = 2\phi(x).$$

Now, left multiplication by $\phi(xy)$ gives

$$\phi(xy)\phi(x) = \phi(x);$$

right multiplication by $\phi(xy)$ gives

$$\phi(x)\phi(xy) = \phi(x)$$

Thus,

$$\phi(xy)\phi(x) = \phi(x)\phi(xy) = \phi(x).$$

Therefore,

$$fef = f\phi(yx) = \phi(y)\phi(x)\phi(yx) = \phi(y)\phi(x) = f,$$

and

$$efe = \phi(yx)e = \phi(yx)\phi(x)\phi(y) = \phi(x)\phi(y) = e.$$

It follows that $fef = f$ and $efe = e$. Thus, since e and f commute,

$$e = efe = fe = fef = f.$$

Consequently, $\phi(x)\phi(y) = \phi(y)\phi(x)$ and (2.2) is proved.

Now, (ii) follows from (2.1) and (2.2).

Suppose that y is the Drazin inverse of x. Then by (2.1) and (2.2), we have $\phi(y)\phi(x)\phi(y) = \phi(y)$, and $\phi(x)\phi(y) = \phi(y)\phi(x)$. It remains to prove that $\phi(x)^k\phi(y)\phi(x) = \phi(x)^k$.

Observe that by elementary verification, we have

$$2x^k yx = (x^k + x)y(x^k + x) - x^k yx^k - xyx.$$

Thus, since ϕ is a Jordan homomorphism, it follows from (2.1) and (2.2)

$$\begin{aligned} 2\phi(x^k yx) &= \phi(x^k + x)\phi(y)\phi(x^k + x) - \phi(x^k)\phi(y)\phi(x^k) - \phi(xyx) \\ &= 2\phi(x^k)\phi(y)\phi(x). \end{aligned}$$

Consequently, $\phi(x)^k = \phi(x^k yx) = \phi(x)^k\phi(y)\phi(x)$, and (iii) is proved.

The proof that (iv) holds can be checked by direct calculation. ■

In the following theorem we establish an extension of Hua's theorem to the case of Banach algebras and characterize the Jordan homomorphisms in terms of generalized inverse, group inverse and Drazin inverse.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear unital continuous map. Then the following conditions are equivalent:*

(i) *ϕ preserves strongly generalized invertibility;*

(ii) *ϕ preserves strongly group invertibility;*

(iii) *ϕ preserves strongly Drazin invertibility;*

(iv) *ϕ preserves strongly invertibility;*

(v) *ϕ is a Jordan homomorphism.*

Proof. The implications (v) $\Longrightarrow$ (i), (ii), (iii) and (iv), follows from the above theorem.

Let us prove first the implication (i) $\Longrightarrow$ (v). Suppose that (i) holds, and let x be an arbitrary element of $\mathcal{A}$. Then for every $\lambda \in \mathbb{C}$, the element $a(\lambda, x) = \exp(\lambda x)$ is invertible with the inverse $b(\lambda, x) = \exp(-\lambda x)$, and so

$$\phi(\exp(\lambda x))\phi(\exp(-\lambda x))\phi(\exp(\lambda x)) = \phi(\exp(\lambda x)).$$

Hence by the continuity of ϕ we get

$$\left[\sum_{n\geq 0}\frac{\phi(x^n)}{n!}\lambda^n\right]\left[\sum_{n\geq 0}\frac{(-1)^n\phi(x^n)}{n!}\lambda^n\right]\left[\sum_{n\geq 0}\frac{\phi(x^n)}{n!}\lambda^n\right]=\sum_{n\geq 0}\frac{\phi(x^n)}{n!}\lambda^n.$$

Now a simple identification of the coefficients of λ^2 in the above equation leads to the following identity

$$\phi(x^2)=\phi(x)^2,$$

which proves that ϕ is a Jordan homomorphism.

The rest of the proof runs as before. ■

Remark 2.3. It would be interesting to know if $\mathcal{A}$ and $\mathcal{B}$ are semi-simples, then Theorem 2.2 holds without the continuity assumption, i.e. the converse of Theorem 2.1 in general.

As an application of the above theorem in the context of the Banach algebra of bounded linear operator on complex Banach space, we derive the following result, which characterizes the unital bijective continuous linear maps from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ that preserve strongly generalized invertibility (see 27; 28).

Corollary 2.4. *Let X and Y be Banach spaces and let $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a unital bijective continuous linear map. Then the following conditions are equivalent:*

(i) *ϕ preserves strongly generalized invertibility;*

(ii) *ϕ preserves strongly group invertibility;*

(iii) *ϕ preserves strongly Drazin invertibility;*

(iv) *ϕ preserves strongly invertibility;*

(v) *ϕ is a Jordan isomorphism;*

(vi) *either*

(a) *there is an isomorphism A from X onto Y such that*

$$\phi(T)=ATA^{-1} \quad \text{for all} \quad T\in\mathcal{B}(X), \text{ or}$$

(b) *there is an isomorphism B from Y onto X^* such that*

$$\phi(T)=B^{-1}T^*B \quad \text{for all} \quad T\in\mathcal{B}(X).$$

In this case X and Y must be reflexive.

For the special case of the complex matrix algebra $\mathcal{A}=\mathcal{M}_n(\mathbb{C})$, we derive the following corollary that provides a more explicit form of the linear maps preserving strongly generalized invertibility.

Following [16] we say that $\phi : \mathcal{A} \to \mathcal{B}$ is the *sum of a homomorphism and anti-homomorphism* if there exist two idempotents e and f in the center of $\mathcal{B}$, such that $1 = e + f$, $ef = fe = 0$ and $\phi = \phi_1 + \phi_2$ where $\phi_1(x) = \phi(x)e$ is a homomorphism and $\phi_2(x) = \phi(x)f$ is an anti-homomorphism.

Corollary 2.5. *Let $\mathcal{B}$ be any unital Banach algebra. Then $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{B}$ is a linear unital map preserving strongly generalized inverses (respectively, group inverses, Drazin inverses, inverses) if and only if ϕ is the sum of a homomorphism and an anti-homomorphism.*

Furthermore, if 0 and 1 are the only central idempotents in $\mathcal{B}$ (in particular, if $\mathcal{B}$ is prime, i.e $a\mathcal{B}b = \{0\}$ implies that $a = 0$ or $b = 0$), then ϕ is either a homomorphism or an anti-homomorphism.

Proof. Since $\mathcal{M}_n(\mathbb{C})$ is finite dimensional, ϕ is automatically continuous, and so by the previous theorem, it is a Jordan homomorphism. Hence [16, theorem 7] ensures that ϕ is the sum of a homomorphism and an anti-homomorphism.
The converse follows from a simple computation.
Finally, if 0 and 1 are the only central idempotents in $\mathcal{B}$, then ϕ is either a homomorphism or an anti-homomorphism. This completes the proof. ∎

For the particular case of the matrix algebras, and ϕ surjective we get the following result

Corollary 2.6. *Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, be a surjective, unital linear map. Then ϕ preserves strongly generalized inverses (respectively, group inverses, Drazin inverses, inverses) if and only if ϕ takes one of the following forms:*

$$\phi(x) = axa^{-1} \quad \text{or} \quad \phi(x) = ax^{tr}a^{-1},$$

for some invertible element $a \in \mathcal{M}_n(\mathbb{C})$.
Here, x^{tr} denotes the transpose of a matrix x.

Proof. This follows from above corollary together with the classical result that every automorphism (respectively, anti-automorphism) of $\mathcal{M}_n(\mathbb{C})$ is inner. ∎

Remark 2.7. The conclusion of Corollary 2.5 and Corollary 2.6 has been obtained respectively by A.R. Sourour ([27; 28]) and J. Dieudonné ([9]), Marcus and Purves

([19]) under a weaker assumption that states that 'ϕ preserves invertibility'. Note that this corollary leads to several different conditions equivalent to their results. In particular, if ϕ is as in the Corollary 2.5 or Corollary 2.6, then the apparently weaker condition 'ϕ preserves invertibility' is actually equivalent to the strongler condition 'ϕ preserves strongly invertibility'.

Remark 2.8. In [6; 7], the authors have been considering linear maps preserving strongly generalized inverse (respectively, group inverses, Drazin inverses) on the algebra of matrices over some fields or rings.

Remark 2.9. In the context of the Kaplansky conjecture (i.e. $\mathcal{A}, \mathcal{B}$ semi-simples and ϕ surjective), observe that the condition (iv) of Theorem 2.2 and [1, theorem 5.5.2] imply that ϕ is automatically continuous. Hence Theorem 2.2 shows that the conjecture of Kaplansky reduces just to the question if his assumption of preserving invertibility actually implies the strong preserving invertibility (as it is in the matrix case mentioned above in Remark 2.7).

3. Linear maps preserving the Moore-Penrose inverses

Throughout this section, $\mathcal{A}$ and $\mathcal{B}$ will denote a C^*-algebras, and we will say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is *C^*-Jordan homomorphism* if it is a Jordan homomorphism that preserves the adjoint operation, i.e. $\phi(x^*) = \phi(x)^*$ for all x in $\mathcal{A}$. The C^*-homomorphism and C^*-anti-homomorphism are analogously defined.

We will say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ *preserves strongly Moore-Penrose invertibility* if $\phi(x^\dagger) = \phi(x)^\dagger$ for all $x \in \mathcal{A}^\dagger$.

Let us introduce the following notion that will play a crucial role in this section. We will say that a C^*-algebra is *of real rank zero* if the set formed by all the real linear combinations of (orthogonal) projections is dense in the set of self-adjoint elements of $\mathcal{A}$ (see [5]). It is well known that this property is satisfied by every von Neumann algebra, and in particular by the C^*-algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H ([5]).

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a linear unital continuous map that preserves strongly the Moore-Penrose invertibility. The following assertions hold:*

(i) *ϕ is a Jordan homomorphism;*

(ii) *ϕ preserves the set of projections (i.e. ϕ maps projections in $\mathcal{A}$ into projections in $\mathcal{B}$).*

PROOF. We prove that ϕ is a Jordan homomorphism. For $\lambda \in \mathbb{C}$ and $x \in \mathcal{A}$, consider the element $y(\lambda, x) = \exp(\lambda x)$ in $\mathcal{A}$. Then $y(\lambda, x)^\dagger = \exp(-\lambda x)$. By the same arguments as in the prove of the Theorem 2.2, we get that ϕ is a Jordan homomorphism, and (i) is proved.
(ii) Let e be an idempotent such that $e^* = e$, in particular we have $e = e^\dagger$ and so

$$\phi(e) = \phi(e^\dagger) = \phi(e)^\dagger.$$

Moreover, because ϕ is a Jordan homomorphism, $\phi(e)$ is an idempotent, and since $\phi(e)\phi(e)^\dagger$ is a projection, we obtain

$$\phi(e) = \phi(e)\phi(e)^\dagger = (\phi(e)\phi(e)^\dagger)^* = \phi(e)^*.$$

Thus $\phi(e)$ is a projection in $\mathcal{B}$, as desired. ■

In the following theorem, we characterize the linear maps preserving strongly Moore-Penrose invertibility when, in addition, $\mathcal{A}$ is of real rank zero and $\mathcal{B}$ is a prime algebra.

Theorem 3.2. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero and $\mathcal{B}$ a prime C^*-algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective, unital and continuous linear map. Then the following conditions are equivalent:*

(i) *ϕ preserves strongly Moore-Penrose invertibility;*
(ii) *ϕ is either a C^*-homomorphism or a C^*-anti-homomorphism.*

PROOF. (i) $\Longrightarrow$ (ii). It follows by the above proposition that ϕ is a Jordan homomorphism and preserves the set of projections. Hence a standard argument shows that ϕ preserves also the orthogonality between projections, i.e. if e and f are mutually orthogonal projections in $\mathcal{A}$, $(ef = 0)$, and then $\phi(e)$ and $\phi(f)$ are mutually orthogonal projections in $\mathcal{B}$.
We claim that $\phi(x^*) = \phi(x)^*$ for every x in $\mathcal{A}$. To this end, let $h = h^*$ be an arbitrary self-adjoint element in $\mathcal{A}$. Since $\mathcal{A}$ is a C^*-algebra of real rank zero,

$$h = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i e_i,$$

where $\sum_{i=1}^{n} \lambda_i e_i$ a finite real linear combination of mutually orthogonal projections. Therefore by the continuity of ϕ we obtain

$$\phi(h)^* = \lim_{n\to\infty} (\sum_{i=1}^{n} \lambda_i \phi(e_i))^* = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i \phi(e_i^*) = \lim_{n\to\infty} \phi(\sum_{i=1}^{n} \lambda_i e_i^*) = \phi(h^*).$$

Hence $\phi(h)^* = \phi(h^*)$. Now, since every element in $\mathcal{A}$ is a linear combination of two self-adjoint elements, we get that

$$\phi(x^*) = \phi(x)^* \quad \text{for all} \quad x \in \mathcal{A}.$$

Thus ϕ preserves the adjoint. Therefore, ϕ is C^*-Jordan homomorphism.
On the other hand, because $\mathcal{B}$ is a prime algebra, then every Jordan homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ is a homomorphism or an anti-homomorphism (see [14]). Consequently ϕ is either a C^*-homomorphism or a C^*-anti-homomorphism.
(ii) $\Longrightarrow$ (i). Without loss of generality we can suppose that ϕ is a C^*-homomorphism. From (2.1), it is easy to see for every Moore-Penrose invertible element x,

$$\phi(x)\phi(x^\dagger)\phi(x) = \phi(x) \quad \text{and} \quad \phi(x^\dagger)\phi(x)\phi(x^\dagger) = \phi(x^\dagger),$$

and moreover

$$(\phi(x)\phi(x^\dagger))^* = (\phi(xx^\dagger))^* = \phi((xx^\dagger)^*) = \phi(xx^\dagger) = \phi(x)\phi(x^\dagger).$$

Thus $\phi(x)\phi(x^\dagger)$ is a projection, and by the same argument we get that $\phi(x^\dagger)\phi(x)$ is also a projection. This shows that $\phi(x^\dagger)$ is the Moore-Penrose inverse of $\phi(x)$, which completes the proof. ■

As an application of the above theorem in the context of the C^*-algebra $\mathcal{B}(H)$ of bounded linear operator on complex separable Hilber space, we derive the following result, which characterizes the bijective unital continuous linear map from $\mathcal{B}(H)$ onto itself that preserve strongly Moore-Penrose inverse.

Let $\mathcal{B}^\dagger(H)$ denote the set of the operators on H that possess a Moore-Penrose inverse.

Theorem 3.3. *Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a bijective unital continuous linear map. Then the following conditions are equivalent:*

(i) $\phi(T^\dagger) = \phi(T)^\dagger$ *for all* $T \in \mathcal{B}^\dagger(H)$;

(ii) *there is a unitary operator U in $\mathcal{B}(H)$ such that ϕ takes one of the following forms:*

$$\phi(T) = UTU^* \quad \text{or} \quad \phi(T) = UT^{tr}U^* \quad \text{for all} \quad T,$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal base of H.

Proof. Suppose that (i) holds. Since $\mathcal{B}(H)$ is a prime C^*-algebra of real rank zero, the above theorem implies that ϕ is either a C^*-automorphism or a C^*-anti-automorphism. Now by the fundamental isomorphism theorem [26, theorem 2.5.19; see also [8]], there exists an invertible operator $A \in \mathcal{B}(H)$ such that either $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(H)$ or $\phi(T) = AT^{tr}A^{-1}$ for every $T \in \mathcal{B}(H)$.

Let us establish that A is unitary. Without loss of generality we can assume that our map ϕ is an automorphism. Therefore for every $T \in \mathcal{B}(H)$, $\phi(T) = ATA^{-1}$, and since ϕ preserves adjoint, we have $ATA^{-1} = A^{*-1}TA^*$. Hence it follows that

$$A^*AT = TA^*A \quad \text{for all } T \in \mathcal{B}(H).$$

Consequently, A^*A is in the center of $\mathcal{B}(H)$. Since A is invertible and ϕ is unital, $A^*A = I$, which implies that A is unitary. Thus ϕ is of the desired form, and (ii) is proved.

The converse can be checked straightforwardly. ■

Remark 3.4. In [29] the authors give a characterization of the linear maps on the algebra of matrices over some fields that preserve Moore-Penrose inverse.

In connection with Theorem 3.2, we conclude the paper by the following conjecture

Conjecture 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective linear map. Then the following conditions are equivalent:*

(i) *ϕ preserves strongly Moore-Penrose invertibility;*

(ii) *ϕ is either a C^*-homomorphism or a C^*-anti-homomorphism.*

ADDED IN PROOF: The reader can find further results, in some cases with weaker hypothesis than in the present paper, in ‘Additive maps preserving strongly generalized inverses’ by N. Boudi and the author, to appear in the *Journal of Operator Theory.*

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References

[1] B. Aupetit, *A primer on spectral theory*, Springer-Verlag, New York, 1991.

[2] B. Aupetit, Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras, *The Journal of the London Mathematical Society* **62** (2000), 917–24.

[3] B. Aupetit, Sur les transformations qui conservent le spectre, in *Banach Algebra'97*, 55–78, Walter de Gruyter, Berlin, 1998.

[4] M. Brešar and P. Šemrl, Linear preservers on $\mathcal{B}(X)$, *Banach Center Publications* **38** (1997), 49–58.

[5] L.G. Brown and G.K. Pedersen, C^*-algebras of real rank zero, *Journal of Functional Analysis* **99** (1991), 131–49.

[6] C. Bu, Linear maps preserving Drazin inverses of matrices over fields, *Linear Algebra and its Applications* **396** (2005), 159–73.

[7] C. Cao, X. Zhang, Linear preservers between matrix modules over connected commutative rings, *Linear Algebra and its Applications* **397** (2005), 355–66.

[8] P.R. Chernoff, Representations, automorphisms, and derivations of some operator algebras, *Journal of Functional Analysis* **12** (1973), 257–89.

[9] J. Dieudonné, Sur une généralisation du groupe orthogonal à quatre variables, *Archiv der Mathematik* (Basel) **1** (1949), 282–7.

[10] M. P. Drazin, Pseudo-inverse in associative rings and semigroups, *The American Mathematical Monthly* **65** (1958), 506–14.

[11] H. Essannouni, A. Kaidi Le théorème de Hua pour les algèbres artiniennes simples, *Linear Algebra and its Applications***297** (1999), 9–22.

[12] R. Harte, M. Mbekhta, On generalized inverses in C*-algebras, *Studia Mathematica***103** (1992), 71–7.

[13] R. Harte and, M. Mbekhta, Generalized inverses in C*-algebras II, *Studia Mathematica* **10** (1993), 129–38.

[14] I.N. Herstein, Jordan homomorphisms, *Transactions of the American Mathematical Society* **81** (1956), 331–41.

[15] L.K. Hua, On the automorphisms of a Sfield, *Proceedings of the National Academy of Sciences of the United States of America* **35** (1949), 386–9.

[16] N. Jacobson and C.E. Rickart, Jordan homomorphism of rings, *Transactions of the American Mathematical Society* **69** (1950), 497–502.

[17] I. Kaplansky, Regular Banach algebras, *The Journal of the Indian Mathematical Society* **12** (1948), 57–62.

[18] I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, American Mathematical Society, Providence, 1970.

[19] M. Marcus and R. Purves, Linear transformations on algebras of matrices: the invariance of the elementary symmetric functions, *Canadian Journal of Mathematics* **11** (1959), 383–96.

[20] M. Mbekhta, Conorme et inverse généralisé dans les C*-algèbres, *Canadian Mathematical Bulletin* **35** (4) (1992) 515–22

[21] M. Mbekhta, L. Rodman and P. Šemrl, Linear maps preserving generalized invertibility, *Integral Equations and Operator Theory* **55** (2006), 93–109.

[22] M. Mbekhta, Linear maps preserving a set of Fredholm operators, *Proceedings of the American Mathematical Society* **135** (2007), 3613–19.

[23] M.Z. Nashed (ed.), *Generalized inverses and applications*, Academic Press, New York – London, 1976.

[24] R. Penrose, A generalized inverse for matrices, *Proceedings of the Cambridge Philosophical Society* **51** (1955), 406–13.

[25] S. Pierce, M.H. Lim, R. Loewy, C.K. Li, N.K. Tsing and L. Beasley, A survey of linear preserver problems, *Linear and Multilinear Algebra* **33** (1992), 1–192.

[26] C.E. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, 1960.

[27] A.R. Sourour, Invertibility preserving linear maps on $\mathcal{L}(X)$, *Transactions of the American Mathematical Society* **348** (1996), 13–30.

[28] A.R. Sourour, *The Gleason-Kahane-Żelazko theorem and its generalizations*, Banach Center Publications **30** (1994), 327–31.

[29] X. Zhang, C. Cao and C. Bu, Additive maps preserving M-P inverses of matrices over fields, *Linear and Multilinear Algebra* **46** (1999), 199–211.